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A FIBERING MAP APPROACH TO A POTENTIAL OPERATOR EQUATION AND ITS APPLICATIONS

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Abstract. In this paper, we study the existence of multiple solutions for operator equations involving homogeneous potential operators. With the help of the Nehari manifold and the fibering method, we prove that some such equations have at least two nonzero solutions. Furthermore, we apply this result to prove the existence of two positive solutions for some quasilinear elliptic problems involving sign-changing weight functions.

1. Introduction

In this paper, we consider multiplicity results for nonzero solutions of the operator equation

$$A\left(u\right)-B\left(u\right)-C\left(u\right)=0,\qquad u\in X, \tag{E}$$

where X is a reflexive Banach space equipped with norm $\|\cdot\|$ and duality pair $\langle\cdot,\cdot\rangle$ between X and X^* and where $A,B,C:X\rightarrow X^*$ are homogeneous potential operators. We shall assume throughout that A,B and C are homogeneous of degrees $p-1,\ q-1,$ and $r-1$, where $1<q<p<r$, and so it is well known that the corresponding functionals are given by

$$a(u)=\tfrac{1}{p}\langle A(u),u\rangle;\qquad b(u)=\tfrac{1}{q}\langle B(u),u\rangle;\qquad c(u)=\tfrac{1}{r}\langle C(u),u\rangle.$$

For the definitions and properties of potential operators we refer the reader to Chabrowski [9, Chapter 1]. Moreover, we shall assume

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(H1) $u \to \langle A(u), u\rangle$ is weakly lower semicontinuous on X and there exists a continuous function $\kappa : [0,\infty) \to [0,\infty)$ with $\kappa(s) > 0$ on $(0,\infty)$ and $\lim_{s\to\infty} \kappa(s) = \infty$ such that $\langle A(u), u\rangle \geq \kappa(\|u\|)\|u\|$ for all $u \in X$;

(H2) there exist $u, v \in X$ such that $\langle B(u), u\rangle > 0$ and $\langle C(v), v\rangle > 0$;

(H3) $B : X \to X^*$ and $C : X \to X^*$ are strongly continuous;

(H4) there exist positive numbers d_1 and d_2 with

$$d_1^{r-p} d_2^{p-q} < (p-q)^{p-q}(r-p)^{r-p}(r-q)^{q-r},$$

such that

$$\langle B(u), u\rangle \leq d_1 [\langle A(u), u\rangle]^{q/p} \text{ and } \langle C(u), u\rangle \leq d_2 [\langle A(u), u\rangle]^{r/p}.$$

The study of (E) is motivated by a number of problems involving nonlinearities which are a combination of concave and convex terms, e.g., quasilinear boundary-value problems (see [1, 2, 4, 7, 8, 10, 11, 13, 17]), problems with nonlinear boundary conditions (see [6, 12, 16]) and quasilinear elliptic systems (see [3, 5, 18]). In this paper, we will use variational methods involving the Nehari manifold and fibering maps to prove the existence of at least two nonzero solutions of equation (E) and show how this provides corresponding existence results in a wide range of applications including those listed above.

This paper is organized as follows: In section 2, we discuss the Nehari manifold and fibering maps for equation (E). In section 3, we prove that equation (E) has at least two nontrivial solutions. In section 4, we apply this result to a wide range of applications with nonlinearities involving both concave and convex terms.

2. The Nehari manifold and fibering maps

A function $u \in X$ is a solution of equation (E) if and only if

$$\langle A(u) - B(u) - C(u), \varphi\rangle = 0 \text{ for all } \varphi \in X.$$

Thus, the energy functional corresponding to equation (E) is defined by

$$J(u) = a(u) - b(u) - c(u) \text{ for } u \in X;$$

i.e., u is a solution of (E) if and only if $J'(u) = 0$; i.e., u is a critical point of J.

As the energy functional J is not bounded below on X, it is useful to consider the functional on the Nehari manifold (see [14])

$$\mathbf{N} = \left\{u \in X \setminus \{0\} : \left\langle J'(u), u\right\rangle = 0\right\}.$$

Thus, $u \in \mathbf{N}$ if and only if

$$\langle A(u), u\rangle - \langle B(u), u\rangle - \langle C(u), u\rangle = 0.$$

Moreover, we have the following result.

Lemma 2.1. *The energy functional J is coercive and bounded below on* $\mathbf{N}$.

Proof. If $u \in \mathbf{N}$, then

$$J(u) = \left(\tfrac{1}{p} - \tfrac{1}{r}\right)\langle A(u), u\rangle - \left(\tfrac{1}{q} - \tfrac{1}{r}\right)\langle B(u), u\rangle, \tag{2.1}$$

and so by (H4)

$$J(u) \geq \left(\tfrac{1}{p} - \tfrac{1}{r}\right)\langle A(u), u\rangle - d_1\left(\tfrac{1}{q} - \tfrac{1}{r}\right)[\langle A(u), u\rangle]^{q/p}.$$

Thus, by condition $(H1)$, we have J coercive and bounded below on $\mathbf{N}$. □

The Nehari manifold $\mathbf{N}$ is closely linked to the behavior of functions of the form $h_u : t \to J(tu)$ for $t > 0$. Such maps are known as fibering maps and were introduced by Drabek-Pohozaev in [10] and are also discussed in Brown-Zhang [8]. If $u \in X$, we have

$$h_u(t) = \frac{t^p}{p}\langle A(u), u\rangle - \frac{t^q}{q}\langle B(u), u\rangle - \frac{t^r}{r}\langle C(u), u\rangle;$$
$$h_u'(t) = t^{p-1}\langle A(u), u\rangle - t^{q-1}\langle B(u), u\rangle - t^{r-1}\langle C(u), u\rangle;$$
$$h_u''(t) = (p-1)t^{p-2}\langle A(u), u\rangle - (q-1)t^{q-2}\langle B(u), u\rangle - (r-1)t^{r-2}\langle C(u), u\rangle.$$

It is easy to see that

$$h_u'(t) = \frac{1}{t}\big(\langle A(tu), tu\rangle - \langle B(tu), tu\rangle - \langle C(tu), tu\rangle\big),$$

and so, for $u \in X\setminus\{0\}$ and $t > 0$, $h_u'(t) = 0$ if and only if $tu \in \mathbf{N}$; i.e., positive critical points of h_u correspond to points on the Nehari manifold. In particular, $h_u'(1) = 0$ if and only if $u \in \mathbf{N}$. Thus, it is natural to split $\mathbf{N}$ into three parts corresponding to local minima, local maxima and points of inflection and so we define

$$\mathbf{N}_+ = \left\{u \in \mathbf{N} : h_u''(1) > 0\right\}; \quad \mathbf{N}_0 = \left\{u \in \mathbf{N} : h_u''(1) = 0\right\};$$
$$\mathbf{N}_- = \left\{u \in \mathbf{N} : h_u''(1) < 0\right\}.$$

We now derive some basic properties of $\mathbf{N}_+, \mathbf{N}_-$ and $\mathbf{N}_0$.

Lemma 2.2. *Suppose that u_0 is a local minimizer for J on $\mathbf{N}$ and that $u_0 \notin \mathbf{N}_0$. Then $J'(u_0) = 0$ in X^*.*

Proof. The proof is essentially the same as that in Brown-Zhang [8, Theorem 2.3] (or see Binding-Drabek-Huang [4]). □

Lemma 2.3. *(i) For any $u \in \mathbf{N}_+$, we have* $\langle B(u), u\rangle > 0$;
(ii) for any $u \in \mathbf{N}_0$, we have $\langle B(u), u\rangle > 0$ and $\langle C(u), u\rangle > 0$;
(iii) For any $u \in \mathbf{N}_-$, we have $\langle C(u), u\rangle > 0$.

Proof. If $u \in \mathbf{N}$, we have

$$\begin{aligned} h_u''(1) &= (p-1)\langle A(u), u\rangle - (q-1)\langle B(u), u\rangle - (r-1)\langle C(u), u\rangle \\ &= (p-r)\langle A(u), u\rangle - (q-r)\langle B(u), u\rangle \\ &= (p-q)\langle A(u), u\rangle - (r-q)\langle C(u), u\rangle. \end{aligned} \tag{2.2}$$

The result now follows immediately from (2) and (3) and the fact that $\langle A(u), u\rangle > 0$. □

Lemma 2.4. *We have $\mathbf{N}_0 = \emptyset$.*

Proof. Suppose otherwise; i.e., $\mathbf{N}_0 \neq \emptyset$. Then, for $u \in \mathbf{N}_0$, by (2) and (H4) we have

$$\langle A(u), u\rangle = \frac{r-q}{r-p}\langle B(u), u\rangle \le d_1 \, \frac{r-q}{r-p}\left[\langle A(u), u\rangle\right]^{\frac{q}{p}},$$

and so

$$\langle A(u), u\rangle^{\frac{1}{p}} \le \left[d_1 \, \frac{r-q}{r-p}\right]^{\frac{1}{p-q}}.$$

Similarly, using (3) and (H4) we have

$$\langle A(u), u\rangle^{\frac{1}{p}} \ge \left[\frac{1}{d_2} \, \frac{p-q}{r-q}\right]^{\frac{1}{r-p}}.$$

Hence, we must have

$$d_1^{r-p} d_2^{p-q} \ge (p-q)^{p-q} \, (r-p)^{r-p} \, (r-q)^{q-r},$$

which contradicts (H4). Hence, $\mathbf{N}_0 = \emptyset$. □

In order to get a better understanding of the Nehari manifold and fibering maps we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_u(t) = t^{p-q}\langle A(u), u\rangle - t^{r-q}\langle C(u), u\rangle \ \text{ for } t > 0. \tag{2.3}$$

Clearly $tu \in \mathbf{N}$ if and only if $m_u(t) = \langle B(u), u\rangle$. Moreover,

$$m_u'(t) = (p-q)t^{p-q-1}\langle A(u), u\rangle - (r-q)t^{r-q-1}\langle C(u), u\rangle, \tag{2.4}$$

and so it is easy to see that, if $tu \in \mathbf{N}$, then $t^{q-1}m_u'(t) = h_u''(t)$. Hence, $tu \in \mathbf{N}_+$ ($\mathbf{N}_-$) if and only if $m_u'(t) > 0$ (< 0).

Suppose $\langle C(u), u\rangle > 0$. Then, by (2.4), m_u has a unique critical point at $t = t_{\max}$ where

$$t_{\max} = \Big(\frac{(p-q)\,\langle A(u), u\rangle}{(r-q)\,\langle C(u), u\rangle}\Big)^{\frac{1}{r-p}} > 0,$$

and clearly m_u is strictly increasing on $(0, t_{\max})$ and strictly decreasing on $(t_{\max}, \infty)$ with $\lim_{t\to\infty} m_u(t) = -\infty$. Moreover,

$$\begin{aligned} m_u(t_{\max}) &= \Big[\Big(\frac{p-q}{r-q}\Big)^{\frac{p-q}{r-p}} - \Big(\frac{p-q}{r-q}\Big)^{\frac{r-q}{r-p}}\Big] \frac{\langle A(u), u\rangle^{\frac{r-q}{r-p}}}{\langle C(u), u\rangle^{\frac{p-q}{r-p}}} \\ &= \langle A(u), u\rangle^{\frac{q}{p}} \Big(\frac{r-p}{r-q}\Big)\Big(\frac{p-q}{r-q}\Big)^{\frac{p-q}{r-p}} \Big(\frac{[\langle A(u), u\rangle]^{\frac{r}{p}}}{\langle C(u), u\rangle}\Big)^{\frac{p-q}{r-p}} \\ &\geq \frac{1}{d_1}\,\langle B(u), u\rangle \Big(\frac{r-p}{r-q}\Big)\Big(\frac{p-q}{r-q}\Big)^{\frac{p-q}{r-p}} \Big(\frac{1}{d_2}\Big)^{\frac{p-q}{r-p}} = \alpha\,\langle B(u), u\rangle\,, \end{aligned}$$

where

$$\begin{aligned} \alpha^{r-p} &= d_1^{p-r} d_2^{q-p} \Big(\frac{r-p}{r-q}\Big)^{r-p} \Big(\frac{p-q}{r-q}\Big)^{p-q} \\ &= d_1^{r-p} d_2^{q-p} (r-p)^{r-p} (p-q)^{p-q} (r-q)^{q-r} > 1, \end{aligned}$$

by (H4). Hence, $m_u(t_{\max}) > \langle B(u), u\rangle$. Thus, we have the following lemma.

Lemma 2.5. *For each $u \in X$ with $\langle C(u), u\rangle > 0$ we have the following.*

(i) If $\langle B(u), u\rangle \leq 0$, then there is unique $t^- > t_{\max}$ such that $t^- u \in \mathbf{N}_-$ and h_u is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover,

$$J(t^- u) = \sup_{t\geq 0} J(tu). \tag{2.5}$$

(ii) If $\langle B(u), u\rangle > 0$, then there are unique $0 < t^+ < t_{\max} < t^-$ such that $t^+ u \in \mathbf{N}_+$, $t^- u \in \mathbf{N}_-$, h_u is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). Moreover,

$$J(t^+ u) = \inf_{0\leq t\leq t_{\max}} J(tu); J(t^- u) = \sup_{t\geq t^+} J(tu). \tag{2.6}$$

Proof. Fix $u \in X$ with $\langle C(u), u\rangle > 0$.

(i) Suppose $\langle B(u), u\rangle \leq 0$. Then $m_u(t) = \langle B(u), u\rangle$ has a unique solution $t^- > t_{\max}$ and $m_u'(t^-) < 0$. Hence h_u has a unique turning point at $t = t^-$ and $h_u''(t^-) < 0$. Thus $t^- u \in \mathbf{N}_-$ and (6) holds.

(ii) Suppose $\langle B(u), u\rangle > 0$. Since $m_u(t_{\max}) > \langle B(u), u\rangle$, the equation $m_u(t) = \langle B(u), u\rangle$ has exactly two solutions $t^+ < t_{\max} < t^-$ such that $m_u'(t^+) > 0$ and $m_u'(t^-) < 0$. Hence, there are exactly two multiples of u

lying in $\mathbf{N}$, viz, $t^+u \in \mathbf{N}_+$ and $t^-u \in \mathbf{N}_-$. Thus, h_u has turning points at $t = t^+$ and $t = t^-$ with $h_u''(t^+) > 0$ and $h_u''(t^-) < 0$. Thus, h_u is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^+, ∞). Hence, (7) must hold. □

Similarly, we define the function $\overline{m}_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$\overline{m}(t) = t^{p-r} \langle A(u), u \rangle - t^{q-r} \langle B(u), u \rangle \text{ for } t > 0.$$

If $\langle B(u), u \rangle > 0$, it is clear that $\overline{m}(t) \to -\infty$ as $t \to 0^+$ and $\overline{m}(t) \to 0^+$ as $t \to \infty$. Moreover, the function attains its maximum at

$$\overline{t}_{\max} = \Big(\frac{(r-q) \langle B(u), u \rangle}{(r-p) \langle A(u), u \rangle} \Big)^{\frac{1}{p-q}}.$$

Using arguments similar to those used in the proof of the previous lemma we have the following.

Lemma 2.6. *For each $u \in X$ with $\langle B(u), u \rangle > 0$ we have the following.*

(i) If $\langle C(u), u \rangle \le 0$, then there is unique $t^+ < \overline{t}_{\max}$ such that $t^+u \in \mathbf{N}_+$, h_u is decreasing on $(0, t^+)$ and increasing on $(t^+, 0)$. Moreover,

$$J(t^+u) = \inf_{t \ge 0} J(tu).$$

(ii) If $\langle C(u), u \rangle > 0$, then there are unique $0 < t^+ < \overline{t}_{\max} < t^-$ such that $t^+u \in \mathbf{N}_+$, $t^-u \in \mathbf{N}_-$ and h_u is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). Moreover,

$$J(t^+u) = \inf_{0 \le t \le \overline{t}_{\max}} J(tu); \;\; J(t^-u) = \sup_{t \ge t^+} J(tu).$$

Finally, we remark that it follows from (H2) and Lemmas 2.5 and 2.6 that $\mathbf{N}_+$ and $\mathbf{N}_-$ are nonempty.

3. EXISTENCE OF NONZERO SOLUTIONS

By Lemma 2.4, we may write $\mathbf{N} = \mathbf{N}_+ \cup \mathbf{N}_-$ and by Lemma 2.1 may define

$$\alpha_+ = \inf_{u \in \mathbf{N}_+} J(u) \text{ and } \alpha_- = \inf_{u \in \mathbf{N}_-} J(u).$$

For $u \in \mathbf{N}_+$, we have by (2)

$$\langle B(u), u \rangle < \frac{r-p}{r-q} \langle A(u), u \rangle . \tag{3.1}$$

Hence, by (1)

$$J(u) = \big(\tfrac{1}{p} - \tfrac{1}{r}\big) \langle A(u), u \rangle - \big(\tfrac{1}{q} - \tfrac{1}{r}\big) \langle B(u), u \rangle$$

$$> \left(\tfrac{1}{p} - \tfrac{1}{r}\right) \langle A(u), u\rangle - \left(\tfrac{1}{q} - \tfrac{1}{r}\right) \frac{r-p}{r-q} \langle A(u), u\rangle$$
$$= \frac{r-p}{rp} \langle A(u), u\rangle - \frac{r-p}{qr} \langle A(u), u\rangle < 0,$$

and so $\alpha_+ < 0$. Furthermore, we have the following result.

Theorem 3.1. *The functional J has a minimizer u_0^+ in $\mathbf{N}_+$ such that $J(u_0^+) = \alpha_+$.*

Proof. Let $\{u_n\}$ be a minimizing sequence for J on $\mathbf{N}_+$. Then by Lemma 2.1 J is coercive on $\mathbf{N}$ and so $\{u_n\}$ is bounded in X. Hence by (H3) there exist a subsequence $\{u_n\}$ and $u_0^+ \in X$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^+ \text{ weakly in } X, \\ \langle B(u_n), u_n\rangle &\to \left\langle B(u_0^+), u_0^+\right\rangle \text{ as } n \to \infty, \\ \langle C(u_n), u_n\rangle &\to \left\langle C(u_0^+), u_0^+\right\rangle \text{ as } n \to \infty. \end{aligned}$$

Since

$$J(u_n) = \frac{r-p}{pr} \langle A(u_n), u_n\rangle - \frac{r-q}{qr} \langle B(u_n), u_n\rangle ,$$

and so

$$\frac{r-q}{qr} \langle B(u_n), u_n\rangle = \frac{r-p}{pr} \langle A(u_n), u_n\rangle - J(u_n),$$

letting $n \to \infty$, we see that $\left\langle B(u_0^+), u_0^+\right\rangle > 0$. Hence, by Lemma 2.6 there is a unique t_0^+ such that $t_0^+ u_0^+ \in \mathbf{N}_+$. Now, we prove that

$$\langle A(u_n), u_n\rangle \to \left\langle A(u_0^+), u_0^+\right\rangle \text{ as } n \to \infty.$$

Supposing the contrary, then

$$\left\langle A(u_0^+), u_0^+\right\rangle < \liminf \langle A(u_n), u_n\rangle . \tag{3.2}$$

Since

$$h'_{u_n}(t) = t^{p-1} \langle A(u_n), u_n\rangle - t^{q-1} \langle B(u_n), u_n\rangle - t^{r-1} \langle C(u_n), u_n\rangle , \tag{3.3}$$

and

$$h'_{u_0^+}(t) = t^{p-1} \left\langle A(u_0^+), u_0^+\right\rangle - t^{q-1} \left\langle B(u_0^+), u_0^+\right\rangle - t^{r-1} \left\langle C(u_0^+), u_0^+\right\rangle ,$$

and $h'_{u_0^+}(t_0) = 0$, it follows that that $h'_{u_n}(t_0) > 0$ for n sufficiently large. Since $\{u_n\} \subseteq \mathbf{N}_+$, $h'_{u_n}(1) = 0$ and by Lemma 2.6 $h'_{u_n}(t) < 0$ for all $t \in (0, 1)$. Hence, $t_0 > 1$. But $t_0^+ u_0^+ \in \mathbf{N}_+$ and so by Lemma 2.6 and (3.2)

$$J(t_0^+ u_0^+) < J(u_0^+) < \lim_{n\to\infty} J(u_n) = \alpha_+.$$

This is impossible and so

$$\langle A(u_n), u_n\rangle \to \left\langle A(u_0^+), u_0^+\right\rangle \text{ as } n \to \infty.$$

Hence, $h'_{u_0^+}(1) = \lim_{n\to\infty} h'_{u_n}(1) = 0$ and $h''_{u_0^+}(1) = \lim_{n\to\infty} h''_{u_n}(1) \geq 0$ and so $u_0^+ \in \mathbf{N}_+$. Moreover, $J(u_n) \to J(u_0^+) = \alpha_+$ as $n \to \infty$ and so u_0^+ is a minimizer for J on $\mathbf{N}_+$. □

Next, we establish the existence of a local minimum for J on $\mathbf{N}_-$.

Theorem 3.2. *J has a minimizer u_0^- in $\mathbf{N}_-$ such that $J(u_0^-) = \alpha_-$.*

Proof. Let $\{u_n\}$ be a minimizing sequence for J on $\mathbf{N}_-$. Then as in the previous proof there exists a subsequence $\{u_n\}$ and $u_0^- \in X$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^- \text{ weakly in } X, \\ \langle B(u_n), u_n\rangle &\to \left\langle B(u_0^-), u_0^-\right\rangle \text{ as } n \to \infty, \\ \langle C(u_n), u_n\rangle &\to \left\langle C(u_0^-), u_0^-\right\rangle \text{ as } n \to \infty. \end{aligned}$$

Moreover, since $u_n \in \mathbf{N}_-$,

$$(p-q)\langle A(u_n), u_n\rangle - (r-q)\langle C(u_n), u_n\rangle < 0,$$

and so

$$\left[\left\langle C(u_0^-), u_0^-\right\rangle\right]^{\frac{r-p}{r}} \geq \frac{p-q}{r-q}\,\left(\frac{1}{d_2}\right)^{\frac{p}{r}} > 0. \tag{3.4}$$

In particular, $u_0^- \neq 0$. Now we prove that

$$\langle A(u_n), u_n\rangle \to \left\langle A(u_0^-), u_0^-\right\rangle \text{ as } n \to \infty.$$

Suppose otherwise, then

$$\left\langle A(u_0^-), u_0^-\right\rangle < \liminf \langle A(u_n), u_n\rangle.$$

By Lemma 2.5, there is a unique t_0^- such that $t_0^- u_0^- \in \mathbf{N}_-$. Then $h'_{u_0^-}(t_0^-) = 0$ and so it follows from (3.3) that $h'_{u_n}(t_0^-) > 0$ for n sufficiently large. Since $u_n \in \mathbf{N}_-$, $h'_{u_n}(1) = 0$ and it is clear from Lemma 2.5 and Lemma 2.6 that h_{u_n} is increasing on $(t_0^-, 1)$. Hence, $h_{u_n}(t_0^-) < h_{u_n}(1)$; i.e., $J(t_0^- u_n) < J(u_n)$. Thus, using (3.2),

$$J(t_0^- u_0^-) < \liminf J(t_0^- u_n) \leq \liminf J(u_n) = \alpha_-,$$

which is a contradiction. Hence, $\langle A(u_n), u_n\rangle \to \left\langle A(u_0^-), u_0^-\right\rangle$ as $n \to \infty$. The proof can now be completed in the same way as that used in Theorem 3.1. □

Furthermore, we have the following result.

Theorem 3.3. *The equation (E) has at least two nonzero solutions.*

Proof. By Theorems 3.1, 3.2 there exist $u_0^+ \in \mathbf{N}_+$ and $u_0^- \in \mathbf{N}_-$ such that $J(u_0^+) = \alpha_+$ and $J(u_0^-) = \alpha_-$. Since $\mathbf{N}_+ \cap \mathbf{N}_- = \emptyset$, this implies that u_0^+ and u_0^- are distinct. Moreover, by Lemma 2.2, u_0^+ and u_0^- are nonzero solutions of equation (E). □

4. Applications

In this section, we discuss a variety of applications of Theorem 3.3.

(I) We study the p–Laplacian elliptic equation

$$\begin{cases} -\Delta_p u = \lambda f(x) |u|^{q-2} u + \mu g(x) |u|^{r-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{$D_{\lambda,\mu}$}$$

where $1 < q < p < r < p^*$ $(p^* = \frac{pN}{N-p}$ if $N > p,\ p^* = \infty$ if $N \leq p),\ \Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, λ and μ are parameters in $\mathbb{R}^+$, and the weight functions $f, g \in C(\overline{\Omega})$ satisfy $f^+ = \max\{f, 0\} \not\equiv 0$ and $g^+ = \max\{g, 0\} \not\equiv 0$. We shall apply Theorem 3.3 with potential operators A, B and C from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ given by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx,$$

$$\langle B(u), v\rangle = \lambda \int_\Omega f |u|^{q-2} uv \, dx,$$

and

$$\langle C(u), v\rangle = \mu \int_\Omega g |u|^{r-2} uv \, dx,$$

for u and v in $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus, the corresponding energy functional of equation $(D_{\lambda,\mu})$ is defined by

$$J(u) = \tfrac{1}{p} \langle A(u), u\rangle - \tfrac{1}{q} \langle B(u), u\rangle - \tfrac{1}{r} \langle C(u), u\rangle \text{ for } u \in W_0^{1,p}(\Omega),$$

and the Nehari manifold is defined by

$$\mathbf{N} = \left\{ u \in W_0^{1,p}(\Omega) \backslash \{0\} : \langle A(u), u\rangle = \langle B(u), u\rangle + \langle C(u), u\rangle \right\}.$$

Clearly,

$$\langle A(u), u\rangle = \int_\Omega |\nabla u|^p \, dx = \|u\|^p_{W_0^{1,p}},$$

and so A satisfies (H1).

Since $f^{+} \not\equiv 0$ and $g^{+} \not\equiv 0$, B and C satisfy (H2). Moreover, by standard compact embedding theorems B and C satisfy (H3),

$$\langle B(u), u\rangle \leq \lambda \left\|f^{+}\right\|_{\infty} S_q^q \left[\langle A(u), u\rangle\right]^{q/p},$$

and

$$\langle C(u), u\rangle \leq \mu \left\|g^{+}\right\|_{\infty} S_r^r \left[\langle A(u), u\rangle\right]^{r/p},$$

where S_l is the best Sobolev constant for the embedding of $W_0^{1,p}(\Omega)$ in $L^l(\Omega)$ for $1 \leq l < p^*$. Thus, if we choose $d_1 = \lambda \left\|f^{+}\right\|_{\infty} S_q^q$ and $d_2 = \mu \left\|g^{+}\right\|_{\infty} S_r^r$, then it is clear that (H4) is satisfied provided that $\lambda\mu$ is chosen to be sufficiently small.

Theorem 4.1. *There exists $\Lambda_0 > 0$ such that for $0 < \lambda\mu < \Lambda_0$ equation $(D_{\lambda,\mu})$ has at least two positive solutions.*

Proof. By Theorem 3.3 there exist $u_0^{+} \in \mathbf{N}_{+}$ and $u_0^{-} \in \mathbf{N}_{-}$ such that $J(u_0^{+}) = \alpha_{+}$ and $J(u_0^{-}) = \alpha_{-}$. Moreover, $J(u_0^{\pm}) = J_\lambda(\left|u_0^{\pm}\right|)$ and $\left|u_0^{\pm}\right| \in \mathbf{N}_{\pm}$ and so we may assume $u_0^{\pm} \geq 0$. By Lemma 2.2 $u_0^{\pm}$ are critical points for J on $W_0^{1,p}(\Omega)$ and hence are weak solutions (and so by standard regularity results classical solutions) of equation $(D_{\lambda,\mu})$. Moreover, by the Harnack inequality due to Trudinger [15], we obtain that $u_0^{\pm}$ are positive solutions of $(D_{\lambda,\mu})$. □

(II) We consider the multiplicity of positive solutions for the following quasilinear equation with nonlinear boundary condition:

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = g(x) |u|^{r-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda f(x) |u|^{q-2} u & \text{on } \partial\Omega, \end{cases} \tag{Γ_λ}$$

where $1 < q < p < r < p^*$, $q < p_*$ ($p_* = \frac{p(N-1)}{N-p}$ if $N > p$, $p_* = \infty$ if $N \leq p$), $\lambda \in \mathbb{R} - \{0\}$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\frac{\partial}{\partial \nu}$ is the outer normal derivative and the weight functions f, g satisfy the following conditions:

(F) $f \in C(\partial\Omega)$ with $f^{\pm} = \max\{\pm f, 0\} \not\equiv 0$;

(G) $g \in C(\overline{\Omega})$ with $g^{+} = \max\{g, 0\} \not\equiv 0$.

We shall apply Theorem 3.3 with potential operators A, B and C from $W^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ given by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx + \int_\Omega |u|^{p-2} uv \, dx,$$

$$\langle B(u), v\rangle = \lambda \int_{\partial\Omega} f |u|^{q-2} uv \, d\sigma,$$

and

$$\langle C(u), v\rangle = \int_\Omega g\,|u|^{r-2}\,uv\,dx,$$

for u and v in $W^{1,p}(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $d\sigma$ is the measure on the boundary.

As before (H1), (H2) and (H3) are satisfied. Moreover, by the Sobolev embedding theorems,

$$\langle B(u), u\rangle \leq |\lambda|\,\|f\|_\infty\,\overline{S}_q^q\,[\langle A(u), u\rangle]^{q/p},$$

and

$$\langle C(u), u\rangle \leq \left\|g^+\right\|_\infty S_r^r\,[\langle A(u), u\rangle]^{r/p},$$

where $\overline{S}_q$ is the best Sobolev trace constant for the embedding of $W^{1,p}(\Omega)$ in $L^q(\partial\Omega)$ and S_r is the best Sobolev constant for the embedding of $W^{1,p}(\Omega)$ in $L^r(\Omega)$. Thus, if we choose $d_1=|\lambda|\,\|f\|_\infty\,\overline{S}_q^q$ and $d_2=\|g^+\|_\infty\,S_r^r$, then (H4) is satisfied provided λ is sufficiently small. Hence, arguing exactly as in the previous example we can prove the following.

Theorem 4.2. *There exists $\Lambda_0>0$ such that, for $0<|\lambda|<\Lambda_0$, equation (Γ_λ) has at least two positive solutions.*

(III) In a very similar way, we can consider the nonlinear boundary-value problem

$$\begin{cases} \Delta_p u - |u|^{p-2}\,u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda f(x)\,|u|^{q-2}\,u + g(x)\,|u|^{r-2}\,u & \text{on } \partial\Omega, \end{cases} \qquad (\Theta_\lambda)$$

where $1<q<p<r<p_*$, $\lambda\in\mathbb{R}-\{0\}$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\frac{\partial}{\partial\nu}$ is the outer normal derivative and the weight functions $f,g\in C(\partial\Omega)$ satisfy $f^\pm\not\equiv 0$ and $g^+\not\equiv 0$. We now apply Theorem 3.3, with potential operators A,B and C from $W^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ given by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\,\nabla u\nabla v\,dx,$$

$$\langle B(u), v\rangle = \lambda\int_{\partial\Omega} f\,|u|^{q-2}\,uv\,d\sigma,$$

and

$$\langle C(u), v\rangle = \int_{\partial\Omega} g\,|u|^{r-2}\,uv\,d\sigma.$$

In this case, by the Sobolev embedding theorem

$$\langle B(u), u\rangle \leq |\lambda|\,\|f\|_\infty\,\overline{S}_q^q\,[\langle A(u), u\rangle]^{q/p},$$

and

$$\langle C(u), u\rangle \leq \left\|g^{+}\right\|_{\infty} \overline{S}_r^r \left[\langle A(u), u\rangle\right]^{r/p}.$$

If we set $d_1 = |\lambda| \, \|f\|_{\infty} \overline{S}_q^q$ and $d_2 = \|g^{+}\|_{\infty} \overline{S}_r^r$, then (H4) is satisfied provided λ is sufficiently small. Thus, arguing exactly as in the previous examples we can prove the following.

Theorem 4.3. *There exists $\Lambda_0 > 0$ such that, for $0 < |\lambda| < \Lambda_0$, equation (Θ_λ) has at least two positive solutions.*

(IV) We now consider multiplicity results for positive solutions of the following quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = \lambda f(x) |u|^{q-2} u + \frac{\alpha}{\alpha+\beta} h(x) |u|^{\alpha-2} u |v|^{\beta} & \text{in } \Omega, \\ -\Delta_p v = \mu g(x) |v|^{q-2} v + \frac{\beta}{\alpha+\beta} h(x) |u|^{\alpha} |v|^{\beta-2} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (L_{\lambda,\mu})$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha > 1, \beta > 1$ satisfy $1 < q < p < \alpha+\beta < p^*$, λ and μ are parameters such that $\lambda \in \mathbb{R} - \{0\}$ and $\mu \in \mathbb{R} - \{0\}$, and the weight functions $f, g, h \in C(\overline{\Omega})$ satisfy

$(U1)$ the intersections of the set $\{x \in \Omega : h(x) > 0\}$ with each of the sets $\{x \in \Omega : f(x) > 0\}$, $\{x \in \Omega : f(x) < 0\}$, $\{x \in \Omega : g(x) > 0\}$, $\{x \in \Omega : g(x) < 0\}$ have positive measures.

We pose Problem $(L_{\lambda,\mu})$ in the framework of the Sobolev space $W = W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ with the standard norm

$$\|(u,v)\|_W = \Big(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |\nabla v|^p \, dx \Big)^{\frac{1}{p}}.$$

Moreover, a pair of functions $(u, v) \in W$ is said to be a weak solution of problem $(L_{\lambda,\mu})$ if

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 \, dx &+ \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi_2 \, dx \\ &- \lambda \int_\Omega f |u|^{q-2} u \varphi_1 dx - \mu \int_\Omega g |v|^{q-2} v \varphi_2 \, dx \\ - \frac{\alpha}{\alpha+\beta} \int_\Omega h |u|^{\alpha-2} u |v|^{\beta} \varphi_1 dx &- \frac{\beta}{\alpha+\beta} \int_\Omega h |u|^{\alpha} |v|^{\beta-2} v \varphi_2 \, dx = 0, \end{aligned}$$

for all $(\varphi_1, \varphi_2) \in W$. We shall apply Theorem 3.3, with potential operators A, B and C from W into W^* given by

$$\langle A(u,v), (\varphi_1, \varphi_2)\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 dx + \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi_2 \, dx,$$

$$\langle B(u,v),(\varphi_1,\varphi_2)\rangle = \lambda\int_\Omega f\,|u|^{q-2}\,u\varphi_1 dx + \mu\int_\Omega g\,|v|^{q-2}\,v\varphi_2\,dx,$$

and

$$\begin{aligned}\langle C(u,v),(\varphi_1,\varphi_2)\rangle &= \frac{\alpha}{\alpha+\beta}\int_\Omega h\,|u|^{\alpha-2}\,u\,|v|^{\beta}\,\varphi_1 dx\\ &\quad+\frac{\beta}{\alpha+\beta}\int_\Omega h\,|u|^{\alpha}\,|v|^{\beta-2}\,v\varphi_2\,dx,\end{aligned}$$

for (u,v) and (φ_1,φ_2) in W. Thus, the corresponding energy functional of equation $(L_{\lambda,\mu})$ is defined by

$$J(u,v) = \frac{1}{p}\langle A(u,v),(u,v)\rangle - \frac{1}{q}\langle B(u,v),(u,v) - \frac{1}{\alpha+\beta}\langle C(u,v),(u,v)\rangle,$$

for $(u,v)\in W$. Moreover, the Nehari manifold is defined by

$$\mathbf{N}=\{u\in W\setminus\{(0,0)\} : \langle A(u,v),(u,v)\rangle=\langle B(u,v),(u,v)\rangle + \langle C(u,v),(u,v)\rangle\}\,.$$

It is easy to see that (H1)and (H2) are satisfied and by standard compact embedding theorems B and C satisfy (H3) and moreover,

$$\begin{aligned}\langle B(u,v),(u,v)\rangle &= \lambda\int_\Omega f|u|^q\,dx + \mu\int_\Omega g|v|^q\,dx\\ &\le |\lambda|\|f\|_\infty\int_\Omega |u|^q dx + |\mu|\|g\|_\infty\int_\Omega |v|^q\,dx\\ &\le S_q^q(|\lambda|\|f\|_\infty + |\mu|\|g\|_\infty)\,\|(u,v)\|^q\\ &\le S_q^q(|\lambda|\|f\|_\infty + |\mu|\|g\|_\infty)\,[\langle A(u,v),(u,v)\rangle]^{q/p}\,,\end{aligned}$$

and

$$\langle C(u,v),(u,v)\rangle = \int_\Omega h|u|^\alpha|v|^\beta\,dx \le S_{\alpha+\beta}^{\alpha+\beta}\left\|h^+\right\|_\infty [\langle A(u,v),(u,v)\rangle]^{(\alpha+\beta)/p}\,,$$

where we have used the fact that $|u|^\alpha|v|^\beta \le |u|^{\alpha+\beta} + |v|^{\alpha+\beta}$. Thus, if we set $d_1 = S_q^q\big(|\lambda|\|f\|_\infty + |\mu|\|g\|_\infty\big)$ and $d_2 = S_{\alpha+\beta}^{\alpha+\beta}\,\|h^+\|_\infty$, it is clear that (H4) is satisfied provided that $|\lambda|+|\mu|$ is sufficiently small.

Theorem 4.4. *There exists $\Lambda_0>0$ such that for $|\lambda|+|\mu|<\Lambda_0$ equation $(L_{\lambda,\mu})$ has at least two positive solutions.*

Proof. By Theorem 3.3 there exist $(u_0^+,v_0^+)\in\mathbf{N}_+$ and $(u_0^-,v_0^-)\in\mathbf{N}_-$ such that $J(u_0^+,v_0^+)=\alpha_+$ and $J(u_0^-,v_0^-)=\alpha_-$. Moreover, $J(u_0^\pm,v_0^\pm) = J(|u_0^\pm|,|v_0^\pm|)$ and $(|u_0^\pm|,|v_0^\pm|)\in\mathbf{N}_\pm$. Thus, we may assume that $u_0^\pm\ge 0$ and $v_0^\pm\ge 0$. By Lemma 2.2 $(u_0^\pm,v_0^\pm)$ are critical points for J on $W_0^{1,p}(\Omega)$

and hence are weak solutions of equation $(L_{\lambda,\mu})$. Moreover, by (3.4), $u_0^- \neq 0, v_0^- \neq 0$. Finally, we prove that $u_0^+ \not\equiv 0, v_0^+ \not\equiv 0$. We assume, without loss of generality, that $v_0^+ \equiv 0$. Then, since $\alpha_+ < 0$, we must have $u_0^+ \not\equiv 0$. Then, as u_0^+ is a solution of

$$-\Delta_p u = \lambda f |u|^{q-2} u \text{ in } \Omega; \qquad u = 0 \text{ on } \partial\Omega,$$

we have

$$\lambda \int_\Omega f |u_0^+|^q \, dx = \int_\Omega |\nabla u_0^+|^p \, dx > 0. \tag{4.1}$$

Moreover by $(U1)$ we may choose $w \in W_0^{1,p}(\Omega) \backslash \{0\}$ such that

$$\int_\Omega |\nabla w|^p \, dx = \mu \int_\Omega g \, |w|^q \, dx, \tag{4.2}$$

and

$$\int_\Omega h \left| u_0^+ \right|^\alpha |w|^\beta \, dx \geq 0. \tag{4.3}$$

Now,

$$\left\langle B(u_0^+, w), (u_0^+, w) \right\rangle = \lambda \int_\Omega f \left| u_0^+ \right|^q \, dx + \mu \int_\Omega g \, |w|^q \, dx > 0$$

and so by Lemma 2.6 there is a unique $t^+ > 0$ such that $0 < t^+ < \bar{t}_{\max}$ and $(t^+ u_0^+, t^+ w) \in \mathbf{N}_+$ where

$$\bar{t}_{\max} = \Big(\frac{(\alpha + \beta - q) \langle B(u_0^+, w), (u_0^+, , w) \rangle}{(\alpha + \beta - p) \langle A(u_0^+, w), (u_0^+, w) \rangle} \Big)^{\frac{1}{p-q}}.$$

By (4.1) and (4.2) $\langle A(u_0^+, w), (u_0^+, w) \rangle = \langle B(u_0^+, w), (u_0^+, w) \rangle$ and so $\bar{t}_{\max} > 1$. By Lemma 2.6

$$J(t^+ u_0^+, t^+ w) = \inf_{0 \leq t \leq \bar{t}_{\max}} J(t u_0^+, t w),$$

and so, by (4.3),

$$J(t^+ u_0^+, t^+ w) \leq J(u_0^+, w) < J(u_0^+, 0) = \alpha_+,$$

which is a contradiction. Moreover, by the Harnack inequality due to Trudinger [15], we may conclude that (u_0^+, v_0^+) and (u_0^-, v_0^-) are positive solutions of equation $(L_{\lambda,\mu})$. □

(V) Similarly, we can consider multiplicity results for positive solutions of the following quasilinear elliptic system with nonlinear boundary conditions:

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = \frac{\alpha}{\alpha+\beta} h(x) |u|^{\alpha-2} u |v|^{\beta} & \text{in } \Omega, \\ -\Delta_p v + |v|^{p-2} v = \frac{\beta}{\alpha+\beta} h(x) |u|^{\alpha} |v|^{\beta-2} v & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda f(x) |u|^{q-2} u, \frac{\partial v}{\partial n} = \mu g(x) |v|^{q-2} v & \text{on } \partial\Omega, \end{cases} \qquad (\Gamma_{\lambda,\mu})$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha > 1, \beta > 1$ satisfy $1 < q < p < \alpha + \beta < p^*$, $q < p_*$, λ and μ are parameters in $\mathbb{R} - \{0\}$, and the weight functions $f, g \in C(\partial\Omega)$ and $h \in C(\overline{\Omega})$ satisfy the following condition:

$(U2)$ $f^{\pm} = \max\{\pm f, 0\} \not\equiv 0$, $g^{\pm} = \max\{\pm g, 0\} \not\equiv 0$ and $h^{+} = \max\{h, 0\} \not\equiv 0$.

We consider problem $(\Gamma_{\lambda,\mu})$ in the framework of the Sobolev space $W = W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ with the standard norm

$$\|(u,v)\|_W = \Big(\int_\Omega \Big(|\nabla u|^p + |u|^p \Big)\, dx + \int_\Omega \Big(|\nabla v|^p + |v|^p \Big)\, dx \Big)^{\frac{1}{p}}.$$

Moreover, a pair of functions $(u,v) \in W$ is said to be a weak solution of problem $(\Gamma_{\lambda,\mu})$ if

$$\begin{aligned} &\int_\Omega \Big(|\nabla u|^{p-2} \nabla u \nabla \varphi_1 + |u|^{p-2} u \varphi_1 \Big)\, dx \\ &+ \int_\Omega \Big(|\nabla v|^{p-2} \nabla v \nabla \varphi_2 + |v|^{p-2} v \varphi_2 \Big)\, dx - \lambda \int_{\partial\Omega} f |u|^{q-2} u \varphi_1 \, d\sigma \\ &- \mu \int_{\partial\Omega} g |v|^{q-2} v \varphi_2 \, d\sigma - \frac{\alpha}{\alpha+\beta} \int_\Omega h |u|^{\alpha-2} u |v|^{\beta} \varphi_1 \, dx \\ &- \frac{\beta}{\alpha+\beta} \int_\Omega h |u|^{\alpha} |v|^{\beta-2} v \varphi_2 \, dx = 0 \quad \forall (\varphi_1, \varphi_2) \in W, \end{aligned}$$

where $d\sigma$ is the measure on the boundary. We shall apply Theorem 3.3, with potential operators A, B and C from W into W^* given by

$$\begin{aligned} \langle A(u,v), (\varphi_1, \varphi_2) \rangle &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 + |u|^{p-2} u \varphi_1 \, dx \\ &\quad + \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi_2 + |v|^{p-2} v \varphi_2 \, dx, \\ \langle B(u,v), (\varphi_1, \varphi_2) \rangle &= \lambda \int_{\partial\Omega} f |u|^{q-2} u \varphi_1 \, d\sigma + \mu \int_{\partial\Omega} g |v|^{q-2} v \varphi_2 \, d\sigma \end{aligned}$$

$$\langle C(u,v),(\varphi_1,\varphi_2)\rangle = \frac{\alpha}{\alpha+\beta}\int_\Omega h\,|u|^{\alpha-2}\,u\,|v|^{\beta}\,\varphi_1\,dx + \frac{\beta}{\alpha+\beta}\int_\Omega h\,|u|^{\alpha}\,|v|^{\beta-2}\,v\varphi_2\,dx,$$

for (u,v) and (φ_1,φ_2) in W.

Again (H1), (H2) and (H3) are satisfied and by standard compact embedding theorems

$$\langle B(u,v),(u,v)\rangle \le \overline{S}_q^q(|\lambda|\,\|f\|_\infty + |\mu|\,\|g\|_\infty)\,[\langle A(u,v),(u,v)\rangle]^{q/p},$$

and

$$\langle C(u,v),(u,v)\rangle \le S_{\alpha+\beta}^{\alpha+\beta}\,\|h^+\|_\infty\,[\langle A(u,v),(u,v)\rangle]^{(\alpha+\beta)/p}.$$

Thus, if we set $d_1 = \overline{S}_q^q(|\lambda|\,\|f\|_\infty + |\mu|\,\|g\|_\infty)$ and $d_2 = S_{\alpha+\beta}^{\alpha+\beta}\,\|h^+\|_\infty$, it is clear (H4) is satisfied provided $|\lambda|+|\mu|$ is sufficiently small. Hence, arguing as in the previous example, we have the following.

Theorem 4.5. *There exists $\Lambda_0 > 0$ such that for $|\lambda|+|\mu| < \Lambda_0$ problem $(\Gamma_{\lambda,\mu})$ has at least two positive solutions.*

(VI) Finally, we consider multiplicity results for positive solutions of the following quasilinear elliptic system with boundary condition containing both convex and concave nonlinearities:

$$\begin{cases} \Delta_p u - |u|^{p-2}\,u = 0 & \text{in } \Omega, \\ \Delta_p v - |v|^{p-2}\,v = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda f(x)\,|u|^{q-2}\,u + \frac{\alpha}{\alpha+\beta}h(x)\,|u|^{\alpha-2}\,u\,|v|^{\beta} & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = \mu g(x)\,|v|^{q-2}\,v + \frac{\beta}{\alpha+\beta}h(x)\,|u|^{\alpha}\,|v|^{\beta-2}\,v & \text{on } \partial\Omega, \end{cases} \qquad (\Theta_{\lambda,\mu})$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha > 1, \beta > 1$ satisfy $1 < q < p < \alpha+\beta < p_*$, λ and μ are parameters in $\mathbb{R} - \{0\}$, and the weight functions $f, g, h \in C(\partial\Omega)$ satisfy the following condition:

$(U3)$ the intersections of the set $\{x \in \partial\Omega : h(x) > 0\}$ with each of the sets $\{x \in \partial\Omega : f(x) > 0\}$, $\{x \in \partial\Omega : f(x) < 0\}$, $\{x \in \partial\Omega : g(x) > 0\}$, $\{x \in \partial\Omega : g(x) < 0\}$ have positive measures.

As in the previous example we consider the problem $(\Theta_{\lambda,\mu})$ in the framework of the Sobolev space $W = W^{1,p}(\Omega)\times W^{1,p}(\Omega)$ with the standard norm. Thus, a pair of functions $(u,v) \in W$ is said to be a weak solution of problem $(\Theta_{\lambda,\mu})$ if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u\nabla\varphi_1 + |u|^{p-2}\,u\varphi_1\right)dx$$

$$
\begin{aligned}
&+ \int_\Omega \Big(|\nabla v|^{p-2} \nabla v \nabla \varphi_2 + |v|^{p-2} v\varphi_2 \Big)\, dx - \lambda \int_{\partial\Omega} f\, |u|^{q-2} u\varphi_1 d\sigma \\
&- \mu \int_{\partial\Omega} g\, |v|^{q-2} v\varphi_2\, d\sigma - \frac{\alpha}{\alpha+\beta} \int_{\partial\Omega} h\, |u|^{\alpha-2} u\, |v|^{\beta} \varphi_1 d\sigma \\
&- \frac{\beta}{\alpha+\beta} \int_{\partial\Omega} h\, |u|^{\alpha} |v|^{\beta-2} v\varphi_2\, d\sigma = 0 \qquad \forall (\varphi_1, \varphi_2) \in W,
\end{aligned}
$$

where $d\sigma$ is the measure on the boundary. We shall apply Theorem 3.3, with potential operators A, B and C from W into W^* given by

$$
\begin{aligned}
\langle A(u,v), (\varphi_1,\varphi_2)\rangle &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 + |u|^{p-2} u\varphi_1 dx \\
&\quad + \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi_2 + |v|^{p-2} v\varphi_2 dx, \\
\langle B(u,v), (\varphi_1,\varphi_2)\rangle &= \lambda \int_{\partial\Omega} f\, |u|^{q-2} u\varphi_1 d\sigma + \mu \int_{\partial\Omega} g\, |v|^{q-2} v\varphi_2 d\sigma \\
\langle C(u,v), (\varphi_1,\varphi_2)\rangle &= \frac{\alpha}{\alpha+\beta} \int_{\partial\Omega} h\, |u|^{\alpha-2} u\, |v|^{\beta} \varphi_1\, d\sigma \\
&\quad + \frac{\beta}{\alpha+\beta} \int_{\partial\Omega} h\, |u|^{\alpha} |v|^{\beta-2} v\varphi_2\, d\sigma.
\end{aligned}
$$

Again, (H1), (H2) and (H3) are satisfied and by standard compact embedding theorems

$$
\langle B(u,v),(u,v)\rangle \le \overline{S}_q^q (|\lambda|\, \|f\|_\infty + |\mu|\, \|g\|_\infty)\, [\langle A(u,v),(u,v)\rangle]^{q/p}
$$

and

$$
\langle C(u,v),(u,v)\rangle \le \overline{S}_{\alpha+\beta}^{\alpha+\beta}\, \|h^+\|_\infty\, [\langle A(u,v),(u,v)\rangle]^{(\alpha+\beta)/p}\,.
$$

If we set $d_1 = \overline{S}_q^q \big(|\lambda|\, \|f\|_\infty + |\mu|\, \|g\|_\infty \big)$ and $d_2 = \overline{S}_{\alpha+\beta}^{\alpha+\beta}\, \|h^+\|_\infty$, then it is clear that (H4) is satisfied provided $|\lambda| + |\mu|$ is sufficiently small. Arguing as before we have the following.

Theorem 4.6. *There exists $\Lambda_0 > 0$ such that for $|\lambda| + |\mu| < \Lambda_0$ problem $(\Theta_{\lambda,\mu})$ has at least two positive solutions.*

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