Differential and Integral Equations Volume 22, Numbers 11-12 (2009), 1125–1139

GENERAL DECAY OF SOLUTIONS OF A NONLINEAR TIMOSHENKO SYSTEM WITH A BOUNDARY CONTROL OF MEMORY TYPE

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(Submitted by: Viorel Barbu)

Abstract. In this paper, we consider a nonlinear Timoshenko system, in a bounded domain, where the memory-type damping is acting on a part of the boundary. We establish a general decay result, from which the usual exponential and polynomial decay rates are only special cases. Our work allows certain relaxation functions which are not necessarily of exponential or polynomial decay and, therefore, generalizes and improves earlier results in the literature.

1. Introduction

In [12], Messaoudi and Soufyane considered

$$\rho_1(x)u_{tt} = \Delta u + \alpha \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} - \beta u - a(x) f_1(u,v), \quad \text{in} \quad \Omega \times \mathbb{R}^+ \tag{1.1}$$

$$\rho_2(x)v_{tt} = \Delta v - \alpha \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} - a(x) f_2(u,v), \quad \text{in} \quad \Omega \times \mathbb{R}^+ \tag{1.2}$$

$$u(x,t) = v(x,t) = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \tag{1.3}$$

Accepted for publication: February 2009.
AMS Subject Classifications: 35L75, 37B25, 93D20.

$$u(x,t) = -\int_0^t g_1(t-s)\frac{\partial u}{\partial \nu}(s)ds \quad on \;\; \Gamma_1 \times \mathbb{R}^+, \tag{1.4}$$

$$v(x,t) = -\int_0^t g_2(t-s)\frac{\partial v}{\partial \nu}(s)ds \quad on \;\; \Gamma_1 \times \mathbb{R}^+, \tag{1.5}$$

$$(u(0), v(0)) = (u^0, v^0) \text{ and } (\sqrt{\rho_1}u_t(0), \sqrt{\rho_2}v_t(0)) = (\sqrt{\rho_1}u^1, \sqrt{\rho_2}v^1), \tag{1.6}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint, with $\text{meas}(\Gamma_0) > 0$, ν is the unit outer normal, u represents the deflection of the beam from its equilibrium and v is the total rotatory angle of the beam (see [22]), and $g_i (i = 1, 2)$ are the relaxation functions considered positive and nonincreasing belonging to $W^{1,2}(0, +\infty)$. Equations (1.4) and (1.5) are the nonlocal boundary conditions responsible for the memory effect. They established a general decay result, which depends on the value of (u^0, v^0) on Γ_0 and the rate of the decay of the kernels k_1 and k_2.

Stabilization of Timoshenko systems or coupled wave systems by frictional boundary damping or by feedback of memory type has also attracted the attention of many researchers. Different mechanisms have been utilized to stabilize such systems and several decay and stability results have been obtained. In this regard, we mention, among many others, the work of Ammar-Khodja et al. [1], Cavalcanti *et al.* ([3] , [4]), De Lima Santos [5], Guesmia and Messaoudi [6], Gugat [7], Kim and Renardy [8], Komornik and Rao [9], Krabs et al. ([10], [11]), Munoz Rivera and Racke [14], Raposo *et al.* [15], Shi and Feng ([16], [17], [18]), Soufyane et al. ([19], [20]), and Taylor [21].

In this paper we consider the system (1.1)-(1.6), under some conditions on a, α, β, f_1, and f_2. Precisely, we suppose that α is a small positive number, $\beta > n\alpha$, $a : \Omega \to \mathbb{R}^+$, and the functions f_i are in $C^1(\mathbb{R})$ (for $i = 1, 2$). We also assume that there exists a function $F(u, v)$ such that

$$f_1(u,v) = \frac{\partial F}{\partial u}, \quad f_2(v,u) = \frac{\partial F}{\partial v}, \tag{1.7}$$

satisfying the following conditions

$$F \geq 0, \quad uf_1(u,v) + vf_2(v,u) - (1+b)F(u,v) \geq 0, \text{ for } b > 1, \tag{1.8}$$

and

$$|F(\xi, \varsigma)| \leq d(|\xi|^p + |\varsigma|^p), \qquad \forall(\xi, \varsigma) \in \mathbb{R}^2, \tag{1.9}$$

for some constant $d > 0$ and $p \geq 1$ such that $(n-2)p \leq n$.

We assume that $\rho_i \in C^1(\overline{\Omega})$ is positive for $i = 1, 2$, and satisfies the following hypothesis:

$$\nabla \rho_i . m \geq 0 \quad \text{for } i = 1, 2, \tag{1.10}$$

where $m(x) = x - x^0$, $x^0 \in \mathbb{R}^n$. Also we shall assume that

$$\Gamma_0 = \{x \in \partial\Omega : \nu . m(x) \leq 0\}, \quad \Gamma_1 = \{x \in \partial\Omega : \nu . m(x) > 0\}.$$

Note that because of condition (1.3) the solution of the system (1.1)-(1.6) must belong to the following space:

$$V = \left\{v \in H^1(\Omega) : v = 0 \ on \ \Gamma_0\right\}.$$

This work is divided into three sections. In section two we state, without proof, an existence result of solutions to system (1.1)-(1.6) and present some material needed for the proof of our main result. In particular, we establish some relations between the relaxation function $g_i (i = 1, 2)$ and the corresponding resolvent kernel k_i $(i = 1, 2)$ similar, but more general than, those usually found in the literature. In section three, our main result is stated and proved. Our argument is close to the one in [13] with the necessary modifications required by the nature of our problem.

2. Notation and Existence

In this section, we introduce some notation and we study the existence of regular and weak solutions to system (1.1)-(1.6). First, we will use equations (1.4) and (1.5) to estimate the boundary terms $\frac{\partial u}{\partial \nu}$ and $\frac{\partial v}{\partial \nu}$.

Defining the convolution product operator by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds,$$

and differentiating equation (1.4) we obtain

$$\frac{\partial u}{\partial \nu} + \frac{1}{g_1(0)}(g_1' * \frac{\partial u}{\partial \nu}) = -\frac{1}{g_1(0)} u_t \quad on \ \ \Gamma_1 \times \mathbb{R}^+.$$

Applying Volterra's inverse operator, we get

$$\frac{\partial u}{\partial \nu} = -\frac{1}{g_1(0)}(u_t + k_1 * u_t) \quad on \ \ \Gamma_1 \times \mathbb{R}^+,$$

where the resolvent kernel k_1 satisfies

$$k_1 + \frac{1}{g_1(0)}(g_1' * k_1) = -\frac{1}{g_1(0)} g_1' \quad on \ \ \Gamma_1 \times \mathbb{R}^+.$$

Denoting by $\eta_1 = \frac{1}{g_1(0)}$, we arrive at

$$\frac{\partial u}{\partial \nu} = -\eta_1(u_t + k_1(0)u - k_1(t)u^0 + k_1' * u) \quad on \ \ \Gamma_1 \times \mathbb{R}^+. \tag{2.1}$$

A similar procedure leads to

$$\frac{\partial v}{\partial \nu} = -\eta_2(v_t + k_2(0)v - k_2(t)v^0 + k_2' * v) \quad on \ \ \Gamma_1 \times \mathbb{R}^+, \tag{2.2}$$

where $\eta_2 = \frac{1}{g_2(0)}$.

Reciprocally, taking, in a natural way, the initial data $u^0 = v^0 = 0$ on Γ_1, the identities (2.1) and (2.2) imply (1.4) and (1.5).

Since we are interested in relaxation functions of exponential or polynomial type and identities (2.1)-(2.2) involve the resolvent kernels k_i $(i = 1, 2)$, we want to know if k_i has the same properties. The following lemma answers this question.

Let $h : [0, +\infty) \to \mathbb{R}_+$ be continuous. Let k be its resolvent; that is,

$$k(t) = h(t) + (k * h)(t). \tag{2.3}$$

It is well known that k is continuous and positive [3].

Lemma 2.1. (See [13].) *Suppose that* $h(t) \le c_0 e^{-\int_0^t \gamma(\zeta) d\zeta}$ *for* $\gamma : [0, +\infty) \to \mathbb{R}_+$, *a nonincreasing function satisfying, for some positive constant* $\varepsilon < 1$,

$$c_1 = \int_0^{+\infty} e^{-\int_0^s (1-\varepsilon)\gamma(\zeta) d\zeta} ds < \frac{1}{c_0}.$$

Then k *satisfies*

$$k(t) \le \frac{c_0}{1 - c_0 c_1} e^{-\varepsilon \int_0^t \gamma(\zeta) d\zeta}.$$

Basing our argument on Lemma 2.1, we will use (2.1)-(2.2) instead of (1.4)-(1.5). We then define

$$(g \ o \ \varphi)(t) := \int_0^t g(t-s) \left|\varphi(t) - \varphi(s)\right|^2 ds, \tag{2.4}$$

and

$$(g \ \odot \ \varphi)(t) := \int_0^t g(t-s)(\varphi(t) - \varphi(s)) ds. \tag{2.5}$$

By using Hölder's inequality for $0 \le \mu \le 1$, we have

$$\left|(g \ \odot \ \varphi)(t)\right|^2 \le \Big(\int_0^t \left|g(s)\right|^{2(1-\mu)} ds \Big) (\left|g\right|^{2\mu} o\varphi)(t). \tag{2.6}$$

Lemma 2.2. (see [3], [4]). *If* $g, \varphi \in C^1(R^+)$, *then*

$$(g * \varphi)\varphi_t = -\frac{1}{2}g(t)\left|\varphi(t)\right|^2 + \frac{1}{2}g^{'} o\ \varphi - \frac{1}{2}\frac{d}{dt}\Big(g\ o\ \varphi - \Big(\int_0^t g(s)ds\Big)\left|\varphi(t)\right|^2\Big). \tag{2.7}$$

The well posedness of system (1.1)-(1.6) is presented in the following theorem.

Theorem 2.3. *Let* $k_i \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $\rho_i \in C^1(\overline{\Omega})$, $i = 1, 2$. *Assume that* $(u^0, v^0) \in (H^2(\Omega) \cap V)^2$ *and* $(u^1, v^1) \in (V)^2$ *with*

$$\begin{cases} \dfrac{\partial u^0}{\partial \nu} + \eta_1 u^1 = 0 \quad \text{on } \Gamma_1 \\ \dfrac{\partial v^0}{\partial \nu} + \eta_2 v^1 = 0 \quad \text{on } \Gamma_1. \end{cases} \tag{2.8}$$

Then there exists a unique strong solution (u, v) *of the system* (1.1)-(1.6) *such that*

$$u, v \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap V), \sqrt{\rho_1}u_t, \sqrt{\rho_2}v_t \ \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \tag{2.9}$$

$$u_t, v_t \in L^\infty(\mathbb{R}^+; V), \sqrt{\rho_1}u_{tt}, \sqrt{\rho_2}v_{tt} \ \in L^\infty(\mathbb{R}^+; L^2(\Omega)). \tag{2.10}$$

Proof. This theorem can be proved, using the Galerkin method and following exactly the procedure of [4].

3. Decay of solutions

In this section, we study the asymptotic behavior of the solutions of system (1.1)-(1.6) when the resolvent kernel k_i $(i = 1, 2)$ satisfies

$$k_i(0) > 0, \quad k_i(t) \geq 0, \quad k_i'(t) \leq 0, \quad k_i^{''}(t) \geq \gamma_i(t)(-k_i'(t)), \tag{3.1}$$

where $\gamma_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the following conditions

$$\gamma_i(t) > 0, \quad \gamma_i'(t) \leq 0, \qquad \text{and} \int_0^{+\infty} \gamma_i(t)dt = +\infty. \tag{3.2}$$

Assume that $\alpha < 1$ and $\beta > n\alpha$. We define the first-order energy of system (1.1)-(1.6) by

$$\begin{aligned} E(t) := \frac{1}{2}\int_\Omega (\rho_1 \left|u_t\right|^2 + (\beta - n\alpha)\left|u\right|^2 + \left|\nabla u\right|^2)dx + \int_\Omega a(x)F(u, v)dx \\ + \frac{1}{2}\int_\Omega \rho_2 \left|v_t\right|^2 dx + \frac{\alpha}{2}\sum_{i=1}^n \int_\Omega \Big|\frac{\partial v}{\partial x_i} - u\Big|^2 dx + \frac{(1-\alpha)}{2}\int_\Omega \left|\nabla v\right|^2)dx \end{aligned} \tag{3.3}$$

$$+\frac{\eta_1}{2}\int_{\Gamma_1}(k_1(t)\left|u\right|^2-k_1'\ o\ u)d\Gamma_1+\frac{\eta_2}{2}\int_{\Gamma_1}(k_2(t)\left|v\right|^2-k_2'\ o\ v)d\Gamma_1.$$

Theorem 3.1. *Given* $((u^0,u^1),(v^0,v^1))\in(V\times L^2(\Omega))^2$, *assume that* (1.7), (1.8), (1.9), (1.10), *and* (3.1) *hold, with*

$$\lim_{t\to\infty}k_i(t)\ =0. \tag{3.4}$$

Assume further that

$$((-n+\varepsilon_0)b+n+\varepsilon_0)a+2m.\nabla a<0,\quad\forall x\in\Omega. \tag{3.5}$$

Then, for some t_0 *large enough, we have*

$$\begin{cases} E(t)\le cE(0)e^{-c_1\int_0^t\gamma_1(s)ds} & If\quad \gamma_1(t)\le\gamma_2(t),\quad\forall t\ge t_0\\ & or\\ E(t)\le cE(0)e^{-c_1\int_0^t\gamma_2(s)ds} & If\quad \gamma_2(t)\le\gamma_1(t),\quad\forall t\ge t_0,\end{cases} \tag{3.6}$$

if $(u^0,v^0)=(0,0)$ on Γ_1, otherwise $(If\ (u^0,v^0)\neq(0,0)\ on\ \Gamma_1)$

$$\begin{cases} E(t)\le c\left\{\begin{array}{c}E(0)+\int_0^t[\int_{\Gamma_1}\left|u^0\right|^2d\Gamma_1.k_1^2(s)\\ +\int_{\Gamma_1}\left|v^0\right|^2d\Gamma_1.k_2^2(s)][1+e^{c_1\int_{t_0}^s\gamma_1(\zeta)d\zeta}]ds\end{array}\right\}e^{-c_1\int_0^t\gamma_1(s)ds},\\ \qquad\qquad\text{If }\gamma_1(t)\le\gamma_2(t),\ \forall t\ge t_0\text{ or}\\ E(t)\le c\left\{\begin{array}{c}E(0)+\int_0^t[\int_{\Gamma_1}\left|u^0\right|^2d\Gamma_1.k_1^2(s)\\ +\int_{\Gamma_1}\left|v^0\right|^2d\Gamma_1.k_2^2(s)][1+e^{c_1\int_{t_0}^s\gamma_2(\zeta)d\zeta}]ds\end{array}\right\}e^{-c_1\int_0^t\gamma_2(s)ds},\\ \qquad\qquad\text{If }\ \gamma_2(t)\le\gamma_1(t),\ \forall t\ge t_0,\end{cases} \tag{3.7}$$

where c_1 *is a fixed positive constant and* c *is a generic positive constant.*

Proof. The main idea is to construct a Lyapunov functional $\mathcal{L}(t)$ equivalent to $E(t)$. To do this we use the multiplier techniques. The proof of Theorem 3.1 will be achieved with the help of two lemmas.

Lemma 3.2. *Under the assumptions of Theorem* 3.1, *the energy of the solution of* (1.1)–(1.6) *satisfies*

$$\begin{aligned}\frac{dE}{dt}\le&-\frac{\eta_1}{2}\int_{\Gamma_1}\left|u_t\right|^2d\Gamma_1+\frac{\eta_1}{2}\int_{\Gamma_1}k_1^2(t)\left|u^0\right|^2d\Gamma_1-\frac{\eta_1}{2}\int_{\Gamma_1}k_1''oud\Gamma_1\\ &-\frac{\eta_2}{2}\int_{\Gamma_1}k_2''ovd\Gamma_1-\frac{\eta_2}{2}\int_{\Gamma_1}\left|v_t\right|^2d\Gamma_1+\frac{\eta_2}{2}\int_{\Gamma_1}k_2^2(t)\left|v^0\right|^2d\Gamma_1.\end{aligned} \tag{3.8}$$

Proof. Multiplying equation (1.1) by u_t and equation (1.2) by v_t, and integrating by parts over Ω, we obtain

$$\frac{d}{2dt}\int_\Omega(|\rho_1 u_t|^2+|\nabla u|^2+\beta\,|u|^2)dx-\alpha\sum_{i=1}^n\int_\Omega\frac{\partial v}{\partial x_i}u_t+\int_\Omega a(x)u_t f_1(u,v)dx$$
$$=\int_{\Gamma_1}\frac{\partial u}{\partial\nu}u_t d\Gamma_1,$$

and

$$\frac{d}{2dt}\int_\Omega(|\rho_2 v_t|^2+|\nabla v|^2)dx+\alpha\sum_{i=1}^n\int_\Omega\frac{\partial u}{\partial x_i}v_t+\int_\Omega a(x)v_t f_2(u,v)dx$$
$$=\int_{\Gamma_1}\frac{\partial v}{\partial\nu}v_t d\Gamma_1.$$

By a summation of these two identities, using (2.1), (2.2), and Lemma 2.2, we obtain the desired result.

Remark 3.3. a) If $(u^0,v^0)=(0,0)$ on Γ_1, then $E(t)\le E(0)$.

b) If ($u^0\neq 0$ or $v^0\neq 0$) on Γ_1, then

$$E(t)\le E(0)+\frac{\eta_1}{2}\int_{\Gamma_1}\left|u^0\right|^2 d\Gamma_1\int_0^t k_1^2(s)ds+\frac{\eta_2}{2}\int_{\Gamma_1}\left|v^0\right|^2 d\Gamma_1\int_0^t k_2^2(s)ds.$$

Lemma 3.4. *Under the assumptions of Theorem* 3.1, *the solution of* (1.1)–(1.6) *satisfies*

$$\frac{d}{dt}\int_\Omega[(2m.\nabla u+(n-\varepsilon_0)u)\rho_1 u_t+(2m.\nabla v+(n-\varepsilon_0)v)\rho_2 v_t]dx$$
$$\le-\varepsilon_0\int_\Omega(\rho_1\,|u_t|^2+\rho_2\,|v_t|^2+|\nabla u|^2+|\nabla v|^2)dx$$
$$-\varepsilon_0\sum_{i=1}^n\int_\Omega\left|\frac{\partial v}{\partial x_i}-u\right|^2dx+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1$$
$$-\int_{\Gamma_1}m.\nu\,|\nabla u|^2\,d\Gamma_1+\int_{\Gamma_1}\frac{\partial v}{\partial\nu}(2m.\nabla v+(n-\varepsilon_0)v)d\Gamma_1$$
$$-\int_{\Gamma_1}m.\nu\,|\nabla v|^2\,d\Gamma_1+\int_{\Gamma_1}m.\nu(\rho_1\,|u_t|^2+\rho_2\,|v_t|^2)d\Gamma_1$$
$$+\int_\Omega[((-n+\varepsilon_0)b+n+\varepsilon_0)a+2m.\nabla a]F(u,v)dx,$$

where ε_0 is a small positive number.

Proof. We multiply equation (1.1) by $2m.\nabla u+(n-\varepsilon_0)u$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (2m.\nabla u+(n-\varepsilon_0)u)\rho_1 u_t dx\\
=\quad & \int_\Omega 2\rho_1 u_t m.\nabla u_t dx+(n-\varepsilon_0)\int_\Omega \rho_1 |u_t|^2 dx+\int_\Omega 2(\Delta u)m.\nabla u dx\\
&+(n-\varepsilon_0)\int_\Omega \Delta u u dx+\alpha\sum_{i=1}^n\int_\Omega \frac{\partial v}{\partial x_i}[2m.\nabla u+(n-\varepsilon_0)u]dx\\
&-\int_\Omega (a(x)f_1(u,v)+\beta u)[2m.\nabla u+(n-\varepsilon_0)u]dx.
\end{aligned}$$

Integrating by parts and using the relation $div(m)=n$ we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (2m.\nabla u+(n-\varepsilon_0)u)\rho_1 u_t dx\\
&\le \int_{\Gamma_1} m.\nu\rho_1 |u_t|^2 d\Gamma_1-\varepsilon_0\int_\Omega \rho_1 |u_t|^2 dx-\int_\Omega \nabla\rho_1.m |u_t|^2 dx\\
&+\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1-\int_{\Gamma_1} m.\nu |\nabla u|^2 d\Gamma_1\\
&-(1-\varepsilon_0)\int_\Omega |\nabla u|^2 dx+\alpha\sum_{i=1}^n\int_\Omega \frac{\partial v}{\partial x_i}[2m.\nabla u+(n-\varepsilon_0)u]dx\\
&-\beta\int_{\Gamma_1} m.\nu |u|^2 d\Gamma_1+\beta\varepsilon_0\int_\Omega |u|^2 dx-\int_\Omega a(x)f_1(u,v)[2m.\nabla u+(n-\varepsilon_0)u]dx.
\end{aligned}$$

Similarly, we multiply equation (1.2) by $2m.\nabla v+(n-\varepsilon_0)v$ and integrate over Ω, using integration by parts, to arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (2m.\nabla v+(n-\varepsilon_0)v)\rho_2 v_t dx\\
&\le \int_{\Gamma_1} m.\nu\rho_2 |v_t|^2 d\Gamma_1-\varepsilon_0\int_\Omega \rho_2 |v_t|^2 dx-\int_\Omega \nabla\rho_2.m |v_t|^2 dx\\
&+\int_{\Gamma_1}\frac{\partial v}{\partial\nu}(2m.\nabla v+(n-\varepsilon_0)v)d\Gamma_1-\int_{\Gamma_1} m.\nu |\nabla v|^2 d\Gamma_1\\
&-(1-\varepsilon_0)\int_\Omega |\nabla v|^2 dx-\alpha\sum_{i=1}^n\int_\Omega \frac{\partial u}{\partial x_i}[2m.\nabla v+(n-\varepsilon_0)v]dx\\
&-\int_\Omega a(x)f_2(u,v)[2m.\nabla v+(n-\varepsilon_0)v]dx.
\end{aligned}$$

Summing the above two inequalities and taking into account (1.10) we easily deduce that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega [(2m.\nabla u + (n-\varepsilon_0)u)\rho_1 u_t + (2m.\nabla v + (n-\varepsilon_0)v)\rho_2 v_t]dx \\
&\leq \int_{\Gamma_1} m.\nu(\rho_1 |u_t|^2 + \rho_2 |v_t|^2)d\Gamma_1 - \varepsilon_0 \int_\Omega (\rho_1 |u_t|^2 + \rho_2 |v_t|^2)dx \\
&+ \int_{\Gamma_1} \frac{\partial u}{\partial \nu}(2m.\nabla u + (n-\varepsilon_0)u)d\Gamma_1 - \int_{\Gamma_1} m.\nu |\nabla u|^2 d\Gamma_1 \\
&- (1-\varepsilon_0)\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx + \int_{\Gamma_1} \frac{\partial v}{\partial \nu}(2m.\nabla v + (n-\varepsilon_0)v)d\Gamma_1 \\
&- \int_{\Gamma_1} m.\nu |\nabla v|^2 d\Gamma_1 + \alpha \sum_{i=1}^n \int_\Omega \frac{\partial v}{\partial x_i}[2m.\nabla u + (n-\varepsilon_0)u]dx \\
&- \beta \int_{\Gamma_1} m.\nu |u|^2 d\Gamma_1 + \beta\varepsilon_0 \int_\Omega |u|^2 dx - \alpha \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}[2m.\nabla v + (n-\varepsilon_0)v]dx \\
&- \int_\Omega a(x) f_1(u,v)[2m.\nabla u + (n-\varepsilon_0)u]dx \\
&- \int_\Omega a(x) f_2(u,v)[2m.\nabla v + (n-\varepsilon_0)v]dx.
\end{aligned}$$

To estimate

$$-\int_\Omega a(x)f_1(u,v)[2m.\nabla u+(n-\varepsilon_0)u]dx - \int_\Omega a(x)f_2(u,v)[2m.\nabla v+(n-\varepsilon_0)v]dx,$$

we exploit condition (1.8). Consequently, we get

$$\begin{aligned}
&-\int_\Omega a(x)f_1(u,v)[2m.\nabla u + (n-\varepsilon_0)u]dx \\
&\quad -\int_\Omega a(x)f_2(u,v)[2m.\nabla v + (n-\varepsilon_0)v]dx \\
&= -\int_\Omega a(x)\frac{\partial F}{\partial u}[2m.\nabla u + (n-\varepsilon_0)u]dx - \int_\Omega a(x)\frac{\partial F}{\partial v}[2m.\nabla v + (n-\varepsilon_0)v]dx \\
&\leq (-n+\varepsilon_0)(1+b)\int_\Omega a(x)F(u,v)dx - 2\int_\Omega a(x)m.\nabla F(u,v)dx \\
&\leq ((-n+\varepsilon_0)b + n + \varepsilon_0)\int_\Omega a(x)F(u,v)dx + \int_\Omega (2m.\nabla a)F(u,v)dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\Gamma_1} a(x)(2m.\nu)F(u,v)d\Gamma_1 \\
&\leq \int_{\Omega}[((-n+\varepsilon_0)b+n+\varepsilon_0)a+2m.\nabla a]F(u,v)dx.
\end{aligned}
$$

Next, we use (3.5) and the fact that there exists a positive constant c such that

$$
\alpha \sum_{i=1}^{n} \int_{\Omega}(\frac{\partial v}{\partial x_i} m.\nabla u + \frac{\partial u}{\partial x_i} m.\nabla v)dx \leq c\alpha[\int_{\Omega}(|\nabla u|^2 + |\nabla v|^2)dx],
$$

to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega}[(2m.\nabla u+(n-\varepsilon_0)u)\rho_1 u_t + (2m.\nabla v+(n-\varepsilon_0)v)\rho_2 v_t]dx \\
\leq\ & \int_{\Gamma_1} m.\nu(\rho_1|u_t|^2+\rho_2|v_t|^2)d\Gamma_1 - \varepsilon_0\int_{\Omega}(\rho_1|u_t|^2+\rho_2|v_t|^2)dx \\
&+\int_{\Gamma_1}\frac{\partial u}{\partial \nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1 - \int_{\Gamma_1} m.\nu\,|\nabla u|^2\,d\Gamma_1 \\
&-\frac{(1-\varepsilon_0)}{2}\int_{\Omega}(|\nabla u|^2+|\nabla v|^2)dx - c\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial v}{\partial x_i}-u\right|^2 dx \\
&+\int_{\Gamma_1}\frac{\partial v}{\partial \nu}(2m.\nabla v+(n-\varepsilon_0)v)d\Gamma_1 - \int_{\Gamma_1} m.\nu\,|\nabla v|^2\,d\Gamma_1 \\
&-\beta\int_{\Gamma_1} m.\nu\,|u|^2\,d\Gamma_1 + \int_{\Omega}[((-n+\varepsilon_0)b+n+\varepsilon_0)a+2m.\nabla a]F(u,v)dx.
\end{aligned}
$$

Using Poincaré's inequality and taking ε_0 small enough, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega}[(2m.\nabla u+(n-\varepsilon_0)u)\rho_1 u_t + (2m.\nabla v+(n-\varepsilon_0)v)\rho_2 v_t]dx \\
\leq\ & -\varepsilon_0\int_{\Omega}(\rho_1|u_t|^2+\rho_2|v_t|^2+|\nabla u|^2+|\nabla v|^2)dx - c\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial v}{\partial x_i}-u\right|^2 dx \\
&+\int_{\Gamma_1}\frac{\partial u}{\partial \nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1 - \int_{\Gamma_1} m.\nu\,|\nabla u|^2\,d\Gamma_1 \\
&+\int_{\Gamma_1}\frac{\partial v}{\partial \nu}(2m.\nabla v+(n-\varepsilon_0)v)d\Gamma_1 - \int_{\Gamma_1} m.\nu\,|\nabla v|^2\,d\Gamma_1 \\
&+\int_{\Gamma_1} m.\nu(\rho_1|u_t|^2+\rho_2|v_t|^2)d\Gamma_1 \qquad (3.9)
\end{aligned}
$$

$$+\int_{\Omega}[((-n+\varepsilon_0)b+n+\varepsilon_0)a+2m.\nabla a]F(u,v)dx.$$

The proof of Lemma 3.4 is completed. □

Now, we introduce the Lyapunov functional. For $N>0$ large enough, let

$$\mathcal{L}(t)=NE(t)+\int_{\Omega}[(2m.\nabla u+(n-\varepsilon_0)u)\rho_1 u_t+(2m.\nabla v+(n-\varepsilon_0)v)\rho_2 v_t]dx. \tag{3.10}$$

It is easy to check that, for N large, we have

$$\frac{N}{2}E(t)\leq\mathcal{L}(t)\leq 2NE(t). \tag{3.11}$$

Applying Young's inequality and Poincaré's inequality to the boundary integrals we have, for $\varepsilon>0$,

$$\begin{aligned}&\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(2m.\nabla u+(n-\varepsilon_0)u)d\Gamma_1\\ \leq\ &\varepsilon C\Big(\int_{\Omega}|\nabla u|^2\,dx+\int_{\Gamma_1}m.\nu\,|\nabla u|^2\,d\Gamma_1\Big)+C_\varepsilon\int_{\Gamma_1}\Big|\frac{\partial u}{\partial\nu}\Big|^2 d\Gamma_1,\end{aligned}$$

and

$$\begin{aligned}&\int_{\Gamma_1}\frac{\partial v}{\partial\nu}(2m.\nabla v+(n-\varepsilon_0)v)d\Gamma_1\\ \leq\ &\varepsilon C\Big(\int_{\Omega}|\nabla v|^2\,dx+\int_{\Gamma_1}m.\nu\,|\nabla v|^2\,d\Gamma_1\Big)+C_\varepsilon\int_{\Gamma_1}\Big|\frac{\partial v}{\partial\nu}\Big|^2 d\Gamma_1.\end{aligned}$$

By rewriting the boundary conditions (2.1)-(2.2) as

$$\frac{\partial u}{\partial\nu}=-\eta_1(u_t+k_1(t)u-k_1(t)u^0-k_1'\odot u)\quad on\ \ \Gamma_1\times\mathbb{R}^+,$$

$$\frac{\partial v}{\partial\nu}=-\eta_2(v_t+k_2(t)v-k_2(t)v^0-k_2'\odot v)\quad on\ \ \Gamma_1\times\mathbb{R}^+,$$

and combining all the above relations, taking ε_0 very small, and $\varepsilon\ll\varepsilon_0$, we obtain

$$\begin{aligned}\frac{d\mathcal{L}(t)}{dt}&\leq(\frac{-N\eta_1}{2}+C_\varepsilon(1+k_1^2(t)))\int_{\Gamma_1}|u_t|^2 d\Gamma_1+(\frac{N\eta_1}{2}+C_\varepsilon)\int_{\Gamma_1}k_1^2(t)\left|u^0\right|^2 d\Gamma_1\\ &-\frac{N\eta_1}{2}\int_{\Gamma_1}k_1''oud\Gamma_1-\frac{N\eta_2}{2}\int_{\Gamma_1}k_2''ovd\Gamma_1(\frac{-N\eta_2}{2}+C_\varepsilon(1+k_2^2(t)))\int_{\Gamma_1}|v_t|^2 d\Gamma_1\\ &-(\frac{\varepsilon_0}{2}-C_\varepsilon(k_1^2+k_2^2)(t))\Big[\int_{\Omega}(\rho_1|u_t|^2+\rho_2|v_t|^2+|\nabla u|^2+|\nabla v|^2)dx\end{aligned}$$

$$+\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial v}{\partial x_i}-u\right|^2 dx\Big]+(\frac{N\eta_2}{2}+C_\varepsilon)\int_{\Gamma_1}k_2^2(t)\left|v^0\right|^2 d\Gamma_1$$
$$+\int_{\Omega}[((-n+\varepsilon_0)b+n+\varepsilon_0)a+2m.\nabla a]F(u,v)dx$$
$$+C_\varepsilon\int_{\Gamma_1}(\left|k_1'\odot u\right|^2)d\Gamma_1+C_\varepsilon\int_{\Gamma_1}(\left|k_2'\odot v\right|^2)d\Gamma_1. \tag{3.12}$$

At this point, we use the fact that $\lim_{t\to\infty}k_i(t)=0$ $(i=1,2)$, (3.5), and then take N large enough to arrive at

$$\begin{aligned}\frac{d\mathcal{L}(t)}{dt} \quad &\leq \quad -c_1E(t)+c_2\Big[\int_{\Gamma_1}k_1^2(t)\left|u^0\right|^2 d\Gamma_1+\int_{\Gamma_1}k_2^2(t)\left|v^0\right|^2 d\Gamma_1\Big]\\ &\quad -C\Big[\int_{\Gamma_1}k_1' oud\Gamma_1+\int_{\Gamma_1}k_2' ovd\Gamma_1\Big],\ \forall t\geq t_0,\end{aligned} \tag{3.13}$$

for some t_0 large enough and some positive constants c_1, c_2 and C.

Case 1: If $\gamma_1(t)\leq\gamma_2(t)$ for $t\geq t_0$, we multiply both sides of (3.13) by $\gamma_1(t)$ to get

$$\begin{aligned}\gamma_1(t)\frac{d\mathcal{L}(t)}{dt}&\leq\gamma_1(t)\Big(-c_1E(t)+c_2\Big[\int_{\Gamma_1}k_1^2(t)\left|u^0\right|^2 d\Gamma_1+\int_{\Gamma_1}k_2^2(t)\left|v^0\right|^2 d\Gamma_1\Big]\Big)\\ &\quad +C\int_{\Gamma_1}\gamma_1(t)(-k_1')oud\Gamma_1+C\int_{\Gamma_1}\gamma_2(t)(-k_2')ovd\Gamma_1,\ \forall t\geq t_0.\end{aligned}$$

By using (3.1), the fact that $\gamma_i(t)$ is nonincreasing and (3.8), we easily see that

$$\begin{aligned}\gamma_1(t)\frac{d\mathcal{L}(t)}{dt}&\leq c_2\gamma_1(t)\Big[\int_{\Gamma_1}k_1^2(t)\left|u^0\right|^2 d\Gamma_1+\int_{\Gamma_1}k_2^2(t)\left|v^0\right|^2 d\Gamma_1\Big]\\ &\quad -c_1\gamma_1(t)E(t)-C\frac{dE}{dt},\ \forall t\geq t_0,\end{aligned}$$

which yields

$$\begin{aligned}&\gamma_1(t)\frac{d\mathcal{L}}{dt}+C\frac{dE}{dt}\leq\\ &c_2\gamma_1(t)\Big[\int_{\Gamma_1}k_1^2(t)\left|u^0\right|^2 d\Gamma_1+\int_{\Gamma_1}k_2^2(t)\left|v^0\right|^2 d\Gamma_1\Big]-c_1\gamma_1(t)E(t),\ \forall t\geq t_0.\end{aligned} \tag{3.14}$$

Again using the fact that $\gamma_1(t)$ is nonincreasing and setting

$$F(t)=\gamma_1(t)\mathcal{L}(t)+CE(t)\sim E(t), \tag{3.15}$$

estimate (3.14) gives

$$\frac{dF}{dt} \leq -c_1\gamma_1(t)F(t) + c_3\Big[\int_{\Gamma_1} k_1^2(t)\left|u^0\right|^2 d\Gamma_1 + \int_{\Gamma_1} k_2^2(t)\left|v^0\right|^2 d\Gamma_1\Big], \quad \forall t \geq t_0. \tag{3.16}$$

Subcase 1: If ($u^0 = 0$ and $v^0 = 0$) on Γ_1, then (3.16) reduces to

$$\frac{dF}{dt} \leq -c_1\gamma(t)F(t), \qquad \forall t \geq t_0.$$

A simple integration over (t_0, t) yields

$$F(t) \leq F(t_0)e^{-c_1\int_{t_0}^t \gamma_1(s)ds}, \qquad \forall t \geq t_0.$$

By using (3.15), we then obtain for some positive constant c_4

$$E(t) \leq c_4E(t_0)e^{-c_1\int_{t_0}^t \gamma_1(s)ds}, \qquad \forall t \geq t_0;$$

using Remark 3.3, then we get

$$E(t) \leq c_4E(0)e^{c_1\int_0^{t_0} \gamma_1(s)ds}e^{-c_1\int_0^t \gamma_1(s)ds}, \qquad \forall t \geq t_0.$$

Subcase 2: If $(u^0,\, v^0) \neq (0,0)$ *on* Γ_1, then (3.16) gives

$$\frac{dF}{dt} \leq -c_1\gamma_1(t)F(t) + C_1k_1^2(t) + C_2k_2^2(t),\ \forall t \geq t_0, \tag{3.17}$$

where

$$C_1 = c_3\int_{\Gamma_1}\left|u^0\right|^2 d\Gamma_1,\ C_2 = c_3\int_{\Gamma_1}\left|v^0\right|^2 d\Gamma_1.$$

In this case we introduce

$$\begin{aligned} H(t) :&= F(t) - C_1e^{-c_1\int_{t_0}^t \gamma_1(s)ds}\int_{t_0}^t k_1^2(s)e^{c_1\int_{t_0}^s \gamma_1(\zeta)d\zeta}ds \\ &- C_2e^{-c_1\int_{t_0}^t \gamma_1(s)ds}\int_{t_0}^t k_2^2(s)e^{c_1\int_{t_0}^s \gamma_1(\zeta)d\zeta}ds. \end{aligned} \tag{3.18}$$

A simple differentiation of H, using (3.17), leads to

$$\frac{dH}{dt} \leq -c_1\gamma_1(t)H(t).\ \ \forall t \geq t_0.$$

Again a simple integration over (t_0, t) yields

$$H(t) \leq H(t_0)e^{-c_1\int_{t_0}^t \gamma_1(s)ds}, \qquad \forall t \geq t_0, \tag{3.19}$$

which implies

$$F(t) \le F(t_0)e^{-c_1\int_{t_0}^t \gamma_1(s)ds} + C_1\int_{t_0}^t k_1^2(s)e^{c_1\int_{t_0}^s \gamma_1(\zeta)d\zeta}ds\ e^{-c_1\int_{t_0}^t \gamma_1(s)ds}$$
$$+C_2\int_{t_0}^t k_2^2(s)e^{c_1\int_{t_0}^s \gamma_1(\zeta)d\zeta}ds\ e^{-c_1\int_{t_0}^t \gamma_1(s)ds}, \qquad \forall t \ge t_0.$$

By using (3.15) and Remark 3.3, we then obtain for some positive constant c_5

$$E(t) \le \Big(c_5E(0) + C_1\int_0^t k_1^2(s)[\frac{c_5\eta_1}{2c_3} + e^{c_1\int_{t_0}^s \gamma_1(\zeta)d\zeta}]ds \tag{3.20}$$
$$+ C_2\int_0^t k_2^2(s)[\frac{c_5\eta_2}{2c_3} + e^{c_1\int_{t_0}^s \gamma_1(\zeta)d\zeta}]ds\Big)e^{-c_1\int_{t_0}^t \gamma_1(s)ds}, \forall t \ge t_0.$$

Case 2: If $\gamma_2(t) \le \gamma_1(t)$, for all $t \ge t_0$, we repeat the same techniques as in case 1, and replacing $\gamma_1(t)$ by $\gamma_2(t)$ in the inequalities, we get

$$E(t) \le c_4E(0)e^{c_1\int_0^{t_0} \gamma_2(s)ds}e^{-c_1\int_0^t \gamma_2(s)ds},\ \forall t \ge t_0,\ \text{ if } (u^0, v^0) = (0,0) \text{ on } \Gamma_1,$$

otherwise

$$E(t) \le \Big(c_5E(0) + C_1\int_0^t k_1^2(s)[\frac{c_5\eta_1}{2c_3} + e^{c_1\int_{t_0}^s \gamma_2(\zeta)d\zeta}]ds$$
$$+ C_2\int_0^t k_2^2(s)[\frac{c_5\eta_2}{2c_3} + e^{c_1\int_{t_0}^s \gamma_2(\zeta)d\zeta}]ds\Big)e^{-c_1\int_{t_0}^t \gamma_2(s)ds}, \forall t \ge t_0.$$

This complete the proof of Theorem 3.1.

Remark 3.5. Estimates (3.6) and (3.7) are also true for $t \in [0, t_0]$ by virtue of continuity and boundedness of $E(t)$ and $\gamma_i(t)$ $(i = 1, 2)$.

Remark 3.6. We note here that our argument is also applied to the systems considered in [2], [4], [6] and [9].

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