

## INVARIANT MEASURES FOR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SUPERLINEAR DRIFT TERM

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**Abstract.** We consider a stochastic functional differential equation with an arbitrary Lipschitz diffusion coefficient depending on the past. The drift part contains a term with superlinear growth and satisfying a dissipativity condition. We prove tightness and Feller property of the segment process to show existence of an invariant measure.

### 1. INTRODUCTION AND PRELIMINARIES

There have been some investigations on stationary solutions of stochastic functional differential equations with nonlinear diffusion coefficients, see for instance [1, 3, 9] and references therein. One approach is to rewrite the functional differential equation as a semilinear infinite-dimensional equation and use results on invariant measures of such equations (see [5]). The operator induced by the linear part of a functional differential equation is often not dissipative. For results on invariant measures for non-dissipative systems, see [2, 11]. These results require that the linear part be exponentially stable and that the Lipschitz constant of the diffusion be small with respect to the decay of the linear part. By means of a finite-dimensional analysis it has been shown that the Lipschitz constant of the diffusion coefficient may be

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arbitrarily large, provided the diffusion coefficient is uniformly bounded (see [8]).

In this paper, we prove existence of an invariant measure for stochastic functional differential equations with no boundedness conditions on the diffusion coefficient nor conditions on the size of its Lipschitz constant. Instead, we consider a stabilizing feedback term in the drift with super-linear growth. Let  $r > 0$  and denote by  $C([-r, 0], \mathbb{R}^d)$  the space of  $\mathbb{R}^d$ -valued continuous functions on  $[-r, 0]$  and let  $g: C([-r, 0], \mathbb{R}^d) \rightarrow \mathbb{R}^d$  and  $h: C([-r, 0], \mathbb{R}^d) \rightarrow \mathbb{R}^{d \times m}$  be Lipschitz functions with respect to the maximum norm. Let  $(B(t))_{t \geq 0}$  denote a standard  $\mathbb{R}^m$ -valued Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ . We will show existence of an invariant measure for the functional differential equation

$$dx(t) = \left( -x(t) \cdot |x(t)|^s + g(x_t) \right) dt + h(x_t) dB(t), \quad t \geq 0, \quad (1.1)$$

where  $s > 0$  and  $x_t$  denotes the segment of  $x$  given by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-r, 0].$$

In order to show existence of an invariant measure, we consider the segments of a solution. In contrast to the scalar solution process, the process of segments is a Markov process. We show that the process of segments is also Feller and that there exists a solution of which the segments are tight. Then, we apply the Krylov-Bogoliubov method.

Since the segment process has values in the infinite-dimensional space  $C([-r, 0], \mathbb{R}^d)$ , boundedness in probability does not automatically imply tightness. For solution processes of infinite-dimensional equations, one often uses compactness of the orbits of the underlying deterministic equation to obtain tightness. For an infinite-dimensional formulation of the functional differential equation, however, such a compactness property does not hold.

Our proof of tightness involves a Lyapunov function technique to obtain boundedness in probability for the segment process  $(x_t)_{t \geq 0}$ . Further, we use the assumption on the coefficients for the deterministic part, and Kolmogorov's criterion for the noise part. By using a monotonicity argument, we prove the Feller property for  $(x_t)_{t \geq 0}$  which implies the existence of an invariant measure by the Krylov-Bogoliubov theorem. Our analysis holds true for the more general equation

$$\begin{cases} dx(t) = (f(x(t)) + g(x_t))dt + h(x_t)dB(t), & \text{for } t \geq 0, \\ x(s) = \varphi(s) & \text{for } s \in [-r, 0], \end{cases} \quad (1.2)$$

where we assume the following hypotheses:

(**H<sub>0</sub>**)  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is continuous and

$$\lim_{|v| \rightarrow +\infty} \frac{\langle f(v), v \rangle}{|v|^2} = -\infty.$$

(**H<sub>1</sub>**)  $g : C([-r, 0], \mathbb{R}^d) \rightarrow \mathbb{R}^d$ ,  $h : C([-r, 0], \mathbb{R}^d) \rightarrow \mathbb{R}^{d \times m}$  are continuous and bounded on bounded subsets of  $C([-r, 0], \mathbb{R}^d)$ .

(**H<sub>2</sub>**) There exists a positive constant  $L$  such that for all  $x, y \in C([-r, 0], \mathbb{R}^d)$

$$\begin{aligned} & \left( 2\langle f(x(0)) - f(y(0)), x(0) - y(0) \rangle^+ + 2\langle g(x) - g(y), x(0) - y(0) \rangle^+ \right. \\ & \quad \left. + \|h(x) - h(y)\|^2 \right) \leq L\|x - y\|^2, \end{aligned}$$

where  $\|M\| := (\text{Tr}(MM^*))^{\frac{1}{2}}$  denotes the trace norm of the matrix  $M$ .

The initial process  $\varphi$  has almost surely continuous paths and is independent of  $(B(t))_{t \geq 0}$  with  $\mathbb{E}\|\varphi(\cdot, \omega)\|^p < \infty$  for all  $p \geq 2$ .

Note that under hypotheses (**H<sub>0</sub>**), (**H<sub>1</sub>**) and (**H<sub>2</sub>**), and thanks to [10, Theorem 2.3], equation (1.2) has a unique global solution given by

$$x(t) = x(0) + \int_0^t f(x(s)) ds + \int_0^t g(x_s) ds + \int_0^t h(x_s) dB(s) \quad \text{for any } t > 0.$$

We will prove existence of an invariant measure  $\mu$  for the segment process  $(x_t)_{t \geq 0}$  associated to the solution  $x(t)_{t \geq 0}$ . Of course, our hypotheses (**H<sub>1</sub>**) and (**H<sub>2</sub>**) allow the coefficient  $h$  to be degenerate which can not guarantee uniqueness of  $\mu$ . For recent results on the uniqueness of invariant measures for stochastic functional differential equations, see [6].

We end this introduction by the following elementary remark which is useful for our arguments in the sequel of this paper.

**Remark 1.1.** Let  $T > 0$ . Consider a stochastic process  $x(t)$ ,  $-r \leq t \leq T$  with continuous paths and let  $x_t$ ,  $t \geq 0$  be its associated segment process on  $[-r, 0]$ . If  $x_0 = \varphi$  and  $p \geq 1$ , then

$$\mathbb{E} \sup_{0 \leq t \leq T} \|x_t\|^p \leq \mathbb{E}\|\varphi\|^p + \mathbb{E} \sup_{0 \leq t \leq T} |x(t)|^p.$$

**Proof.** We have

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \|x_t\|^p &= \mathbb{E} \sup_{0 \leq t \leq T} \sup_{-r \leq s \leq 0} |x(t+s)|^p = \mathbb{E} \sup_{0 \leq t \leq T} \sup_{t-r \leq s \leq t} |x(s)|^p \\ &= \mathbb{E} \sup_{-r \leq s \leq T} |x(s)|^p \leq \mathbb{E}\|\varphi\|^p + \mathbb{E} \sup_{0 \leq s \leq T} |x(s)|^p. \quad \square \end{aligned}$$

2. TIGHTNESS OF THE SEGMENT PROCESS  $(x_t)_{t \geq 0}$ 

In this section, we will prove tightness of the family  $\{x_t : t \geq 0\}$ . To this end we shall prove first boundedness in probability.

We fix the initial process  $\varphi$  and consider the solution of (1.2).

**Proposition 2.1.** *Under hypotheses  $(\mathbf{H}_0)$ ,  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  the process  $(x_t)_{t \geq 0}$  is bounded in probability.*

For the proof of the proposition we need some preparation. Let  $2 < p < +\infty$  and  $\eta: [0, \infty) \times \Omega \rightarrow \mathbb{R}$  be a progressively measurable process such that

$$\mathbb{E} \int_0^T |\eta(s, \omega)|^p ds < +\infty.$$

Consider a one-dimensional Brownian motion  $(\beta(t))_{t \geq 0}$  and for  $\mu > 0$  let us introduce the following equation:

$$\begin{cases} dv(t) = -\mu v(t)dt + \eta(t, \omega)d\beta(t), & t \geq 0 \\ v(0) = 0. \end{cases}$$

If we denote by  $(v_\mu(\cdot))$  its solution we have

$$v_\mu(t) = \int_0^t e^{-\mu(t-s)} \eta(s, \omega) d\beta(s).$$

The following lemma gives an estimate for the process  $v_\mu(\cdot)$ .

**Lemma 2.2.** *For  $2 < p < +\infty$  and  $\mu > 0$  there exists a positive constant  $a_{p, \mu}$  such that*

$$\lim_{\mu \rightarrow +\infty} a_{p, \mu} = 0,$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} |v_\mu(t)|^p \leq a_{p, \mu} \cdot \mathbb{E} \int_0^T |\eta(s, \omega)|^p ds, \text{ for every } T > 0. \quad (2.1)$$

**Proof.** Fix  $2 < p < \infty$ ,  $T > 0$  and assume that

$$\mathbb{E} \int_0^T |\eta(s, \omega)|^p ds < \infty.$$

Let  $\frac{1}{p} < \alpha < \frac{1}{2}$  and define

$$y(t) := \int_0^t (t-s)^{-\alpha} e^{-\mu(t-s)} \eta(s, \omega) d\beta(s), \quad t \geq 0.$$

Using the factorization formula (see [4, Section 7.1])

$$\int_0^t e^{-\mu(t-s)} \eta(s, \omega) d\beta(s) = \frac{\sin \pi \alpha}{\pi} R_\alpha y(t),$$

where

$$R_\alpha f(t) = \int_0^t (t-s)^{\alpha-1} e^{-\mu(t-s)} f(s) ds,$$

defines a bounded linear operator from  $L^p([0, T], \mathbb{R})$  into  $C([0, T], \mathbb{R})$ . Indeed, take a function  $f$  in  $L^p([0, T], \mathbb{R})$ ; then we have

$$\begin{aligned} |R_\alpha f(t)| &\leq \int_0^t (t-s)^{\alpha-1} e^{-\mu(t-s)} |f(s)| ds \\ &\leq \|f\|_{L^p([0, T], \mathbb{R})} \left( \int_0^t (t-s)^{(\alpha-1)p/(p-1)} e^{-\mu p(t-s)/(p-1)} ds \right)^{\frac{p-1}{p}} \\ &\leq \|f\|_{L^p([0, T], \mathbb{R})} \left( \int_0^{+\infty} s^{(\alpha-1)p/(p-1)} e^{-\mu p s/(p-1)} ds \right)^{\frac{p-1}{p}} \\ &= \|f\|_{L^p([0, T], \mathbb{R})} \left( \frac{p-1}{\mu p} \right)^{\alpha-\frac{1}{p}} \Gamma\left(\frac{\alpha p-1}{p-1}\right)^{1-\frac{1}{p}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t e^{-\mu(t-s)} \eta(s, \omega) d\beta(s) \right|^p \right)^{\frac{1}{p}} \\ &= \left( \mathbb{E} \sup_{0 \leq t \leq T} \left| \frac{\sin \pi \alpha}{\pi} R_\alpha y(t) \right|^p \right)^{\frac{1}{p}} \leq \|R_\alpha\| \left( \mathbb{E} \|y(\cdot)\|_{L^p([0, T], \mathbb{R})}^p \right)^{\frac{1}{p}} \\ &\leq \left( \frac{p-1}{\mu p} \right)^{\alpha-\frac{1}{p}} \Gamma\left(\frac{\alpha p-1}{p-1}\right)^{1-\frac{1}{p}} \left( \mathbb{E} \|y(\cdot)\|_{L^p([0, T], \mathbb{R})}^p \right)^{\frac{1}{p}}. \end{aligned}$$

Using Burkholder-Davis-Gundy's inequality we obtain

$$\begin{aligned} \mathbb{E} \|y\|_{L^p([0, T], \mathbb{R})}^p &= \mathbb{E} \int_0^T |y(t)|^p dt \\ &= \int_0^T \mathbb{E} \left| \int_0^t (t-s)^{-\alpha} e^{-\mu(t-s)} \eta(s, \omega) d\beta(s) \right|^p dt \\ &\leq c_p \mathbb{E} \int_0^T \left( \int_0^t (t-s)^{-2\alpha} |e^{-\mu(t-s)} \eta(s, \omega)|^2 ds \right)^{\frac{p}{2}} dt \\ (\text{Young's inequality}) &\leq c_p \left( \int_0^T s^{-2\alpha} e^{-2\mu s} ds \right)^{\frac{p}{2}} \cdot \mathbb{E} \int_0^T |\eta(s, \omega)|^p ds \end{aligned}$$

$$\leq c_p \left( \frac{1}{2\mu} \int_0^{+\infty} \left( \frac{t}{2\mu} \right)^{-2\alpha} e^{-t} dt \right)^{\frac{p}{2}} \cdot \mathbb{E} \int_0^T |\eta(s, \omega)|^p ds.$$

Hence, we have

$$\begin{aligned} \mathbb{E} \|y\|_{L^p([0, T], \mathbb{R})}^p &\leq c_p \cdot \left( \frac{1}{(2\mu)^{1-2\alpha}} \Gamma(1-2\alpha) \right)^{\frac{p}{2}} \mathbb{E} \int_0^T |\eta(s, \omega)|^p ds \\ &= c_{p, \mu} \cdot \mathbb{E} \int_0^T |\eta(s, \omega)|^p ds, \end{aligned}$$

where  $c_{p, \mu} := c_p \left( \frac{1}{(2\mu)^{1-2\alpha}} \Gamma(1-2\alpha) \right)^{\frac{p}{2}}$ . Therefore, we deduce

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \int_0^t e^{-\mu(t-s)} \eta(s, \omega) d\beta(s) \right|^p \right) \leq a_{p, \mu} \mathbb{E} \int_0^T |\eta(s, \omega)|^p ds,$$

where

$$a_{p, \mu} := c_{p, \mu} \cdot \left( \frac{p-1}{\mu p} \right)^{p\alpha-1} \Gamma \left( \frac{\alpha p - 1}{p-1} \right)^{p-1}. \quad \square$$

We are now in the position to complete the proof of the proposition.

**Proof.** Let  $\lambda \geq 1$ . For  $x \in \mathbb{R}^d$  we define  $R_\lambda(x) := 2\langle f(x), x \rangle + \lambda|x|^2$ . By hypothesis **(H<sub>0</sub>)** there exists  $A_\lambda > 0$  such that

$$\frac{\langle f(x), x \rangle}{|x|^2} \leq -\lambda, \quad |x| \geq A_\lambda.$$

Again, by **(H<sub>0</sub>)** we can find  $B \geq 0$  independent of  $\lambda$  such that

$$R_\lambda(x) \leq B + \lambda A_\lambda^2 \quad \text{for all } x \in \mathbb{R}^d. \quad (2.2)$$

We now consider the solution  $x(\cdot)$  of equation (1.2) and set  $z(t) := |x(t)|^2$ ,  $t \geq 0$ . Then Itô's formula implies that for fixed  $t \geq 0$  we have

$$\begin{aligned} dz(t) &= 2\langle f(x(t)), x(t) \rangle dt + 2\langle g(x_t), x(t) \rangle dt + \|h(x_t)\|^2 dt \\ &\quad + 2\langle x(t), h(x_t) dB(t) \rangle \\ &= \left( -\lambda z(t) + R_\lambda(x(t)) + 2\langle g(x_t), x(t) \rangle dt + \|h(x_t)\|^2 \right) dt \\ &\quad + 2\langle x(t), h(x_t) dB(t) \rangle \\ &\leq \left( -\lambda z(t) + R_\lambda(x(t)) + 2\langle g(x_t) - g(0), x(t) \rangle + 2\langle g(0), x(t) \rangle \right. \\ &\quad \left. + 2\|h(x_t) - h(0)\|^2 + 2\|h(0)\|^2 \right) dt + 2\langle x(t), h(x_t) dB(t) \rangle \\ &\leq \left( -\lambda z(t) + R_\lambda(x(t)) + 2L\|x_t\|^2 + 2\langle g(0), x(t) \rangle + 2\|h(0)\|^2 \right) dt \end{aligned} \quad (2.3)$$

$$\begin{aligned}
& + 2\langle x(t), h(x_t)dB(t) \rangle \\
\leq & \left( -\lambda z(t) + R_\lambda(x(t)) + 3L\|x_t\|^2 + \frac{1}{L}|g(0)|^2 + 2\|h(0)\|^2 \right) dt \\
& + 2\langle x(t), h(x_t)dB(t) \rangle,
\end{aligned}$$

where we used the estimate

$$\langle g(0), x(t) \rangle \leq \frac{L}{2}|x(t)|^2 + \frac{1}{2L}|g(0)|^2 \leq \frac{L}{2}\|x_t\|^2 + \frac{1}{2L}|g(0)|^2.$$

Set  $D := \frac{1}{L}|g(0)|^2 + 2\|h(0)\|^2$ , so that the variation of constants formula yields

$$\begin{aligned}
z(t) & \leq z(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} \left( R_\lambda(x(s)) + 3L\|x_s\|^2 + D \right) ds \\
& \quad + 2 \int_0^t e^{-\lambda(t-s)} \langle x(s), h(x_s)dB(s) \rangle \\
& \leq z(0)e^{-\lambda t} + A_\lambda^2 + \frac{B+D}{\lambda} + \frac{3L}{\lambda} \sup_{-r \leq s \leq t} |x(s)|^2 \\
& \quad + 2 \int_0^t e^{-\lambda(t-s)} \langle x(s), h(x_s)dB(s) \rangle.
\end{aligned}$$

There exists a one-dimensional Brownian motion  $\beta$  with respect to the same filtration such that

$$\langle x(s), h(x_s)dB(s) \rangle = \eta(s, \omega) d\beta(s),$$

where

$$\eta(s, \omega) = \left( \sum_{j=1}^m \left( \sum_{i=1}^d x_i(s) h_{ij}(x_s) \right)^2 \right)^{1/2}.$$

By **(H<sub>2</sub>)**, we get

$$|\eta(s, \omega)|^3 \leq |x(s)|^3 \|h(x_s)\|^3 \leq 4|x(s)|^3 (L^{3/2}\|x_s\|^3 + D^{3/2}). \quad (2.4)$$

Hence, for  $0 \leq t \leq r$  we obtain

$$\begin{aligned}
e^t z(t) & \leq z(0) + e^r \left( A_\lambda^2 + \frac{B+D}{\lambda} \right) + \frac{3L}{\lambda} e^r \sup_{-r \leq s \leq t} |x(s)|^2 \\
& \quad + 2e^r \sup_{0 \leq t \leq r} \left| \int_0^t e^{-\lambda(t-s)} \eta(s) d\beta(s) \right|.
\end{aligned}$$

Now, using Lemma 2.2 and (2.4) we get

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq r} \left| \int_0^t e^{-\lambda(t-s)} \eta(s) d\beta(s) \right|^3 &\leq a_{3,\lambda} r \mathbb{E} \|\eta_r\|^3 \\ &\leq 4 a_{3,\lambda} r \left( D^{3/2} \mathbb{E} \|x_r\|^3 + L^{3/2} (\mathbb{E} \|x_r\|^6 + \mathbb{E} (\|\varphi\|^3 \|x_r\|^3)) \right) \\ &\leq 2 a_{3,\lambda} r \left( D^{3/2} (\mathbb{E} \|x_r\|^6 + 1) + L^{3/2} (3 \mathbb{E} \|x_r\|^6 + \mathbb{E} \|\varphi\|^6) \right). \end{aligned}$$

If we choose  $\kappa \in (1, e^{3r})$  and  $\gamma > 1$  such that  $(a + b + c + d)^3 \leq \kappa a^3 + \gamma(b^3 + c^3 + d^3)$  for all  $a, b, c, d \geq 0$ , we have

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq r} |e^t z(t)|^3 &\leq \kappa \mathbb{E} |z(0)|^3 + \gamma e^{3r} \left( A_\lambda^2 + \frac{B+D}{\lambda} \right)^3 \\ &\quad + \gamma \frac{27L^3}{\lambda^3} e^{3r} (\mathbb{E} \|\varphi\|^6 + \mathbb{E} \sup_{0 \leq s \leq r} |e^s z(s)|^3) \\ &\quad + \gamma 16 a_{3,\lambda} r e^{3r} \left( D^{\frac{3}{2}} (\mathbb{E} \sup_{0 \leq t \leq r} |e^t z(t)|^3 + 1) + L^{\frac{3}{2}} (3 \mathbb{E} \sup_{0 \leq t \leq r} |e^t z(t)|^3 + \mathbb{E} \|\varphi\|^6) \right). \end{aligned}$$

Let  $\psi(s) := |\varphi(s)|^2$ ,  $s \in [-r, 0]$ . We define the function  $V : C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}^+$  by

$$V(\zeta) := \sup_{-r \leq s \leq 0} (e^{3s} |\zeta(s)|^3).$$

We deduce from the above calculation that

$$\begin{aligned} \mathbb{E} V(z_r) &\leq \kappa e^{-3r} \mathbb{E} V(\psi) + \gamma \left( A_\lambda^2 + \frac{B+D}{\lambda} \right)^3 + \gamma \frac{27L^3}{\lambda^3} e^{3r} (\mathbb{E} V(\psi) + \mathbb{E} V(z_r)) \\ &\quad + 16 \gamma a_{3,\lambda} r \left( \mathbb{E} V(z_r) e^{3r} (D^{3/2} + 3L^{3/2}) + e^{3r} L^{3/2} \mathbb{E} V(\psi) + D^{3/2} \right). \quad (2.5) \end{aligned}$$

Hence, for  $\lambda_*$  sufficiently large, we get

$$\mathbb{E} V(z_r) \leq \delta \mathbb{E} V(\psi) + \rho, \quad (2.6)$$

where

$$\begin{aligned} \delta &:= \frac{\kappa e^{-3r} + \gamma \frac{27L^3}{\lambda_*^3} e^{3r} + 16 \gamma a_{3,\lambda_*} r e^{3r} L^{3/2}}{1 - \gamma e^{3r} \left( \frac{27L^3}{\lambda_*^3} + 16 a_{3,\lambda_*} r (D^{3/2} + 3L^{3/2}) \right)} < 1, \\ \rho &:= \frac{\gamma \left( A_{\lambda_*}^2 + \frac{B+D}{\lambda_*} \right)^3 + 16 \gamma a_{3,\lambda_*} r D^{3/2}}{1 - \gamma e^{3r} \left( \frac{27L^3}{\lambda_*^3} + 16 a_{3,\lambda_*} r (D^{3/2} + 3L^{3/2}) \right)}, \end{aligned}$$



provided that  $\mathbb{E}V(z_r) < \infty$  ( $\mathbb{E}V(\psi)$  is finite by assumption). To see that this property holds, apply the previous calculation to the process  $|x(t)|$  stopped as soon as it reaches level  $N$  and then let  $N \rightarrow \infty$ . Iterating (2.6) we get

$$\mathbb{E}V(z_{kr}) \leq \delta^k \mathbb{E}V(\psi) + \frac{\rho}{1-\delta} \leq \mathbb{E}V(\psi) + \frac{\rho}{1-\delta}, \quad \text{for all } k \in \mathbb{N}. \quad (2.7)$$

Let  $t \geq 0$ . Then, there exists  $k \in \mathbb{N}_0$  such that  $kr \leq t \leq (k+1)r$  and we have

$$\mathbb{E}\|z_t\|^3 \leq \mathbb{E}\|z_{kr}\|^3 + \mathbb{E}\|z_{(k+1)r}\|^3. \quad (2.8)$$

Using (2.7) we obtain

$$\mathbb{E}\|z_{kr}\|^3 = \mathbb{E} \sup_{-r \leq s \leq 0} |z_{kr}(s)|^3 \leq e^{3r} \mathbb{E}V(z_{kr}) \leq e^{3r} \left( \mathbb{E}V(\psi) + \frac{\rho}{1-\delta} \right).$$

Combining this with (2.8) yields

$$\sup_{t \geq 0} \mathbb{E}\|x_t\|^6 < +\infty. \quad (2.9)$$

This implies the boundedness in probability of the segment process  $(x_t)_{t \geq 0}$  and the proposition is proved.  $\square$

The following theorem is our main result in this section.

**Theorem 2.3.** *Under hypotheses  $(\mathbf{H}_0)$ ,  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  the family  $\{\mathcal{L}(x_t), t \geq 0\}$  is tight.*

**Proof.** From (2.9), we have in particular the boundedness in probability of the finite-dimensional process  $(x(t))_{t \geq 0}$  and hence the family  $\{\mathcal{L}(x(t)), t \geq 0\}$  is tight. To prove the theorem, it is sufficient to show that

$$\lim_{\delta \rightarrow 0} \sup_{t \geq 0} \mathbb{P} \left( \sup_{\substack{t \leq u \leq v \leq t+r \\ v-u \leq \delta}} |x(v) - x(u)| \geq \gamma \right) = 0 \quad \text{for any } \gamma > 0. \quad (2.10)$$

To shorten notation let

$$\tilde{g}(\eta) := g(\eta) + f(\eta(0)), \quad \eta \in C([-r, 0], \mathbb{R}^d).$$

Thus, we can write

$$x(t) = x(0) + \int_0^t \tilde{g}(x_s) ds + \int_0^t h(x_s) dB(s),$$

and we have

$$\mathbb{P} \left( \sup_{\substack{t \leq u \leq v \leq t+r \\ v-u \leq \delta}} |x(v) - x(u)| \geq \gamma \right) \leq \mathbb{P} \left( \sup_{\substack{t \leq u \leq v \leq t+r \\ v-u \leq \delta}} \int_u^v |\tilde{g}(x_s)| ds \geq \frac{\gamma}{2} \right) \quad (2.11)$$

$$+ \mathbb{P}\left(\sup_{\substack{t \leq u \leq v \leq t+r \\ v-u \leq \delta}} \left| \int_u^v h(x_s) dB(s) \right| \geq \frac{\gamma}{2}\right) = M_t + N_t.$$

Let  $\varepsilon, R > 0$ . For the term  $M_t$  we have

$$\begin{aligned} M_t &\leq \mathbb{P}\left(\sup_{\substack{t \leq u \leq v \leq t+r \\ v-u \leq \delta}} \int_u^v |\tilde{g}(x_s)| ds \geq \frac{\gamma}{2} \mid \|x_t\| \leq R, \|x_{t+r}\| \leq R\right) \\ &\quad + \mathbb{P}(\{\|x_t\| > R\}) + \mathbb{P}(\{\|x_{t+r}\| > R\}). \end{aligned}$$

Since the process  $(x_t)_{t \geq 0}$  is bounded in probability we can choose  $R$  so large that

$$\mathbb{P}(\{\|x_t\| > R\}) + \mathbb{P}(\{\|x_{t+r}\| > R\}) \leq \frac{\varepsilon}{2} \quad \text{for all } t \geq 0.$$

By **(H<sub>1</sub>)**,  $\tilde{g}(x_s)$ ,  $s \in [t-r, t+r]$  is bounded on the set  $\{\|x_t\| \leq R\} \cap \{\|x_{t+r}\| \leq R\}$ , so it follows that there exists  $\delta_0 > 0$  such that

$$\mathbb{P}\left(\sup_{\substack{t \leq u \leq v \leq t+r \\ v-u \leq \delta}} \int_u^v |\tilde{g}(x_s)| ds \geq \frac{\gamma}{2} \mid \|x_t\| \leq R, \|x_{t+r}\| \leq R\right) = 0 \quad \text{for any } \delta < \delta_0.$$

Therefore, we get  $\lim_{\delta \rightarrow 0} \sup_{t \geq 0} M_t = 0$ . For the term  $N_t$  we define

$$J(t) := \int_0^t h(x_s) dB(s).$$

Using Burkholder's inequality and **(H<sub>2</sub>)**, we get

$$\begin{aligned} \mathbb{E}|J(t) - J(s)|^6 &= \mathbb{E}\left|\int_s^t h(x_u) dB(u)\right|^6 \leq c\mathbb{E}\left(\int_s^t \|h(x_u)\|^2 du\right)^3 \\ \text{(Jensen's inequality)} &\leq \bar{c}|t-s|^3\left(\sup_{u \geq 0} \mathbb{E}\|x_u\|^6 + 1\right), \end{aligned}$$

where  $\bar{c}$  depends on  $L$  and  $D$ . Using (2.9) and Kolmogorov's tightness criterion (see [7, 2.4.11]) we infer that

$$\lim_{\delta \rightarrow 0} \sup_{t \geq 0} N_t = \lim_{\delta \rightarrow 0} \sup_{t \geq 0} \mathbb{P}\left(\sup_{\substack{t \leq u \leq v \leq t+r \\ v-u \leq \delta}} \left| \int_u^v h(x_s) dB(s) \right| \geq \frac{\gamma}{2}\right) = 0.$$

This establishes (2.10) and the proof is complete.  $\square$

## 3. INVARIANT MEASURES

In this section, we discuss the existence of an invariant measure  $\mu$  for the segment process  $(x_t)_{t \geq 0}$ . Since, in the last section, we proved tightness of this process, in order to apply Krylov-Bogoliubov's theorem we need to prove the Feller property of  $(x_t)_{t \geq 0}$ .

**Proposition 3.1.** *Assume hypotheses  $(\mathbf{H}_0)$ ,  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$ . Let  $(\varphi_m)_{m \in \mathbb{N}}$  be a sequence in  $C([-r, 0], \mathbb{R}^d)$  such that  $\varphi_m \xrightarrow[m \rightarrow +\infty]{\|\cdot\|} \varphi$ . Let  $x^m$  (respectively  $x$ ) be the solutions to (1.2) with initial condition  $\varphi_m$  (respectively  $\varphi$ ). Then, for any  $t > 0$ ,*

$$\mathbb{E} \sup_{t-r \leq s \leq t} |x^m(s) - x(s)|^4 \rightarrow 0 \quad \text{as } m \rightarrow +\infty. \quad (3.1)$$

In particular,  $(x_t)_{t \geq 0}$  is a Feller process.

**Proof.** Using Itô's formula we can write

$$\begin{aligned} d|x^m(t) - x(t)|^2 &= 2\langle f(x^m(t)) - f(x(t)) + g(x_t^m) - g(x_t), x^m(t) - x(t) \rangle dt \\ &\quad + \|h(x_t^m) - h(x_t)\|^2 dt + dM(t), \end{aligned} \quad (3.2)$$

where

$$M(t) := \int_0^t 2\langle x^m(s) - x(s), (h(x_s^m) - h(x_s)) dB(s) \rangle$$

is a martingale with quadratic variation process bounded by  $4L \int_0^t \|x_s^m - x_s\|^4 ds$ . Thus if we define  $M^*(t) = \sup_{s \leq t} M(s)$  we obtain

$$\|x_t^m - x_t\|^2 \leq \|\varphi_m - \varphi\|^2 + L \int_0^t \|x_s^m - x_s\|^2 ds + M^*(t).$$

This implies

$$\begin{aligned} \mathbb{E}\|x_t^m - x_t\|^4 &\leq 3\mathbb{E}\left(\|\varphi_m - \varphi\|^4 + L^2\left(\int_0^t \|x_s^m - x_s\|^2 ds\right)^2 + \left(M^*(t)\right)^2\right) \\ &\leq 3\left(\|\varphi_m - \varphi\|^4 + L^2 t \int_0^t \mathbb{E}\|x_s^m - x_s\|^4 ds + 4L \int_0^t \mathbb{E}\|x_t^m - x_t\|^4 ds\right). \end{aligned} \quad (3.3)$$

Hence, by Gronwall's inequality, we obtain

$$\mathbb{E}\|x_t^m - x_t\|^4 \leq 3\|\varphi_m - \varphi\|^4 e^{12Lt + 3L^2 t^2}.$$

This implies in particular that for  $\psi : C([-r, 0], \mathbb{R}^d) \rightarrow \mathbb{R}^d$  bounded and continuous we have

$$\lim_{m \rightarrow +\infty} \mathbb{E}\psi((x_m)_t) = \mathbb{E}\psi(x_t) \quad \text{for any } t > 0,$$

which yields the Feller property.  $\square$

Now, by the Krylov-Bogoliubov Theorem (see Section 3.1 in [5]) we have the following result.

**Theorem 3.2.** *Under hypotheses  $(\mathbf{H}_0)$ ,  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$  the segment process  $(x_t)_{t \geq 0}$  corresponding to (1.2) has an invariant measure.*

**Remark 3.3.** Our proofs show that hypothesis  $(\mathbf{H}_0)$  can be weakened by requiring that  $\limsup_{|v| \rightarrow +\infty} \frac{\langle f(v), v \rangle}{|v|^2} < -\lambda$ , for  $\lambda$  a sufficiently large positive constant (which depends on  $L$ ,  $r$  and  $h(0)$ ).

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