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EXISTENCE OF POSITIVE ALMOST PERIODIC OR ERGODIC SOLUTIONS FOR SOME NEUTRAL NONLINEAR INTEGRAL EQUATIONS

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We state sufficient conditions for the existence of the positive, almost periodic or ergodic solutions of the following neutral integral equation:

$$x(t) = \gamma x(t-\sigma) + (1-\gamma)\int_{t-\sigma}^{t} f(s, x(s))ds,$$

where $0 \leq \gamma < 1$ and $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous map. We also treat the asymptotically, weakly and pseudo almost periodic solutions. Our results do not need the monotonicity of $f(t, .)$.

1. Introduction

As we all know, the existence of periodic solutions of functional differential equations (FDE) has great theoretical and practical significance, and is one of the problems of great interest to scholars in the field. Since Yoshizawa [28] presented an excellent result for the existence of periodic solutions to FDE with bounded delay, Cooke and Huang [10], Burton and Hatvani [6] generalized Yoshizawa's result to FDE with infinite delay. We remark that, in nature, there is no phenomenon which is purely periodic; this motivates the idea of considering the almost periodic situation.

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In this paper, we consider the following neutral non-linear integral equation:

$$x(t) = \gamma x(t-\sigma) + (1-\gamma)\int_{t-\sigma}^{t} f(s, x(s))\ ds, \tag{1.1}$$

where $0 \leq \gamma < 1$, $\sigma > 0$ and $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous map.

We give sufficient conditions which guarantee the existence of almost periodic solutions for Equation (1.1). We also treat the ergodic solutions; that means the asymptotically almost periodic, the weakly almost periodic, and pseudo almost periodic solutions. The hypotheses of our results do not require that the function $f(t, .)$ be monotone. To state our results, we use a variant of Hilbert's projective metric on a subset of a space of continuous and bounded functions.

Let us briefly describe the meaning of the equation in the context of epidemics. The number σ can be interpreted as the duration of an infection. In such a model, it is assumed that the total population is constant, $x(t)$ is the population at time t of infectious individuals, $f(t, x(t))$ is the instantaneous rate of infection, $f(t, x(t))dt$ is the fraction of individuals infected within the period $[t, t + dt]$. The number of infectious individuals at time t is obtained by summing up all individuals infected between $t - \sigma$ and t, thus leading to Equation (1.1).

Similar equations were considered, notably in connection with epidemic problems, by Cooke and Kaplan [9], Smith [22], Nussbaum [21], Busenberg and Cooke [7], Kaplan, Sorg and Yorke [18], Leggett and Williams [19], [20], Guo and Lakshmikantham [17]; all those authors considered the periodic case. The extensions of the periodic case were treated by Fink and Gatica [16], Torrejón [24], Chen and Torrejón [8], Ait Dads et al [1], [2], [3], [4] and [5], Ezzinbi and Hachimi [15], more recently by Xu and Yuan [25], [26]. All those works are concerned with almost periodic type solutions.

In recent years, we have witnessed considerable progress in the development of the theoretical tools of population dynamics. Some of the most important results include qualitative study, asymptotic behavior, and the existence of periodic, almost periodic solutions, and stationary solutions. The foundations and basic results of this theory can be found in the paper of Cooke and Kaplan [9]. Let us now give a brief account of what has been done in the literature.

Fink and Gatica [16] were the first to consider the positive almost periodic solutions of

$$x(t) = \int_{t-\sigma}^{t} f(s, x(s))\ ds, \tag{1.2}$$

in the case of delay σ being constant. Torrejón [24] studies the positive almost periodic solution of Equation (1.2) in the case when the delay is state dependent, namely the following:

$$x(t) = \int_{t-\sigma(x(t))}^{t} f(s, x(s))\, ds. \tag{1.3}$$

For Equation (1.2), Ezzinbi and Hachimi [15] state sufficient conditions for the existence of positive almost periodic solutions. For that they use Hilbert's projective metric on a subset of the space of almost periodic solutions. In this work, it is assumed that the function $f(t,.)$ is non-decreasing on $\mathbb{R}^+$ for every $t \in \mathbb{R}$. For Equation (1.2), Ait Dads, Arino and Ezzinbi [3] give similar results by using an argument of minimal and maximal solutions. Ait Dads and Ezzinbi [2] also used Hilbert's projective metric to present conditions for the existence of positive pseudo almost periodic solutions for Equation (1.1). In this work, it is also assumed that $f(t,.)$ is non-decreasing.

Xu and Yuan [26] study the existence of positive pseudo almost periodic solutions for Equation (1.1); they do not assume that $f(t,.)$ is non-decreasing, but only that $f(t,x) = f_1(t,x) + f_2(t,x)$ where $f_1(t,.)$ (respectively $f_2(t,.)$) is non-decreasing (respectively non-increasing). To state these results, the authors construct a new fixed-point theorem in a cone. Similarly, for Equation (1.3), Xu and Yuan [25] state the existence of positive, almost periodic, asymptotically almost periodic, weakly almost periodic and pseudo almost periodic solutions.

The paper is organized as follows: in Section 2, we recall some notation and definitions about almost periodic, asymptotically almost periodic, weakly almost periodic and pseudo almost periodic functions and we recall the main ideas related to Hilbert's projective metric. The results are announced and discussed in Section 3 and compared with those of Xu and Yuan [26] and Ait Dads and Ezzinbi [2]. In Section 4, we prove the main result and in Section 5 we give the proofs of consequences regarding asymptotically, weakly and pseudo almost periodic solutions.

2. Notation and definitions

2.1. **Some results on pseudo almost periodic functions.** Let E be a metric set, $C(E)$ (respectively $C_b(E)$) denotes the space of continuous (respectively continuous and bounded) functions defined on E with values in $\mathbb{R}$. For $f \in \mathcal{C}(\mathbb{R})$ (respectively $\mathcal{C}(\mathbb{R} \times \mathbb{R}^+)$) and $s \in \mathbb{R}$, the translation of f is the function $\tau_s f(t) = f(t-s)$, $t \in \mathbb{R}$, (respectively $\tau_s f(t,x) = f(t-s,x)$),

$(t,x) \in \mathbb{R} \times \mathbb{R}^+$). A subset X of $\mathcal{C}(\mathbb{R})$ is said to be translation invariant if $\tau_s X \subset X$ for all $s \in \mathbb{R}$. $\mathcal{C}_b(\mathbb{R})$ is endowed with the uniform topology.

Definition 1. [11] *A function $g \in \mathcal{C}(\mathbb{R})$ (respectively $\mathcal{C}(\mathbb{R} \times \mathbb{R}^+)$) is called almost periodic (respectively almost periodic in t uniformly with respect to $x \in \mathbb{R}^+$), if for each $\varepsilon > 0$ (respectively $\varepsilon > 0$ and compact $K \subset \mathbb{R}^+$), there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number μ with the property that*

$$\sup_{s \in \mathbb{R}} |\tau_\mu g(s) - g(s)| < \varepsilon,$$

respectively

$$\sup_{(s,x) \in \mathbb{R} \times K} |\tau_\mu g(s,x) - g(s,x)| < \varepsilon.$$

Denote $\mathcal{AP}(\mathbb{R})$ (respectively $\mathcal{AP}(\mathbb{R} \times \mathbb{R}^+)$) the set of all such functions. Every $g \in \mathcal{AP}(\mathbb{R})$ possesses a *mean value*

$$\mathcal{M}\{g(t)\}_t := \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} g(t)dt.$$

For each $\omega \in \mathbb{R}$, $a(g,\omega) := \mathcal{M}\{g(t)e^{-i\omega t}\}_t$ is the *Fourier-Bohr coefficient* of g associated at ω and $\Lambda(g) := \{\omega \in \mathbb{R} : a(g,\omega) \neq 0\}$ is the set of *exponents* of g. The *module* of g, denoted by $mod(g)$, is the additive group generated by $\Lambda(g)$. Similarly, if $g \in \mathcal{AP}(\mathbb{R} \times \mathbb{R}^+)$, the module of g, denoted also by $mod(g)$, is the additive group generated by

$$\Lambda(g) := \bigcup_{x \in \mathbb{R}^+} \{\omega \in \mathbb{R} : \mathcal{M}\{g(t,x)e^{-i\omega t}\}_t \neq 0\}.$$

Set

$$\mathcal{AAP}_0(\mathbb{R}) = \left\{ \varphi \in C(\mathbb{R}) : \lim_{|s| \to +\infty} \varphi(s) = 0 \right\},$$

and

$$\mathcal{AAP}_0(\mathbb{R} \times \mathbb{R}^+) = \left\{ \begin{array}{l} \varphi \in C(\mathbb{R} \times \mathbb{R}^+) : \text{ for any} \\ \text{compact subset } K \text{ of } \mathbb{R}^+ \lim\limits_{|t| \to \infty} \sup\limits_{x \in K} |\varphi(t,x)| = 0 \end{array} \right\}.$$

Definition 2. [29] *A function $f : \mathbb{R} \to \mathbb{R}$ (respectively $\mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$) is called asymptotically almost periodic (respectively asymptotically almost periodic in t uniformly with respect to $x \in \mathbb{R}^+$) if $f = g + \varphi$, where $g \in \mathcal{AP}(\mathbb{R})$ (respectively $\mathcal{AP}(\mathbb{R} \times \mathbb{R}^+)$) and $\varphi \in \mathcal{AAP}_0(\mathbb{R})$ (respectively $\mathcal{AAP}_0(\mathbb{R} \times \mathbb{R}^+)$). g and φ are called the principal and the corrective terms of f respectively. We denote $g = f^{ap}$ and $\varphi = f^e$.*

Denote $\mathcal{AAP}(\mathbb{R})$ (respectively $\mathcal{AAP}(\mathbb{R}\times\mathbb{R}^+)$) to be the set of all such functions.

Definition 3. [14] *A function $f\in C_b(\mathbb{R})$ is called weakly almost periodic if the set $\{\tau_s f:\ s\in\mathbb{R}\}$ is weakly compact in $C_b(\mathbb{R})$.*

Denote by $\mathcal{WAP}(\mathbb{R})$ the set of all such functions. $\mathcal{WAP}_0(\mathbb{R})$ is the set of $\varphi\in\mathcal{WAP}(\mathbb{R})$ such that there exists a sequence $(t_n)_n$ satisfying $\tau_{t_n}\varphi\rightharpoonup 0$ weakly in $C_b(\mathbb{R})$.

Definition 4. [14] *A function $f\in\mathcal{C}(\mathbb{R}\times\mathbb{R}^+,\mathbb{R})$ is called weakly almost periodic in t uniformly with respect to $x\in\mathbb{R}^+$ if*

i) for each $x\in\mathbb{R}^+$, $f(.,x)\in\mathcal{WAP}(\mathbb{R})$;

ii) for each compact subset K of $\mathbb{R}^+$, the map $I_K:K\to\mathcal{WAP}(\mathbb{R})$ defined by $I_K(x)=f(.,x)$ is continuous.

Denote by $\mathcal{WAP}(\mathbb{R}\times\mathbb{R}^+)$ the set of all such functions. Set

$$\mathcal{PAP}_0(\mathbb{R})=\left\{\varphi\in C_b(\mathbb{R}):\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}|\varphi(s)|\,ds=0\right\},$$

and

$$\mathcal{PAP}_0(\mathbb{R}\times\mathbb{R}^+)=\left\{\begin{array}{l}\varphi\in C(\mathbb{R}\times\mathbb{R}^+):\ \text{for any}\\ \text{compact subset } K \text{ of } \mathbb{R}^+ \sup\limits_{(t,x)\in R\times K}|\varphi(t,x)|<+\infty\\ \text{and } \lim\limits_{r\to\infty}\frac{1}{2r}\displaystyle\int_{-r}^{r}|\varphi(s,x)|ds=0 \text{ uniformly in } x\in K\end{array}\right\}.$$

Definition 5. [29] *A function $f:\mathbb{R}\to\mathbb{R}$ (respectively $\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$) is called pseudo almost periodic (respectively pseudo almost periodic in t uniformly with respect to $x\in\mathbb{R}^+$) if $f=g+\varphi$, where $g\in\mathcal{AP}(\mathbb{R})$ (respectively $\mathcal{AP}(\mathbb{R}\times\mathbb{R}^+)$) and $\varphi\in\mathcal{PAP}_0(\mathbb{R})$ (respectively $\mathcal{PAP}_0(\mathbb{R}\times\mathbb{R}^+)$). g and φ are called the almost periodic component and the ergodic perturbation of f respectively. We denote $g=f^{ap}$ and $\varphi=f^e$.*

Denote by $\mathcal{PAP}(\mathbb{R})$ (respectively $\mathcal{PAP}(\mathbb{R}\times\mathbb{R}^+)$) the set of all such functions. With these definitions, we have the following relations:

$$\mathcal{WAP}_0(\mathbb{R})=\mathcal{WAP}(\mathbb{R})\cap\mathcal{PAP}_0(\mathbb{R}),$$

$$\mathcal{WAP}_0(\mathbb{R}\times\mathbb{R}^+)=\mathcal{WAP}(\mathbb{R}\times\mathbb{R}^+)\cap\mathcal{PAP}_0(\mathbb{R}\times\mathbb{R}^+),$$

$$\mathcal{AP}(\mathbb{R})\subset\mathcal{AAP}(\mathbb{R})\subset\mathcal{WAP}(\mathbb{R})\subset\mathcal{PAP}(\mathbb{R}),$$

$$\mathcal{AP}(\mathbb{R}\times\mathbb{R}^+)\subset\mathcal{AAP}(\mathbb{R}\times\mathbb{R}^+)\subset\mathcal{WAP}(\mathbb{R}\times\mathbb{R}^+)\subset\mathcal{PAP}(\mathbb{R}\times\mathbb{R}^+).$$

2.2. **Hilbert's Projective Metric.** Let X be a real Banach space. A closed convex set K in X is called a convex cone if the following conditions are satisfied:

$$\begin{array}{lll} (i) & if\ x \in K, \text{ then } \lambda x \in K & for\ \lambda \geq 0 \\ (ii) & if\ x \in K, \text{ and } -x \in K, \text{ then } x = 0. \end{array} \tag{2.1}$$

A cone K induces a partial ordering $\leq$ in X by

$$x \leq y \qquad if \text{ and only if } y - x \in K. \tag{2.2}$$

A cone K is called "normal" if there exists a constant k such that

$$0 \leq x \leq y \qquad \text{implies that } \|x\| \leq k \|y\|, \tag{2.3}$$

where $\|.\|$ is the norm on X. If K is now a general cone in a Banach space X and x and y are elements of $K^* = K - \{0\}$, we say that x and y are "comparable" if there exist real numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha x \leq y \leq \beta x. \tag{2.4}$$

This defines an equivalence relation on K^* and divides K^* into disjoint subsets which we call "components of K". If x and y are comparable, we define the numbers $m(y/x)$ and $M(y/x)$ by

$$m(y/x) = \sup\{\alpha > 0 : \alpha x \leq y\} \tag{2.5}$$

$$M(y/x) = \inf\{\beta > 0 : y \leq \beta x\}. \tag{2.6}$$

We define a metric which was introduced by Thompson [23]. If $x, y \in K^*$ are comparable, define $d(x, y)$ by

$$\begin{aligned} d(x,y) &= \max(\log\ M(y/x), \log M(x/y)) \\ &= \max(\log(M(y/x), -\log m(y/x)). \end{aligned} \tag{2.7}$$

If C is a component of K, one can easily prove (see [23]) that d gives a metric on C. Moreover, Thompson proves the following result.

Theorem 6. [23] *Let K be a normal cone in a Banach space X and let C be a component of K. Then C is a complete metric space with respect to the metric d.*

Proposition 7. [23] *Let K be a normal cone in a Banach space X with non-empty interior $\overset{o}{K}$. Then $\overset{o}{K}$ is a component of K.*

It follows that, if K is a normal cone with non-empty interior, then $\overset{o}{K}$ is a complete metric space with respect to the metric d.

Theorem 8. (see page 190 in [13]) *Let E be a complete space with respect to the metric d. If f is a mapping from E into E satisfying*

$$d(f(x), f(y)) \leq \phi(d(x, y)) \qquad \textit{for all } x \textit{ and } y \textit{ in } E,$$

where ϕ is a positive non-decreasing function continuous on $[0, \infty)$, satisfying $\phi(r) < r$ for every $r > 0$ and $\phi(0) = 0$, then f has exactly one fixed point in E.

3. Statement of the results

Let X be a closed subspace of $C_b(\mathbb{R})$ that is invariant under translation. For Equation (1.1), we state a result of existence and uniqueness of the positive solution x in X. For that we formulate the following hypotheses about Equation (1.1). In the sequel, we denote

$$K_0 = \big\{x \in X : \inf_{t \in R} x(t) > 0\big\}.$$

We recall that $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous map and $0 \leq \gamma < 1$.

$H_1)$ $\forall \varepsilon \in (0, 1)$ there exists a continuous map $\varphi : (0, 1) \to \mathbb{R}$, satisfying $\varphi(\lambda) > \lambda$ and

$$\begin{aligned} \forall (x, y, t, \lambda) &\in [\varepsilon, \varepsilon^{-1}] \times [\varepsilon, \varepsilon^{-1}] \times \mathbb{R} \times (0, 1) \\ \lambda x &\leq y \leq \lambda^{-1} x \Rightarrow f(t, y) \geq \varphi(\lambda) f(t, x); \end{aligned}$$

$H_2)$ $\forall \sigma > 0$, $\forall x \in K_0$, $\Big(t \to \displaystyle\int_{t-\sigma}^{t} f(s, x(s))\, ds\Big) \in X$;

$H_3)$ $t \to f(t, 1)$ is bounded on $\mathbb{R}$;

$H_4)$ There exists $\sigma_0 > 0$ such that, $\inf_{t \in \mathbb{R}} \displaystyle\int_{t-\sigma_0}^{t} f(s, 1)\, ds > 0$;

$H_5)$ $\lim_{x \to +\infty} \frac{1}{x} f(t, x) = \lim_{x \to 0} x f(t, x) = 0$ uniformly with respect to t in $\mathbb{R}$.

Theorem 9. *Suppose that $H_1) - H_5)$ hold. Then there exists $\sigma_1 > 0$, such that, for all $\sigma \geq \sigma_1$, Equation (1.1) has a unique solution in K_0.*

The proof of Theorem 9 will be given in Section 4.

Remark 10. $H_1)$ implies that the function $x \to x\, f(t, x)$ is non-decreasing and $x \to \frac{1}{x}\, f(t, x)$ is non-increasing for each t in $\mathbb{R}$ (cf. Lemma 23).

Remark 11. For $X = C_b(\mathbb{R})$ and $f(t, x) = 1 + x$, f satisfies $H_1) - H_4)$ and for all $\sigma \geq 1$, Equation (1.1) has no solution, which imposes the hypothesis $H_5)$. In fact, for all $(x, y, t, \lambda) \in [\varepsilon, \varepsilon^{-1}] \times [\varepsilon, \varepsilon^{-1}] \times \mathbb{R} \times (0, 1)$, for

$$\lambda x \leq y \leq \lambda^{-1} x,$$

we have $f(t,y) \geq 1+\lambda x$, if in addition $\lambda < 1$, then $x \to \frac{1+\lambda x}{1+x}$ is non-increasing on $[\varepsilon, \varepsilon^{-1}]$; it follows that

$$\frac{1+\lambda x}{1+x} \geq \frac{1+\lambda\varepsilon^{-1}}{1+\varepsilon^{-1}} = \frac{\varepsilon+\lambda}{\varepsilon+1}.$$

Taking $\varphi(\lambda) = \frac{\varepsilon+\lambda}{\varepsilon+1}$, we obtain H_1). It is easy to verify $H_2) - H_4)$.

If x is a positive solution of Equation (1.2) for $\sigma \geq 1$, denoting $m = \inf_{t\in\mathbb{R}} x(t)$, we have

$$x(t) = \int_{t-\sigma}^{t} 1 + x(s)\ ds \geq \sigma(1+m) \geq 1+m,$$

therefore, $m \geq 1+m$, which is a contradiction.

An easy application of Theorem 9 to the bounded case is the following result.

Corollary 12. *Suppose that $H_1), H_3) - H_5)$ hold. Then there exists $\sigma_1 > 0$ such that, for all $\sigma \geq \sigma_1$, Equation* (1.1) *has a unique bounded solution such that*

$$\inf_{t\in R} x(t) > 0. \tag{3.1}$$

Proof. Hypothesis H_2) follows from iii) of Lemma 23. □

Now, we give an application of Theorem 9 to the almost periodic case.

Proposition 13. *Let $f : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ be a nonzero function almost periodic in t uniformly with respect to $x \in \mathbb{R}^+$. We assume that*

i) f satisfies H_1);

ii) $\lim_{x\to+\infty} \frac{1}{x} f(t,x) = 0$ uniformly with respect to t in $\mathbb{R}$.

Then there exists $\sigma_1 > 0$ such that, for all $\sigma \geq \sigma_1$, Equation (1.1) *has a unique almost periodic solution x satisfying* (3.1). *Furthermore, we have $mod(x) \subset mod(f)$.*

The proof of Proposition 13 will be given in Section 5. From Proposition 13, we deduce the following two corollaries in the periodic case.

Corollary 14. *Let $f : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ be a non-zero function and a continuous map and $\omega-$periodic with respect of t. We assume that*

i) f satisfies H_1);

ii) $\lim_{x\to+\infty} \frac{1}{x} f(t,x) = 0$ for each t in $\mathbb{R}$.

Then there exists $\sigma_1 > 0$ such that, for all $\sigma \geq \sigma_1$, Equation (1.1) *has a unique $\omega-$periodic solution x satisfying* (3.1).

Proof. Since the function $x \to \frac{f(t,x)}{x}$ is non-increasing on $\mathbb{R}^+$ for each t in the compact $[0,\omega]$ (cf. Remark 10) and $\frac{f(t,x)}{x} \underset{x\to\infty}{\to} 0$ pointwise on $[0,\omega]$, by using Dini's theorem, we deduce that $\frac{f(t,x)}{x} \underset{x\to\infty}{\to} 0$ uniformly on $[0,\omega]$. By periodicity of the function $f(.,x)$, hypothesis ii) of Proposition 13 is satisfied. By Proposition 13 , we obtain the existence and uniqueness of the almost periodic solution of Equation (1.1) satisfying (3.1). The formula of modules implies that this last solution is $\omega-$periodic. □

Corollary 15. *Let $f : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ be a continuous map $\omega-$periodic with respect to t. We assume that*

i) $\forall t \in R$, $x \to x\ f(t,x)$ is strictly increasing and $x \to \frac{1}{x}\ f(t,x)$ is strictly decreasing;

ii) $\lim\limits_{x\to+\infty} \frac{1}{x} f(t,x) = 0$ for each t in $\mathbb{R}$.

Then there exists $\sigma_1 > 0$, such that for all $\sigma \geq \sigma_1$, Equation (1.1) *has a unique $\omega-$periodic solution x satisfying* (3.1).

Proof. By Corollary 14, it suffices to prove that i) implies H_1). Since $f \geq 0$ and for all $t \in \mathbb{R}$, $x \to xf(t,x)$ is strictly increasing, we have $f(t,x) > 0$ for all $t \in \mathbb{R}$ and $x > 0$. Let $\varepsilon > 0$. If we choose the function $\varphi : (0,1) \to \mathbb{R}$ defined by

$$\varphi(\lambda) = \min\left\{\frac{f(t,y)}{f(t,x)} : t \in [0,\omega],\ (x,y) \in [\varepsilon,\varepsilon^{-1}],\ \lambda x \leq y \leq \lambda^{-1}x\right\},$$

then φ is evidently continuous. It remains to prove that $\varphi(\lambda) > \lambda$ for all $\lambda \in (0,1)$. By definition of φ, one has $\varphi(\lambda) = \frac{f(t_0,y_0)}{f(t_0,x_0)}$ with $t_0 \in [0,\omega]$, x_0, $y_0 \in [\varepsilon,\varepsilon^{-1}]$ and $\lambda x_0 \leq y_0 \leq \lambda^{-1}x_0$. The result is trivial if $x_0 = y_0$, because $\varphi(\lambda) = 1$. If $y_0 < x_0$, one has $\varphi(\lambda) > \frac{y_0}{x_0} \geq \lambda$, because the function $x \to \frac{1}{x}f(t,x)$ is strictly decreasing. If $x_0 < y_0$, we use the fact that the function $x \to xf(t,x)$ is strictly increasing; so H_1) holds. □

Remark 16. In the periodic case Hypothesis i) is a sufficient condition to have Hypothesis H_1), but not a necessary one. For example let us consider $f(t,x) = x\left(\frac{1+\cos t}{1+x}\right)$, which satisfies H_1) but not i).

Now, we give an application of Theorem 9 to the pseudo almost periodic case.

Proposition 17. *Let $f : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ be a pseudo almost periodic function in t uniformly with respect to $x \in \mathbb{R}^+$ such that f^{ap} is not the zero function. We assume that*

i) f satisfies H_1;
ii) $\lim\limits_{x\to+\infty}\frac{1}{x}f(t,x)=0$ uniformly with respect to t in $\mathbb{R}$;
iii) $\lim\limits_{\sigma\to+\infty}\inf\sup\limits_{t\in\mathbb{R}}\frac{1}{\sigma}\int_{t-\sigma}^{t}\left(f^e(s,1)\right)^-\ ds<m$, where $x^-:=\max(-x,0)$ and

$$m:=\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}f^{ap}(s,1)\ ds.$$

Then there exists $\sigma_1>0$ such that, for all $\sigma\geq\sigma_1$, Equation (1.1) *has a unique pseudo almost periodic solution x satisfying* (3.1). *Furthermore, if we denote by x^{ap} the almost periodic component of the solution x, then x^{ap} is the unique almost periodic solution of*

$$x^{ap}(t)=\gamma x^{ap}(t-\sigma)+(1-\gamma)\int_{t-\sigma}^{t}f^{ap}(s,x^{ap}(s))\ ds, \tag{3.2}$$

satisfying (3.1). *Furthermore, we have $mod(x^{ap})\subset mod(f^{ap})$.*

The proof of Proposition 17 will be given in Section 5.

Remark 18. With the hypotheses of Proposition 17, we have $m>0$. Hypothesis iii) is satisfied in particular if

$$\frac{1}{\sigma}\int_{t-\sigma}^{t}\left(f^e(s,1)\right)^-\ ds\ \ \text{or}\ \ \frac{1}{\sigma}\int_{t-\sigma}^{t}|f^e(s,1)|\ ds$$

converges uniformly on $\mathbb{R}$ to 0. Hypothesis iii) is also satisfied if $\inf_{t\in\mathbb{R}}f^e(t,1)>-m$ (in this case one has $\sup_{t\in\mathbb{R}}\left(f^e(t,1)\right)^-<m$).

Now, we give an application of Theorem 9 to the weakly almost periodic case.

Proposition 19. *Let $f:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+$ be a weakly almost periodic function in t uniformly with respect to $x\in\mathbb{R}^+$ such that f^{ap} is not the zero function. We assume that*
i) f satisfies H_1);
ii) $\lim\limits_{x\to+\infty}\frac{1}{x}f(t,x)=0$ uniformly with respect to t in $\mathbb{R}$.

Then there exists $\sigma_1>0$ such that, for all $\sigma\geq\sigma_1$, Equation (1.1) *has a unique weakly almost periodic solution x satisfying* (3.1). *Furthermore, if we denote by x^{ap} the almost periodic component of the solution x, then x^{ap} is the unique almost periodic solution of Equation* (3.2) *satisfying* (3.1). *Furthermore, we have $mod(x^{ap})\subset mod(f^{ap})$.*

The proof of Proposition 19 will be given in Section 5. Now, we give an application of Theorem 9 to the asymptotically almost periodic case.

Proposition 20. *Let $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ be an asymptotically almost periodic function in t uniformly with respect to $x \in \mathbb{R}^+$ such that f^{ap} is not the zero function. We assume that*

i) f satisfies H_1);

ii) $\lim_{x \to +\infty} \frac{1}{x} f(t, x) = 0$ uniformly with respect to t in $\mathbb{R}$.

Then there exists $\sigma_1 > 0$ such that, for all $\sigma \geq \sigma_1$, Equation (1.1) *has a unique asymptotically almost periodic solution x satisfying* (3.1). *Furthermore, if we denote by x^{ap} the principal term of the solution x, then x^{ap} is the unique almost periodic solution of Equation* (3.2) *satisfying* (3.1). *Furthermore, we have $mod(x^{ap}) \subset mod(f^{ap})$.*

The proof of Proposition 20 will be given in Section 5.

To close this section, we compare our results (Propositions 13 and 17) firstly with some results of Ait Dads and Ezzinbi [2], then with those of Xu and Yuan [26].

Remark 21. On results of Ait Dads and Ezzinbi [2]**.** For Equation (1.1) they give in Theorem 2.1 sufficient conditions for the existence and uniqueness of the positive pseudo almost periodic solution. Assume the following hypotheses:

I) $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a pseudo almost periodic function in t uniformly with respect to $x \in \mathbb{R}^+$, f^{ap} is not the zero function and $f^e \geq 0$;

II) $x \to f(t, x)$ is non-decreasing;

III) there exists $\varphi : (0, 1) \to \mathbb{R}$ such that $\forall \lambda \in (0, 1)$, $\varphi(\lambda) > \lambda$, $\forall x > 0$, $\forall t \in \mathbb{R}$, $f(t, \lambda x) \geq \varphi(\lambda) f(t, x)$;

IV) f is L-lipschitzian in x uniformly with respect to t in $\mathbb{R}$.

This result is as follows: "If Conditions $I) - IV$) hold, then Equation (1.1) has a unique pseudo almost periodic solution satisfying (3.1)."

II) and III) imply H_1). For ii) of Proposition 17, the function $x \to \sup_{t \in \mathbb{R}} \frac{f(t,x)}{x}$ is positive and non-increasing, thus it admits a limit $\ell \geq 0$ as x goes to infinity. Letting $\lambda \in (0, 1)$, one has for $x > 0$

$$\sup_{t \in \mathbb{R}} \frac{f(t, \lambda x)}{\lambda x} \geq \frac{\varphi(\lambda)}{\lambda} \sup_{t \in \mathbb{R}} \frac{f(t, x)}{x};$$

as x tends to $+\infty$, we obtain $\ell \geq \frac{\varphi(\lambda)}{\lambda} \ell$ and as $\frac{\varphi(\lambda)}{\lambda} > 1$, we deduce that $\ell = 0$; so ii) holds. It is easy to see that I) implies iii).

Then Proposition 17 gives existence and uniqueness of the pseudo almost periodic solution satisfying (3.1).

If we consider the function f defined by $f(t,x) = h(t)\frac{x}{1+x}$ where h is pseudo almost periodic, $h(t) \geq 0$ for each $t \in \mathbb{R}$,

$$m := \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} h^{ap}(s)\ ds > 0,$$

and $\inf_{t\in\mathbb{R}} h^e(t) > -m$, then hypotheses $ii)$ and $iii)$ of Proposition 17 hold. Hypothesis $H_1)$ is satisfied with $\phi(\lambda) = \lambda\frac{1+\epsilon}{1+\lambda\epsilon}$; in fact, for $\epsilon > 0$, $t \in \mathbb{R}$, x, $y \in [\epsilon, \epsilon^{-1}]$, $\lambda \in (0,1)$, by monotonicity of $x \longrightarrow f(t,x)$

$$f(t,y) \geq f(t,\lambda x) \geq \inf_{\epsilon\leq u\leq\epsilon^{-1}} \left(\frac{f(t,\lambda u)}{f(t,u)}\right) f(t,x),$$

where

$$\phi(\lambda) = \inf_{\epsilon\leq u\leq\epsilon^{-1}} \frac{f(t,\lambda u)}{f(t,u)} = \lambda \inf_{\epsilon\leq u\leq\epsilon^{-1}} \frac{1+u}{1+\lambda u} = \lambda\frac{1+\epsilon}{1+\lambda\epsilon} > \lambda.$$

Proposition 17 permits us to give existence and uniqueness of the positive pseudo almost periodic solution satisfying (3.1). Although the function $x \to f(t,x)$ is non-decreasing, we cannot conclude with ([2], Theorem 2.1), because

$$\inf_{u>0} \frac{f(t,\lambda u)}{f(t,u)} = \lambda \inf_{u>0} \frac{1+u}{1+\lambda u} = \lambda,$$

thus $III)$ is not valid. In conclusion, Proposition 17 is an improvement of ([2], Theorem 2.1) and also a generalization, because it does not need the monotonicity of $x \to f(t,x)$.

Remark 22. On results of Xu and Yuan [26]**.** For Equation (1.1) they give in Theorem 1 sufficient conditions for the existence and uniqueness of the positive almost periodic solution. Assume the following hypotheses on $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ with $f(t,x) = f_1(t,x) + f_2(t,x)$:

$Y_1)$ for every $t \in \mathbb{R}$, $f_1(t,.)$ is non-decreasing and $f_2(t,.)$ non-increasing on $\mathbb{R}^+$;

$Y_2)$ there exist two functions φ_1 and $\varphi_2 : (0,1) \to \mathbb{R}$, such that, for every $t \in \mathbb{R}$ and $\lambda \in (0,1)$, one has

i) $\varphi_1(\lambda,x)$, $\varphi_2(\lambda,x) > \lambda$, $f_1(t,\lambda x) \geq \varphi_1(\lambda,x)f_1(t,x)$ and $f_2(t,\lambda^{-1}y) \geq \varphi_2(\lambda,y)f_2(t,y)$, for every x, $y > 0$;

ii) $\varphi_1(t,.)$ is non-decreasing and $\varphi_2(t,.)$ non-increasing on $\mathbb{R}^+$;

$Y_3)$ f_1 and $f_2 : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ are almost periodic functions in t uniformly with respect to $x \in \mathbb{R}^+$ and f_1 is not the zero function;

$Y_4)$ $\lim_{x\to+\infty} \frac{f_1(t,x)}{x} = 0$ uniformly with respect to t.

This result is as follows: "If Conditions $Y_1) - Y_4)$ hold, then Equation (1.1) has a unique almost periodic solution satisfying (3.1) and the formula of modules."

For $H_1)$: for $x, y \in [\varepsilon, \varepsilon^{-1}]$ with $\lambda x \le y \le \lambda^{-1} x$ one has:

$$f_1(t,y) \ge f_1(t,\lambda x) \ge \varphi_1(\lambda,\varepsilon) f_1(t,x)$$
$$f_2(t,y) \ge f_2(t,\lambda^{-1} x) \ge \varphi_2(\lambda,\varepsilon^{-1}) f_2(t,x).$$

If we consider $\varphi(\lambda) = \min(\varphi_1(\lambda,\varepsilon), \varphi_2(\lambda,\varepsilon^{-1}))$, we deduce that

$$f(t,y) \ge \varphi(\lambda) f(t,x) \;\text{ and }\; \varphi(\lambda) > \lambda,$$

therefore $H_1)$ holds.

Then Proposition 13 gives existence and uniqueness of the almost periodic solution satisfying (3.1) and the formula of modules.

If we consider the function f defined by $f(t,x) = \frac{x \sin^2 t}{1+x}$, for $0 \le x \le 1$ and $f(t,x) = \frac{\sin^2 t}{1+x}$ for $x \ge 1$, the hypotheses of Proposition 13 hold, thus Proposition 13 permits us to give existence and uniqueness of the almost periodic solution satisfying (3.1). On the other hand, we cannot conclude with ([26], Theorem 1), because $Y_1)$ is not satisfied; in fact $Y_1)$ and f_1, $f_2 \ge 0$ imply that $f(t,.)$ is non-decreasing if $f(t,0) = 0$ and $f(t,.)$ is non-increasing if $\lim\limits_{x\to\infty} f(t,x) = 0$.

In conclusion, Proposition 13 is an improvement of ([26], Theorem 1).

For the same reasons, in the pseudo almost periodic case, Proposition 17 is an improvement and a generalization of the results of ([2], Theorem 2.1) and ([26], Theorem 3).

4. Proof of the Main Result

For the proof of Theorem 9, we need the following preliminary results.

Lemma 23. *i) For all* $x, y > 0, t \in \mathbb{R}$, $f(t,y) \ge \min(\frac{x}{y}, \frac{y}{x}) f(t,x)$.

ii) For all $t \in \mathbb{R}$, *the function* $x \to x f(t,x)$ *is non-decreasing and* $x \to \frac{f(t,x)}{x}$ *is nonincreasing on* $(0,+\infty)$.

iii) For all $[a,b] \subset (0,+\infty)$, f *is bounded on* $\mathbb{R} \times [a,b]$.

iv) For all $x, y > 0$, $t \in \mathbb{R}$,

$$|f(t,x) - f(t,y)| \le \frac{\max(f(t,x), f(t,y))}{\max(x,y)} |x-y|.$$

Proof. *i*) Let $x, y > 0$ and $\varepsilon > 0$ such that $x, y \in [\varepsilon, \varepsilon^{-1}]$. We can assume $x \neq y$, by taking $\lambda = \min(\frac{x}{y}, \frac{y}{x})$; we obtain $\lambda x \leq y \leq \lambda^{-1}x$, then

$$f(t,y) \geq \min(\frac{x}{y}, \frac{y}{x}) f(t,x).$$

ii) follows from *i*).

iii) By *i*) one has, for $x \in [a,b]$,

$$f(t,1) \geq \min(x, \tfrac{1}{x}) f(t,x) \geq \min(a, \tfrac{1}{b}) f(t,x);$$

the result is a consequence of H_3).

iv) With *i*) of this lemma and $\min(\frac{x}{y}, \frac{y}{x}) - 1 = -\frac{|x-y|}{\max(x,y)}$, we deduce that $f(t,y) - f(t,x) \geq -\frac{|x-y|}{\max(x,y)} f(t,x)$. By changing the role of x and y, the result follows. □

Lemma 24. $\lim\limits_{R \to +\infty} \frac{1}{R} \sup\limits_{R^{-1} \leq x \leq R,\ t \in \mathbb{R}} f(t,x) = 0.$

Proof. Let $\varepsilon > 0$, by H_5); there exists $a > 1$, such that, for all $x > a$, $t \in \mathbb{R}$,

$$\frac{f(t,x)}{x} < \varepsilon \quad \text{and} \quad \frac{1}{x} f(t, \tfrac{1}{x}) < \varepsilon.$$

For $R > a$ one has

$$\frac{1}{R} \sup_{a \leq x \leq R, t \in \mathbb{R}} f(t,x) \leq \sup_{a \leq x \leq R, t \in \mathbb{R}} \frac{f(t,x)}{x} < \varepsilon,$$

and

$$\frac{1}{R} \sup_{R^{-1} \leq x \leq a^{-1}, t \in \mathbb{R}} f(t,x) \leq \sup_{R^{-1} \leq x \leq a^{-1}, t \in \mathbb{R}} x f(t,x) = \sup_{a \leq x \leq R, t \in \mathbb{R}} \frac{1}{x} f(t, \tfrac{1}{x}) < \varepsilon,$$

and by *iii*) of Lemma 23, we have $\frac{1}{R} \sup_{a^{-1} \leq x \leq a, t \in \mathbb{R}} f(t,x)$ which goes to 0, thus there exists $R_0 > a$ such that, for all $R > R_0$,

$$\frac{1}{R} \sup_{R^{-1} \leq x \leq R, t \in \mathbb{R}} f(t,x) < \varepsilon,$$

which gives the result. □

Now, we go back to the problem of the neutral integral equation (1.1). For $\gamma \in [0, 1[$, the operator $Id - \gamma \tau_\sigma$ is a bicontinuous isomorphism on X with inverse L given by

$$L = \sum_{n=0}^{+\infty} \gamma^n \tau_{n\sigma}. \tag{4.1}$$

From (4.1), we deduce the following lemma.

Lemma 25. *The operator L is non-decreasing in the sense*

$$x \leq y \Rightarrow Lx \leq Ly.$$

In the sequel, making the following variable change:

$$y = (Id - \gamma\tau_\sigma)(x),$$

Equation (1.1) will be rewritten

$$y(t) = (1-\gamma)\int_{t-\sigma}^{t} f(s,(Ly)(s))\ ds. \tag{4.2}$$

Denote

$$(Tx)(t) = (1-\gamma)\int_{t-\sigma}^{t} f(s, Lx(s))\ ds$$

and $K_\varepsilon = \{x \in X :\ \forall t \in \mathbb{R},\ \varepsilon \leq x(t) \leq \varepsilon^{-1}\}$.

Lemma 26. *There exists $\sigma_1 > 0$ such that, for all $\sigma \geq \sigma_1$, there exists $\varepsilon_\sigma \in (0,1)$ such that, for all $\varepsilon \in (0, \varepsilon_\sigma)$, $TK_\varepsilon \subset K_\varepsilon$.*

Proof. From H_4), there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$,

$$\int_{t-\sigma_0}^{t} f(s,1)\ ds \geq \delta.$$

Let $n \in \mathbb{N}$ with $n\delta \geq \frac{1}{(1-\gamma)^2}$. By letting $\sigma_1 = n\sigma_0$, one has

$$\forall t \in \mathbb{R}, \forall \sigma \geq \sigma_1, \int_{t-\sigma}^{t} f(s,1)\ ds \geq \frac{1}{(1-\gamma)^2}. \tag{4.3}$$

Letting $\sigma \geq \sigma_1$ and $x \in K_\varepsilon$, one has $\varepsilon \leq x \leq \varepsilon^{-1}$ and by Lemma 25 we obtain

$$\varepsilon(1-\gamma) \leq \frac{\varepsilon}{1-\gamma} \leq Lx(s) \leq \frac{1}{(1-\gamma)\varepsilon}. \tag{4.4}$$

It then follows from Lemma 24 that

$$(Tx)(t) \leq (1-\gamma)\sigma \sup_{s\in\mathbb{R}} \sup_{\varepsilon(1-\gamma) \leq u \leq \frac{\varepsilon^{-1}}{1-\gamma}} f(s,u) = o(\frac{1}{\varepsilon}),$$

thus there exists $\varepsilon_\sigma \in (0,1)$ such that, for all $\varepsilon \in (0, \varepsilon_\sigma)$,

$$(Tx)(t) \leq \frac{1}{\varepsilon}.$$

Regarding the other part, from i) of Lemma 23, one has

$$\int_{t-\sigma}^{t} f(s, Lx(s))\, ds \geq \int_{t-\sigma}^{t} \min(Lx(s), \frac{1}{Lx(s)}) f(s,1)\ ds,$$

and by (4.3) and (4.4) one has for all $\sigma \geq \sigma_1$,

$$(1-\gamma)\int_{t-\sigma}^{t} f(s, Lx(s))\ ds \geq \varepsilon(1-\gamma)^2 \int_{t-\sigma}^{t} f(s,1)\ ds \geq \varepsilon,$$

thus $(Tx)(t) \geq \varepsilon$. By H_2), furthermore, one has $Tx \in X$, thus $Tx \in K_\varepsilon$. □

Proof of Theorem 9. Using the variable change $y = (Id - \gamma\tau_\sigma)x$, we have that x is a solution of Equation (1.1) if and only if y is a solution of Equation (4.2) and by invariance of X under translation, we obtain $x \in X$ if and only if $y \in X$. From Lemma 25, we deduce that, if $\inf\limits_{t\in\mathbb{R}} y(t) > 0$, then $\inf\limits_{t\in\mathbb{R}} x(t) > 0$. Denote by $\mathcal{P}$ and $\mathcal{Q}$ the two following problems:

$$\mathcal{P} : x \in K_0 \text{ and } x \text{ is a solution of Equation (1.1)}$$

and

$$\mathcal{Q} : y \in K_0 \text{ and } y \text{ is a solution of Equation (4.2).}$$

Evidently $\mathcal{Q} \subset \mathcal{P}$. For the inverse inclusion, it suffices to prove that, if x is a solution of $\mathcal{P}$, then $\inf\limits_{t\in\mathbb{R}} y(t) > 0$. For that, we use Lemma 23 and Hypothesis H_4) :

$$y(t) = x(t) - \gamma x(t-\sigma) = (1-\gamma)\int_{t-\sigma}^{t} f(s, x(s))\ ds,$$

and we deduce that

$$y(t) \geq (1-\gamma)\min(\inf_{t\in\mathbb{R}} x(t), \frac{1}{\sup\limits_{t\in\mathbb{R}} x(t)})\inf_{t\in\mathbb{R}}\int_{t-\sigma_0}^{t} f(s,1)\ ds,$$

for $\sigma \geq \sigma_0$. In conclusion, for $\sigma \geq \sigma_0$, the two problems $\mathcal{P}$ and $\mathcal{Q}$ are equivalent.

The problem $\mathcal{Q}$ may be formulated as

$$\mathcal{Q}\colon\ y \in K_0 \quad \text{and} \quad Ty = y.$$

For $z \in K_0$, we denote $M(z) = \sup_{t\in\mathbb{R}} z(t)$. K_0 is endowed with the metric

$$d(x,y) = \ln\max(M(\tfrac{x}{y}), M(\tfrac{y}{x})).$$

We know that (K_0, d) is a complete metric space. By H_4), there exists $t_0 \in \mathbb{R}$ such that $f(t_0, 1) > 0$ and by H_1), one has $f(t_0, 1) \geq \phi(\lambda)f(t_0, 1)$ and $\phi(\lambda) > \lambda$ for all $\lambda \in (0,1)$, thus $\lim_{\lambda\to 1}\phi(\lambda) = 1$. Now, we have that the function ϕ is defined and continuous on $(0,1]$. Let $\sigma \geq \sigma_1$. By Lemma 26, there exists $\varepsilon_\sigma \in (0,1)$ such that K_ε is invariant under T for every $\varepsilon \in (0, \varepsilon_\sigma)$. Since K_ε is a closed subset of K_0, it follows that (K_ε, d) is a complete metric space. To obtain a fixed point of T in K_ε, we use Theorem 8. We can assume that ϕ is non-decreasing (for that replace ϕ with $\phi_1(\lambda) = \inf_{\mu\in[\lambda,1]} \phi(\mu)$).

Let $x, y \in K_\varepsilon$, $\lambda \in (0,1)$ such that $\lambda x \le y \le \lambda^{-1}x$. By Lemma 25, one has $\lambda Lx \le Ly \le \lambda^{-1}Lx$, then, by H_1), we obtain

$$\forall t \in \mathbb{R},\ f(t, Ly(t)) \ge \varphi(\lambda) f(t, Lx(t)).$$

As we have also $\lambda y \le x \le \lambda^{-1}y$, we deduce that

$$\forall t \in \mathbb{R}, \qquad \varphi(\lambda) f(t, Lx(t)) \le f(t, Ly(t)) \le \varphi(\lambda)^{-1} f(t, Lx(t)),$$

thus,

$$\varphi(\lambda)(Tx)(t) \le (Ty)(t) \le \varphi(\lambda)^{-1}(Tx)(t),$$

and

$$d(Ty, Tx) \le \ln(\varphi(\lambda)^{-1}).$$

For $\lambda = [\max(M(\frac{y}{x}), M(\frac{x}{y}))]^{-1}$ we have $d(x,y) = \ln(\lambda^{-1})$. If we choose the function $\Phi(u) = -\ln(\varphi(e^{-u}))$ we deduce that

$$d(Ty, Tx) \le \Phi(d(x,y)),$$

and, by H_1), one has $\varphi(\lambda) > \lambda$, then $\forall u > 0$, $\Phi(u) < u$. By Theorem 8, we obtain the existence and uniqueness of the fixed point of T, therefore Equation (4.2) admits a unique solution in K_ε.

Uniqueness of the solution in K_0. Let x, y be two solutions in K_0 and $\sigma \ge \sigma_1$. Then, there exists $\varepsilon \in (0, \varepsilon_\sigma)$ such that $x, y \in K_\varepsilon$, consequently $x = y$. □

5. Proofs of Consequences of Theorem 9

In this section, we prove Propositions 13, 17, 19 and 20.

Proof of Proposition 13. We apply Theorem 9 to the space $X = AP(\mathbb{R})$ and f. X is a subspace of $C_b(\mathbb{R})$ invariant under translation. By the theorem of the composition of almost periodic functions, we obtain H_2). f is almost periodic thus it satisfies H_3). Let us verify H_4). Since f is not the zero function, there exists $(t_0, x_0) \in \mathbb{R}\times]0, +\infty[$, such that $f(t_0, x_0) > 0$. Then we have

$$f(t_0, 1) \ge \min(x_0, \frac{1}{x_0}) f(t_0, x_0) > 0, \tag{5.1}$$

thus $f(., 1)$ is not the zero function; as f is almost periodic, its mean is strictly positive, thus there exist $L > 0$ and $\sigma_0 > 0$ such that

$$\forall \sigma \ge \sigma_0,\ \forall t \in \mathbb{R},\ \frac{1}{\sigma}\int_{t-\sigma}^{t} f(s, 1)\ ds \ge L,$$

so one has H_4). For H_5) we have

$$\lim_{x\to+\infty}\frac{1}{x}f(t,x)=0 \text{ uniformly with respect to } t \text{ in } \mathbb{R}.$$

f is almost periodic thus it is bounded on $\mathbb{R}\times[0,1]$, thus

$$\lim_{x\to 0} xf(t,x)=0 \text{ uniformly with respect to } t \text{ in } \mathbb{R}.$$

Formula of Modules. There exist a and $b\in\mathbb{R}$ such that $0<a<b$ and the almost periodic solution x satisfies $a\le x(t)\le b$ for each $t\in\mathbb{R}$. Let $(t_n)_n$ be a numerical sequence such that

$$f(t+t_n,x)\to g(t,x) \quad \text{if} \quad n\to+\infty, \tag{5.2}$$

uniformly on $\mathbb{R}\times[a,b]$. To state the formula of modules, it suffices to show that the sequence $(x(.+t_n))_n$ converges uniformly on $\mathbb{R}$ (see [27] Theorem 2.8, page 18). Note that $g:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+$ is almost periodic in t uniformly with respect to $x\in\mathbb{R}^+$ and satisfies Hypothesis H_1). From the definition of g we deduce that $\sup_{t\in\mathbb{R}} g(t,x)=\sup_{t\in\mathbb{R}} f(t,x)$; thus g satisfies all hypotheses of Proposition 13, so the equation

$$y(t)=\gamma y(t-\sigma)+(1-\gamma)\int_{t-\sigma}^{t} g(s,y(s))\,ds \tag{5.3}$$

has a unique almost periodic solution y such that $\inf_{t\in\mathbb{R}} y(t)>0$. Let us consider a subsequence of $(x(.+t_n))_n$ which we denote in a similar manner. Since this last subsequence has values in $AP(\mathbb{R})$, it has a cluster point x_* in $AP(\mathbb{R})$, so we have

$$x(t+t_n)\to x_*(t) \quad \text{if} \quad n\to+\infty, \tag{5.4}$$

uniformly on $\mathbb{R}$. Since x is a solution of Equation (1.1), $x(t+t_n)$ is a solution of

$$x(t+t_n)=\gamma x(t+t_n-\sigma)+(1-\gamma)\int_{t-\sigma}^{t} f(s+t_n,x(s+t_n))\,ds. \tag{5.5}$$

We deduce from (5.2) and (5.4) that

$$\lim_{n\to\infty}\int_{t-\sigma}^{t} f(s+t_n,x(s+t_n))\,ds=\int_{t-\sigma}^{t} g(s,x_*(s))\,ds,$$

so, with (5.4) and (5.5), we obtain that x_* is an almost periodic solution of (5.3) satisfying (3.1). By uniqueness of this last solution, we have $x_*=y$. We deduce that $(x(t+t_n))_n$ converges uniformly on $\mathbb{R}$. In conclusion, we have the desired result. $\square$

Proof of Proposition 17. The almost periodic component f^{ap} of f satisfies the hypotheses of Proposition 13. In fact $f^{ap} : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$, f^{ap} is not the zero function and $f^{ap} \geq 0$, since $f \geq 0$. Furthermore, f satisfies H_1), thus, for x and $y \in [\epsilon, \epsilon^{-1}]$, $\lambda \in (0,1)$ and $\lambda x \leq y \leq \lambda^{-1}x$, the map $t \to f(t,y) - \phi(\lambda)f(t,x)$ is pseudo almost periodic and non-negative, thus its almost periodic component is nonnegative: $f^{ap}(t,y) - \phi(\lambda)f^{ap}(t,x) \geq 0$, consequently f^{ap} verifies H_1). Let $\epsilon > 0$. Since $\lim_{x\to+\infty} \frac{f(t,x)}{x} = 0$ uniformly with respect to t, there exists $R > 0$ such that $0 \leq \frac{f(t,x)}{x} \leq \epsilon$, for each $x \geq R$ and $t \in \mathbb{R}$. For $x \geq R$, the map $t \to \frac{f(t,x)}{x}$ is pseudo almost periodic and with values in $[0,\epsilon]$, thus its almost periodic component also has values in $[0,\epsilon]$: $\frac{f^{ap}(t,x)}{x} \in [0,\epsilon]$; so $\lim\limits_{x\to+\infty} \frac{f^{ap}(t,x)}{x} = 0$ uniformly with respect to t. Since f^{ap} satisfies the hypotheses of Proposition 13, we deduce that Equation (3.2) has a unique almost periodic solution y_* which satisfies (3.1). Furthermore, one has $mod(y_*) \subset mod(f^{ap})$. We apply Theorem 9 to the space $X = PAP(\mathbb{R})$ and the function f. As f is in $PAP(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$, it follows that $t \to f(t,1)$ is bounded, thus f satisfies H_1) and H_3) so that we may apply Lemma 23 and obtain, for all a and $b \in \mathbb{R}$ such that $0 < a < b$, that f is bounded on $\mathbb{R} \times [a,b]$ and $\mid f(t,x) - f(t,y) \mid \leq \frac{M}{a} \mid x - y \mid$ for all x and $y \in [a,b]$ where $M := \sup_{s\in\mathbb{R}} \sup_{a\leq u\leq b} f(s,u) < +\infty$. From the composition theorem of pseudo almost periodic functions [12], we obtain that $t \to f(t,x(t))$ is in $PAP(\mathbb{R})$, for each $x \in K_0$. We deduce that $t \to \int_{t-\sigma}^{t} f(s,x(s))\ ds$ is also in $PAP(\mathbb{R})$, thus H_2) is satisfied. H_5) is satisfied since f is pseudo almost periodic, so that f is bounded on $\mathbb{R} \times [0,1]$, thus $\lim\limits_{x\to 0} x f(t,x) = 0$ uniformly with respect to t. For H_4), since f^{ap} satisfies the hypotheses of Proposition 13, we prove as in the same manner in the proof of Proposition 13 that

$$m := \lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} f^{ap}(s,1)\ ds > 0; \tag{5.6}$$

so if we denote

$$l := \liminf_{\sigma\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{\sigma} \int_{t-\sigma}^{t} \left(f^{e}(s,1)\right)^{-}\ ds,$$

one has $0 \leq l < m$. With (5.6), we deduce that there exists $\sigma_* > 0$ such that, for each $\sigma \geq \sigma_*$, one has

$$\inf_{t\in\mathbb{R}} \frac{1}{\sigma} \int_{t-\sigma}^{t} f^{ap}(s,1)\ ds \geq \frac{m+l}{2}.$$

With the inequality $f(t,1) \geq f^{ap}(t,1) - (f^e(t,1))^-$, we deduce that

$$\inf_{t\in\mathbb{R}} \frac{1}{\sigma}\int_{t-\sigma}^{t} f(s,1)\ ds \geq \inf_{t\in\mathbb{R}} \frac{1}{\sigma}\int_{t-\sigma}^{t} f^{ap}(s,1)\ ds - \sup_{t\in\mathbb{R}} \frac{1}{\sigma}\int_{t-\sigma}^{t} (f^e(s,1))^-\ ds.$$

For $\sigma \geq \sigma_*$ we obtain

$$\inf_{t\in\mathbb{R}} \frac{1}{\sigma}\int_{t-\sigma}^{t} f(s,1)\ ds \geq \frac{m+l}{2} - \sup_{t\in\mathbb{R}} \frac{1}{\sigma}\int_{t-\sigma}^{t} (f^e(s,1))^-\ ds,$$

and by taking the superior limit, one has

$$\limsup_{\sigma\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{\sigma}\int_{t-\sigma}^{t} f(s,1)\ ds \geq \frac{m+l}{2} - l = \frac{m-l}{2} > 0;$$

thus there exists $\sigma_0 > 0$ such that

$$\frac{1}{\sigma_0}\int_{t-\sigma_0}^{t} f(s,1)\ ds \geq \frac{m-l}{4} > 0.$$

By Theorem 9, we obtain that Equation (1.1) has a unique pseudo almost periodic solution x satisfying (3.1). To conclude, it suffices to show that the almost periodic solution y_* of Equation (3.2) is equal to x^{ap}. From the composition theorem of pseudo almost periodic functions we obtain that $t \to f(t,x(t))$ is in $PAP(\mathbb{R})$ and $(f(t,x(t)))^{ap} = f^{ap}(t,x^{ap}(t))$, thus x^{ap} is a solution of Equation (3.2). Furthermore the solution x^{ap} satisfies

$$\inf_{t\in\mathbb{R}} x^{ap}(t) \geq \inf_{t\in\mathbb{R}} x(t) > 0.$$

By Proposition 13 we have the uniqueness of such a solution, thus $x^{ap} = y_*$. □

Proof of Proposition 19. We apply Theorem 9 to the space $X = WAP(\mathbb{R})$ and the function f. From the composition theorem of weakly almost periodic functions [14], we obtain that $t \to f(t,x(t))$ is in $WAP(\mathbb{R})$, for each $x \in K_0$. By noting that the map $I : C_b(\mathbb{R}) \longrightarrow C_b(\mathbb{R})$ defined by $(Iu)(t) = \int_{t-\sigma}^{t} u(s)\ ds$ is linear and bounded, we deduce that $t \to \int_{t-\sigma}^{t} f(s,x(s))\ ds$ is also weakly almost periodic, so H_2) is satisfied. Since

$$WAP(\mathbb{R}) \subset PAP(\mathbb{R}) \quad \text{and} \quad WAP(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}) \subset PAP(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}),$$

we obtain in a similar manner as in the proof of Proposition 17 that H_1), H_3) and H_5) are satisfied and f^{ap} satisfies the hypotheses of Proposition 13 and (5.6). Let us verify H_4). Since $t \to f(t,1)$ is in $WAP(\mathbb{R})$ and $t \to f^{ap}(t,1)$ is its almost periodic component, it follows that

$$\lim_{r\to+\infty} \frac{1}{r}\int_{t-r}^{t} f(s,1)\ ds = \mathcal{M}\{f^{ap}(t,x_0)\}_t = m > 0, \tag{5.7}$$

uniformly with respect to t. From (5.7), we deduce the existence of $\sigma_0 > 0$ such that

$$\inf_{t\in\mathbb{R}} \frac{1}{\sigma_0} \int_{t-\sigma_0}^{t} f(s,1)\ ds \geq \frac{m}{2} \geq 0.$$

By Theorem 9, we obtain that Equation (1.1) has a unique weakly almost periodic solution x satisfying (3.1). Since $WAP_0(\mathbb{R}) \subset PAP_0(\mathbb{R})$, the end of the proof is similar to the one of Proposition 17, consequently we deduce that $(f(t,x(t)))^{ap} = f^{ap}(t,x^{ap}(t))$ and x^{ap} is the unique almost periodic solution of Equation (3.2). □

Proof of Proposition 20. We apply Theorem 9 to the space $X = AAP(\mathbb{R})$ and the function f. It is easy to see that the function $t \to \int_{t-\sigma}^{t} f(s,x(s))\ ds$ is in $AAP(\mathbb{R})$, for each $x \in K_0$, so $H_2)$ is satisfied. By using the proof of Proposition 19 and the fact that

$$AAP(\mathbb{R}) \subset WAP(\mathbb{R}) \quad \text{and} \quad AAP(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}) \subset WAP(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}),$$

we deduce the result. □

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REFERENCES

[1] E. Ait Dads and O. Arino, *A nonlinear delay differential equation whose solutions are asymptotically sums of periodic functions*, Funkcial. Ekvac., 32 (1989), 81–89.

[2] E. Ait Dads and K. Ezzinbi, *Existence of positive pseudo almost periodic solution for a class of functional equations arising in epidemic problems*, Cybernet. Systems Anal., 30 (1994), 900–910.

[3] E. Ait Dads, O. Arino, and K. Ezzinbi, *Positive almost periodic solution for some nonlinear delay integral equation*, Nonlinear Stud., 3 (1996), 85–101.

[4] E. Ait Dads and O. Arino, *Exponential dichotomy and existence of pseudo almost-periodic solutions of some differential equations*, Nonlinear Anal., 27 (1996), 369–386.

[5] E. Ait Dads, O. Arino, and K. Ezzinbi, *Pseudo almost periodic solutions for some differential equations in a Banach space*, Nonlinear Anal., (1997), 1141–1155.

[6] T.A. Burton and L. Hatvani, *On the existence of periodic solutions of some nonlinear functional-differential equations with unbounded delay*, Nonlinear Anal., 16 (1991), 389–396.

[7] S. Busenberg and K. Cooke, *Periodic solutions of delay differential equations arising in some models of epidemics*, Applied Nonlinear Analysis (Proc. Third Internat. Conf., Univ. Texas, Arlington, Tex., 1978), pp. 67–78, Academic Press, New York, 1979.

[8] S. Chen and R. Torrejón, *Bifurcation of almost periodic solutions for a nonlinear integral equation with delay*, Nonlinear Anal., 27 (1996), 863–877.
[9] K. Cooke and J. Kaplan, *A periodicity threshold theorem for epidemics and population growth*, Math. Biosci., 31 (1976), 87–104.
[10] K.L. Cooke and W.Z. Huang, *On the problem for linearization for state-dependent delay differential equations*, Proc. Amer. Math. Soc., 124 (1996), 1417–1426.
[11] C. Corduneanu, "Almost Periodic Functions," Wiley, New York, 1968. Reprinted, Chelsea, New York, 1989.
[12] C. Cuevas, and M. Pinto, *Existence and uniqueness of pseudo almost periodic solutions of semilinear Cauchy problems with non dense domain*, Nonlinear Anal., 45 (2001), 73–83.
[13] K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, Berlin, 1985.
[14] W.F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc., 67 (1949), 217–240.
[15] K. Ezzinbi and M.A. Hachimi, *Existence of positive almost periodic solutions of functional equations via Hilbert's projective metric*, Nonlinear Anal., 26 (1996), 1169–1176.
[16] A. Fink and J. Gatica, *Positive almost periodic solutions of some delay integral equations*, J. Differential Equations, 83 (1990), 166–178.
[17] D. Guo and V. Lakshmikantham, *Positive solutions of nonlinear integral equations arising in infectious diseases*, J. Math. Anal. Appl., 134 (1988), 1–8.
[18] J. Kaplan, M. Sorg, and J. Yorke, *Solutions of $x'(t) = f(x(t), x(t-L))$ have limits when f is an order relation*, Nonlinear Anal., 3 (1978), 53–58.
[19] R. Leggett and L. Williams, *A fixed point theorem with application to an infectious disease model*, J. Math. Anal. Appl., 76 (1980), 91–97.
[20] R. Leggett and L. Williams, *Nonzero solutions of nonlinear integral equations modeling infectious disease*, SIAM J. Math. Anal., 13 (1982), 112–121.
[21] R. Nussbaum, *A periodicity threshold theorem for some nonlinear integral equations*, SIAM J. Math. Anal., 9 (1978), 356–376.
[22] H. Smith, *An abstract threshold theorem for one parameter families of positive noncompact operators*, Funkcial. Ekvac., 24 (1981), 141–153.
[23] A.C. Thompson, *On certain contraction mappings in a partially ordered vector space*, Proc. Amer. Math. Soc., 14 (1963), 438–443.
[24] R . Torrejón, *Positive almost periodic solutions of a state-dependent delay nonlinear integral equation*, Nonlinear Anal., (1993), 1383–1416.
[25] B. Xu and R. Yuan, *On the positive almost periodic type solutions for some nonlinear delay integral equations*, J. Math. Anal. Appl., 304 (2005), 249–268.
[26] B. Xu and R. Yuan, *The existence of positive almost periodic type solutions for some neutral nonlinear integral equation*, Nonlinear Anal., 60 (2005), 669–684.
[27] T. Yoshizawa, "Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions," Springer, New-york, 1975.
[28] T. Yoshizawa, "Stability Theory by Liapunov's Second Method," Publications of the Mathematical Society of Japan, No. 9 *The Mathematical Society of Japan*, Tokio, 1966.
[29] C. Zhang, "Almost Periodic Type Functions and Ergodicity," Science Press, Beijing, Kluwer Academic Publishers, Dordrecht, 2003.