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A NON-MONOTONE NONLOCAL CONSERVATION LAW FOR DUNE MORPHODYNAMICS

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Abstract. We investigate a non-local, non-linear conservation law, first introduced by A.C. Fowler to describe morphodynamics of dunes, see [6, 7]. A remarkable feature is the violation of the maximum principle, which allows for erosion phenomenon. We prove well posedness for initial data in L^2 and give an explicit counterexample for the maximum principle. We also provide numerical simulations corroborating our theoretical results.

1. Introduction

We investigate the following Cauchy problem:

$$\begin{cases} \partial_t u(t,x) + \partial_x\left(\frac{u^2}{2}\right)(t,x) + \mathcal{I}[u(t,.)](x) - \partial_{xx}^2 u(t,x) = 0 & t \in (0,T), x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where T is any given positive time, $u_0 \in L^2(\mathbb{R})$ and $\mathcal{I}$ is a non-local operator defined as follows: for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$ and any $x \in \mathbb{R}$,

$$\mathcal{I}[\varphi](x) := \int_0^{+\infty} |\zeta|^{-\frac{1}{3}} \varphi''(x-\zeta)\, d\zeta.$$

Remark 1. Equation (1.1) can also be written in conservative form

$$\partial_t u + \partial_x \left(\tfrac{1}{2} u^2 + \mathcal{L}[u] - \partial_x u \right) = 0,$$

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where

$$\mathcal{L}[\varphi](x) := \int_0^{+\infty} |\zeta|^{-\frac{1}{3}} \varphi'(x-\zeta) d\zeta.$$

Equation (1.1) appears in the work of Fowler [6, 7] on the evolution of *dunes*; the term dunes refers to instabilities in landforms, which occur through the interaction of a turbulent flow with an erodible substrate. Equation (1.1) is valid for a river flow (from left to right) over an erodible bottom $u(t,x)$ with slow variation. For more details on the physical background, we refer the reader to [6, 7].

Roughly speaking, $\mathcal{I}[u]$ is a weighted mean of second derivatives of u with the bad sign; hence, this term has a deregularizing effect and the main consequence is probably the fact that (1.1) does not satisfy the maximum principle (see below for more details). Nevertheless, one can see that the diffusive operator $-\partial^2_{xx}$ controls the instabilities produced by $\mathcal{I}$ and ensures the existence and the uniqueness of a smooth solution for positive times. The starting point to establish these facts is the derivation of a new formula for the operator $\mathcal{I}$, namely (2.1). This result allows us to show easily that $\mathcal{I}-\partial^2_{xx}$ is a pseudo-differential operator with symbol $\psi_{\mathcal{I}}(\xi) = 4\pi^2\xi^2 - a_{\mathcal{I}}|\xi|^{\frac{4}{3}} + i\, b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}}$, where $a_{\mathcal{I}}$ and $b_{\mathcal{I}}$ are positive constants (see (2.2)). The symbol $4\pi^2\xi^2$ corresponds to the diffusive operator $-\partial^2_{xx}$ and $-a_{\mathcal{I}}|\xi|^{\frac{4}{3}} + i\, b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}}$ is the symbol of the nonlocal operator $\mathcal{I}$.

Notice that $-a_{\mathcal{I}}|\xi|^{\frac{4}{3}}$ corresponds to a fractional anti-diffusion and $i b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}}$ to a fractional drift. Because of the fact that the fractional anti-diffusion is of order $\frac{4}{3}$, the real part of $\psi_{\mathcal{I}}(\xi)$ behaves as ξ^2, up to a positive multiplicative constant, as $|\xi| \to +\infty$. A consequence is that Equation (1.1) has a regularizing effect on the initial data: even if u_0 is only L^2, the solution u becomes C^∞ for positive times. The uniqueness of a $L^\infty((0,T);L^2)$ solution is obtained by the use of a mild formulation (see Definition 1) based on Duhamel's formula (3.1), in which appears the kernel K of $\mathcal{I} - \partial^2_{xx}$. The use of such a formula also allows us to prove local-in-time existence with the help of a contracting fixed-point theorem. Such an approach is quite classical; we refer the reader, for instance, to the book of Pazy [9] and the references therein on the application of the theory of semi-groups of linear operators to partial differential equations. We also refer the reader to the work of Droniou *et al.* in [4] for fractal conservation laws of the form

$$\partial_t u + \partial_x(f(u)) + (-\partial^2_{xx})^{\frac{\lambda}{2}}[u] = 0, \tag{1.2}$$

where f is locally Lipschitz continuous and $\lambda \in (1, 2]$, and to the work of Tadmor [10] on the Kuramoto-Sivashinsky equation:

$$\partial_t u + \frac{1}{2}|\partial_x u|^2 - \partial^2_{xx} u = (-\partial^2_{xx})^2[u].$$

In fact, the fractal conservation law (1.2) is monotone and the global existence of an L^∞ solution is based on the fact that the L^∞ norm of u does not increase. In our case, this is not true and we have to use a classical energy estimate to get a global L^2 estimate. The regularizing effect on the initial data are first proved by a fixed-point theorem on Duhamel's formula to get H^1 regularity in space and next by a bootstrap method to get further regularity. This technique has already been used in [4].

On the other hand, one of our main results is probably the proof of the failure of the maximum principle for (1.1); more precisely, we exhibit positive dunes which take negative values in finite time, since we establish that the bottom is eroded downstream from the dune. We also give some numerical results that illustrate this fact (for more precision, see Remark 2 and Section 7). The proof of the failure of the maximum principle is based on the integral formula (2.1), which roughly speaking means that $\mathcal{I}$ is a Lévy operator with a bad sign, see [2]. Notice that the Kuramoto-Sivashinsky equation is also non-monotone, but no proof of the failure of the maximum principle is given in [10].

The paper is organized as follows. In Section 2, we give the integral and pseudo-differential formula for $\mathcal{I}$; we also establish the properties of the kernel K of $\mathcal{I} - \partial^2_{xx}$ that will be needed. In Section 3, we define the notion of mild solution for (1.1). Sections 4 and 5 are, respectively, devoted to the proof of the uniqueness and the existence of a mild solution; Section 5 also contains the proof of the regularity of the solution. The proof of the failure of the maximum principle is given in Section 6. Finally, we give in Section 7 some numerical simulations that illustrate the theory of the preceding sections.

Here are our main results.

Theorem 1. *Let* $T > 0$ *and* $u_0 \in L^2(\mathbb{R})$. *There exists a unique mild solution* $u \in L^\infty((0, T); L^2(\mathbb{R}))$ *of* (1.1) (*see Definition* 1). *Moreover,*

i) $u \in C^\infty((0, T] \times \mathbb{R})$ *and for all* $t_0 \in (0, T]$, u *and all its derivatives belong to* $C([t_0, T]; L^2(\mathbb{R}))$;

ii) u *satisfies* $\partial_t u + \partial_x(\frac{u^2}{2}) + \mathcal{I}[u] - \partial^2_{xx} u = 0$, *on* $(0, T] \times \mathbb{R}$, *in the classical sense* ($\mathcal{I}[u]$ *being properly defined by* (2.1) *and* (2.2));

iii) $u \in C([0, T]; L^2(\mathbb{R}))$ *and* $u(0, .) = u_0$ *almost everywhere* (*a.e. for short*).

Proposition 1 (L^2-stability)**.** *Let* (u,v) *be solutions to* (1.1) *with respective* L^2 *initial data* (u_0,v_0)*. We have*

$$\|u-v\|_{C([0,T];L^2(\mathbb{R}))} \le C\left(T,\|u_0\|_{L^2(\mathbb{R})},\|v_0\|_{L^2(\mathbb{R})}\right)\|u_0-v_0\|_{L^2(\mathbb{R})}.$$

Theorem 2 (Failure of the maximum principle)**.** *Assume that* $u_0 \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ *is non-negative and such that there exist* $x_* \in \mathbb{R}$ *with* $u_0(x_*) = u_0'(x_*) = u_0''(x_*) = 0$ *and*

$$\int_{-\infty}^{0} \frac{u_0(x_*+z)}{|z|^{7/3}}\,dz > 0.$$

Then, there exists $t_* > 0$ *with* $u(t_*,x_*) < 0$*.*

Remark 2. The hypotheses of the theorem above are satisfied, for instance, for non-negative $u_0 \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ such that there exists $x_* \in \mathbb{R}$ with $u_0(x_*) = u_0'(x_*) = u_0''(x_*) = 0$ and

$$\forall x \le x_*,\ u_0(x) \ge 0 \quad \text{and} \quad \exists x_0 < x_* \text{ s.t. } u_0(x_0) > 0.$$

A simple example of such an initial dune is shown in Figure 1. Observe that the bottom is eroded downstream from the dune (recall that the nonlinear convective term propagates a positive dune from the left to the right).

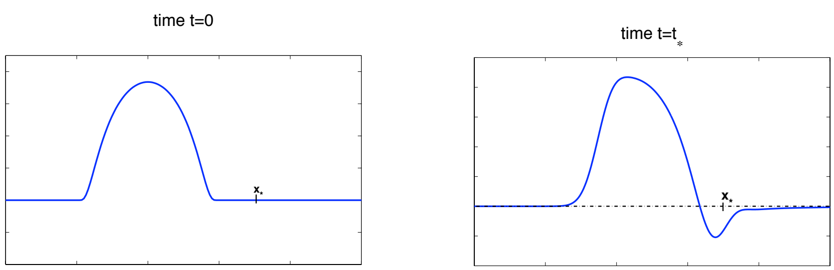

FIGURE 1. Evolution of a dune, at $t = 0$ and $t = t_*$. We can observe that $u(t_*,x_*) < 0$ and that $\int u(t,x)\,dx$ remains constant.

Notation: In the following, we let $\mathcal{F}$ denote the Fourier transform defined for $f \in L^1(\mathbb{R})$ by: for all $\xi \in \mathbb{R}$,

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}} e^{-2i\,\pi x\xi} f(x)dx.$$

We also let $\mathcal{F}$ define the extension of the preceding operator from L^2 to L^2. In the sequel, we only consider the Fourier transform with respect to (w.r.t.

for short) the space variable; in order to simplify the presentation, for any $u \in C([0,T];L^2(\mathbb{R}))$, we let $\mathcal{F}u \in C([0,T];L^2(\mathbb{R},\mathbb{C}))$ denote the function $t \in [0,T] \to \mathcal{F}(u(t,\cdot)) \in L^2(\mathbb{R},\mathbb{C})$.

2. Preliminaries

In Subsection 2.1, we give the integral and the pseudo-differential formula for $\mathcal{I}$ and, in Subsection 2.2, we give the properties on the kernel of $\mathcal{I} - \partial^2_{xx}$.

2.1. Integral formula for $\mathcal{I}$.

Proposition 2. *For all $\varphi \in \mathcal{S}(\mathbb{R})$ and all $x \in \mathbb{R}$,*

$$\mathcal{I}[\varphi](x) = C_{\mathcal{I}} \int_{-\infty}^{0} \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^{7/3}}\, dz, \tag{2.1}$$

with $C_{\mathcal{I}} = \frac{4}{9}$.

Proof. The proof is an easy consequence of Taylor-Poisson's formula and Fubini's theorem; notice that the regularity of φ ensures the validity of the computations that follow. We have

$$\begin{aligned}
&\int_{-\infty}^{0} \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^{7/3}}\, dz \\
&= \int_{-\infty}^{0} |z|^{-\frac{7}{3}} \Big(\int_0^1 (1-\tau)\varphi''(x+\tau z) z^2 d\tau \Big) dz, \\
&= \int_0^1 (1-\tau) \Big(\int_{-\infty}^{0} |z|^{-\frac{1}{3}} \varphi''(x+\tau z) dz \Big) d\tau, \\
&= \int_0^1 (1-\tau)\tau^{-\frac{2}{3}} \Big(\int_0^{+\infty} |\zeta|^{-\frac{1}{3}} \varphi''(x-\zeta) d\zeta \Big) d\tau,
\end{aligned}$$

thanks to the change of variable $\tau z = -\zeta$. Thus,

$$\int_{-\infty}^{0} \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^{7/3}}\, dz = \int_0^1 (1-\tau)\tau^{-\frac{2}{3}} d\tau\, \mathcal{I}[\varphi](x) = \frac{9}{4}\mathcal{I}[\varphi](x).$$

The proof is now complete. □

Corollary 1. *There are positive constants $a_{\mathcal{I}}$ and $b_{\mathcal{I}}$ such that, for all $\varphi \in \mathcal{S}(\mathbb{R})$ and all $\xi \in \mathbb{R}$,*

$$\mathcal{F}\big(\mathcal{I}[\varphi] - \varphi''\big)(\xi) = \psi_{\mathcal{I}}(\xi)\mathcal{F}\varphi(\xi), \tag{2.2}$$

where $\psi_{\mathcal{I}}(\xi) = 4\pi^2\xi^2 - a_{\mathcal{I}}|\xi|^{\frac{4}{3}} + i\, b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}}$.

Proof. We have

$$\mathcal{F}\big(\mathcal{I}[\varphi]\big)(\xi) = C_{\mathcal{I}} \int_{\mathbb{R}} \int_{-\infty}^{0} e^{-2i\pi x\xi} \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^{7/3}}\, dzdx.$$

Notice that Proposition 2 ensures that for $\varphi \in \mathcal{S}(\mathbb{R})$, $\mathcal{I}[\varphi] \in L^1(\mathbb{R})$ and, thus, its Fourier transform is well-defined. By Fubini's theorem, we can first integrate w.r.t. x to deduce that

$$\mathcal{F}\big(\mathcal{I}[\varphi]\big)(\xi) = C_{\mathcal{I}} \int_{-\infty}^{0} \frac{\mathcal{F}(\mathcal{T}_{-z}\varphi)(\xi) - \mathcal{F}\varphi(\xi) - \mathcal{F}(\varphi')(\xi)z}{|z|^{7/3}}\, dz,$$

where we let $\mathcal{T}_{-z}\varphi$ denote the (translated) function $x \to \varphi(x+z)$. Classical formulae on Fourier transform imply that $\mathcal{F}(\mathcal{I}[\varphi])(\xi) = \psi(\xi)\mathcal{F}\varphi(\xi)$, where

$$\psi(\xi) = C_{\mathcal{I}} \int_{-\infty}^{0} \frac{e^{2i\pi\xi z} - 1 - 2i\pi\xi z}{|z|^{7/3}}\, dz.$$

Simple computations show that

$$\psi(\xi) = C_{\mathcal{I}} \int_{-\infty}^{0} \frac{\cos(2\pi\xi z) - 1}{|z|^{7/3}}\, dz + i\, C_{\mathcal{I}} \int_{-\infty}^{0} \frac{\sin(2\pi\xi z) - 2\pi\xi z}{|z|^{7/3}}\, dz.$$

It is immediate that the real part of $\psi(\xi)$ is even, non-positive, non-identically equal to 0 and homogeneous of degree $\frac{4}{3}$ (the last property can be seen by changing the variable by $z' = \xi z$). Moreover, the imaginary part of $\psi(\xi)$ is odd, negative and homogeneous of degree $\frac{4}{3}$ on $\mathbb{R}_*^-$. There thus exist positive constants $a_{\mathcal{I}}$ and $b_{\mathcal{I}}$ such that

$$\psi(\xi) = -a_{\mathcal{I}}|\xi|^{\frac{4}{3}} + i\, b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}}, \tag{2.3}$$

and, in particular, $\mathcal{F}(\mathcal{I}[\varphi])(\xi) = (-a_{\mathcal{I}}|\xi|^{\frac{4}{3}} + ib_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}})\mathcal{F}\varphi(\xi)$. Since $\mathcal{F}(-\varphi'')(\xi) = 4\pi^2\xi^2\mathcal{F}\varphi(\xi)$, the proof of Corollary 1 is complete. □

Remark 3. (1) Since $\mathcal{I}[\varphi] = (\mathbf{1}_{\mathbb{R}_+}|\cdot|^{-\frac{1}{3}}) * \varphi''$, we have

$$\mathcal{F}(\mathcal{I}[\varphi]) = \mathcal{F}(\mathbf{1}_{\mathbb{R}_+}|\cdot|^{-\frac{1}{3}}) \cdot (-4\pi^2|\xi|^2) \cdot \mathcal{F}(\varphi).$$

Elementary computations give

$$\mathcal{F}(\mathbf{1}_{\mathbb{R}_+}|\cdot|^{-\frac{1}{3}}) = \Gamma(\tfrac{2}{3})\Big(\tfrac{1}{2} - i\,\mathrm{sign}(\xi)\tfrac{\sqrt{3}}{2}\Big)|\xi|^{-\frac{2}{3}}.$$

Hence, $a_{\mathcal{I}} = \Gamma(\frac{2}{3})\frac{1}{2}$ and $b_{\mathcal{I}} = \Gamma(\frac{2}{3})\frac{\sqrt{3}}{2}$.

(2) Let $s \geq \frac{4}{3}$. If $\varphi \in H^s(\mathbb{R})$, one can also define $\mathcal{I}[\varphi]$ through its Fourier transform by

$$\mathcal{F}(\mathcal{I}[\varphi])(\xi) := -4\pi^2\Gamma(\tfrac{2}{3})\Big(\tfrac{1}{2} - i\,\mathrm{sign}(\xi)\tfrac{\sqrt{3}}{2}\Big)|\xi|^{\frac{4}{3}} \cdot \mathcal{F}(\varphi).$$

Thus, if $\varphi \in H^s$, we have that $\mathcal{I}[\varphi] \in H^{s-\frac{4}{3}}$ and

$$\|\mathcal{I}[\varphi]\|_{H^{s-\frac{4}{3}}} \leq 4\pi^2\Gamma(\tfrac{2}{3})\|\varphi\|_{H^s}.$$

This implies in particular that $\mathcal{I} : H^2(\mathbb{R}) \to C_b(\mathbb{R}) \cap L^2(\mathbb{R})$, since by a Sobolev embedding $H^{\frac{2}{3}} \hookrightarrow C_b(\mathbb{R}) \cap L^2(\mathbb{R})$.

(3) Corollary 1 implies that $\mathcal{I} - \partial^2_{xx} : C^2(\mathbb{R}) \cap H^2(\mathbb{R}) \to C(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\mathcal{I}$ which satisfies both Formula (2.1) and (2.2).

2.2. **Main properties of the kernel K of $\mathcal{I} - \partial^2_{xx}$.** By Corollary 1, we see that the semi-group generated by $\mathcal{I} - \partial^2_{xx}$ is formally given by the convolution with the kernel (defined for $t > 0$ and $x \in \mathbb{R}$) $K(t, x) = \mathcal{F}^{-1}(e^{-t\psi_{\mathcal{I}}})(x)$.

Proposition 3. *$K(t, \cdot)$ is a L^1 real-valued continuous function.*

Proof. $K(t, \cdot)$ is a L^1 real-valued continuous function as an inverse Fourier transform of a $W^{2,1}$ function with an even real part and an odd imaginary part. □

In the sequel, we only consider real-valued solutions of (1.1). We expose in Figure 2 the evolution of $K(t, \cdot)$ for different times. Note that $K(t, \cdot)$ is not compactly supported but that $K(t, x) \leq \frac{C(t)}{x^2}$, for $|x| \geq 1$ with $C(t) = \frac{1}{4\pi^2}\|\partial^2_{\xi\xi}\mathcal{F}(K(t, .))(\xi)\|_{L^1}$

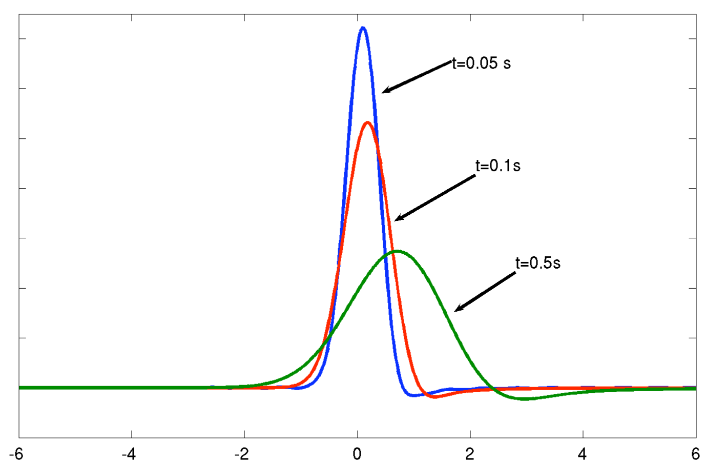

FIGURE 2. The kernel of $\mathcal{I} - \partial^2_{xx}$ for $t = 0.05, 0.1$ and 0.5 s.

Proposition 4. *The kernel K has a non-zero negative part.*

Proof. Let us assume that K is non-negative; then

$$\begin{aligned}|e^{-t\psi_{\mathcal{I}}(\xi)}| \leq \|\mathcal{F}^{-1}(e^{-t\psi_{\mathcal{I}}})\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |K(t,.)| \\ = \int_{\mathbb{R}} K(t,.) = \mathcal{F}\big(\mathcal{F}^{-1}(e^{-t\psi_{\mathcal{I}}})\big)(0) = e^{-t\psi_{\mathcal{I}}(0)} = 1,\end{aligned}$$

for all $\xi \in \mathbb{R}$; hence, since

$$|e^{-t\psi_{\mathcal{I}}(\xi)}| = e^{-t(4\pi^2|\xi|^2 - a_{\mathcal{I}}|\xi|^{\frac{4}{3}})} > 1 \quad \text{for } 0 < |\xi| < \frac{a_{\mathcal{I}}^{\frac{3}{2}}}{8\pi^3},$$

this gives us a contradiction. □

The main consequence of this is the failure of the maximum principle for the equation

$$\partial_t u + \mathcal{I}[u] - \partial^2_{xx} u = 0; \tag{2.4}$$

that is to say, there exists a non-negative initial condition u_0 such that, for some $t > 0$, $u(t,.) := K(t,.) * u_0$ has a non-zero negative part, see Section 6 below. Nevertheless, K enjoys many properties similar to those satisfied by the kernel of the heat equation and that ensure that Equation (2.4) has a regularizing effect on the initial condition: if $u_0 \in L^p(\mathbb{R})$ for some $p \in [1, +\infty)$, then u is C^∞ for positive times, see Section 5.

Let us make precise here the properties that will be needed in this paper. Since $K(t,\cdot) \in L^1(\mathbb{R})$, the family of bounded linear operators $\{u_0 \in L^2(\mathbb{R}) \to K(t,\cdot) * u_0 \in L^2(\mathbb{R})\}_{t>0}$ is well defined. Moreover, it is a strongly continuous semi-group of convolutions; that is to say,

$$\begin{gathered}\forall t,s > 0,\ K(s,\cdot) * K(t,\cdot) = K(s+t,\cdot), \\ \forall u_0 \in L^2(\mathbb{R}),\ \lim_{t\to 0} K(t,\cdot) * u_0 = u_0 \text{ in } L^2(\mathbb{R}).\end{gathered} \tag{2.5}$$

Next, the kernel K is smooth on $(0, +\infty) \times \mathbb{R}$ and we have

$$\forall T > 0,\ \exists \mathcal{K}_0 \text{ s.t. } \forall t \in (0,T],\ \|\partial_x K(t,.)\|_{L^2(\mathbb{R})} \leq \mathcal{K}_0 t^{-\frac{3}{4}}, \tag{2.6}$$

$$\forall T > 0,\ \exists \mathcal{K}_1 \text{ s.t. } \forall t \in (0,T],\ \|\partial_x K(t,.)\|_{L^1(\mathbb{R})} \leq \mathcal{K}_1 t^{-\frac{1}{2}}, \tag{2.7}$$

$$\forall t,s > 0,\ K(s,\cdot) * \partial_x K(t,\cdot) = \partial_x K(s+t,\cdot). \tag{2.8}$$

Proof of these properties. The semi-group property (2.5) and (2.8) are immediate consequences of the Fourier formula. Let us prove the strong continuity. By Plancherel's formula,

$$\|K(t,\cdot) * u_0 - u_0\|^2_{L^2(\mathbb{R})} = \|\mathcal{F}(K(t,\cdot) * u_0) - \mathcal{F}u_0\|^2_{L^2(\mathbb{R})} \tag{2.9}$$

$$= \|e^{-t\psi_{\mathcal{I}}}\mathcal{F}u_0 - \mathcal{F}u_0\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |e^{-t\psi_{\mathcal{I}}} - 1|^2\,|\mathcal{F}u_0|^2.$$

The function $|e^{-t\psi_{\mathcal{I}}} - 1|^2\,|\mathcal{F}u_0|^2$ converges pointwise to 0 on $\mathbb{R}$, as $t \to 0$. Using the fact that $\min \operatorname{Re}(\psi_{\mathcal{I}})$ is finite, we infer that $|e^{-t\psi_{\mathcal{I}}} - 1|^2\,|\mathcal{F}u_0|^2 \leq C|\mathcal{F}u_0|^2$ and the dominated convergence theorem implies that the last term of (2.9) tends to 0 as $t \to 0$. This completes the proof of (2.5). Let us now prove the estimates on the gradient. The smoothness of K is an immediate consequence of the theorem of derivation under the integral sign applied to the definition of K by the Fourier transform. We get in particular

$$\partial_x K(t,\cdot) = \partial_x \mathcal{F}^{-1}(e^{-t\psi_{\mathcal{I}}}) = \mathcal{F}^{-1}\big(\xi \to 2i\pi\xi e^{-t\psi_{\mathcal{I}}(\xi)}\big).$$

Since the function $\xi \to 2i\pi\xi e^{-t\psi_{\mathcal{I}}(\xi)}$ is L^2, $\partial_x K(t,\cdot)$ is L^2 and we have

$$\|\partial_x K(t,.)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} 4\pi^2\xi^2 |e^{-t\psi_{\mathcal{I}}(\xi)}|^2 d\xi = \int_{\mathbb{R}} 4\pi^2\xi^2 e^{-2t(4\pi^2|\xi|^2 - a_{\mathcal{I}}|\xi|^{\frac{4}{3}})} d\xi.$$

Let us change the variable by $\xi' = t^{\frac{1}{2}}\xi$. We get

$$\begin{aligned}\|\partial_x K(t,.)\|^2_{L^2(\mathbb{R})} &= t^{-\frac{3}{2}} \int_{\mathbb{R}} 4\pi^2|\xi'|^2 e^{-2(4\pi^2|\xi'|^2 - t^{\frac{1}{3}} a_{\mathcal{I}}|\xi'|^{\frac{4}{3}})} d\xi',\\ &\leq t^{-\frac{3}{2}} \int_{\mathbb{R}} 4\pi^2|\xi'|^2 e^{-2(4\pi^2|\xi'|^2 - T^{\frac{1}{3}} a_{\mathcal{I}}|\xi'|^{\frac{4}{3}})} d\xi',\end{aligned}$$

for all $t \in (0,T]$. The proof of (2.6) is now complete. To prove (2.7), we have to derive a "homogeneity-like" property for K. Easy computations show that

$$\begin{aligned}K(t,x) &= \int_{\mathbb{R}} e^{2i\,\pi x\xi} e^{-t\psi_{\mathcal{I}}(\xi)} d\xi,\\ &= \int_{\mathbb{R}} e^{2i\,\pi x\xi} e^{-t(4\pi^2|\xi|^2 - a_{\mathcal{I}}|\xi|^{\frac{4}{3}} + i\,b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}})} d\xi,\\ &= t^{-\frac{1}{2}} \int_{\mathbb{R}} e^{2i\,\pi(t^{-\frac{1}{2}}x)\xi'} e^{-(4\pi^2|\xi'|^2 - t^{\frac{1}{3}} a_{\mathcal{I}}|\xi'|^{\frac{4}{3}} + i\,t^{\frac{1}{3}} b_{\mathcal{I}}\xi'|\xi'|^{\frac{1}{3}})} d\xi',\end{aligned}$$

by changing the variable by $\xi' = t^{\frac{1}{2}}\xi$. Then

$$\begin{aligned}K(t,x) &= t^{-\frac{1}{2}} \int_{\mathbb{R}} e^{2i\,\pi(t^{-\frac{1}{2}}x)\xi'} e^{-(4\pi^2|\xi'|^2 - a_{\mathcal{I}}|\xi'|^{\frac{4}{3}} + i\,b_{\mathcal{I}}\xi'|\xi'|^{\frac{1}{3}})}\\ &\quad\times e^{-(1-t^{\frac{1}{3}})(a_{\mathcal{I}}|\xi'|^{\frac{4}{3}} - i\,b_{\mathcal{I}}\xi'|\xi'|^{\frac{1}{3}})} d\xi',\\ &= t^{-\frac{1}{2}} \int_{\mathbb{R}} e^{2i\,\pi(t^{-\frac{1}{2}}x)\xi'} e^{-\psi_{\mathcal{I}}(\xi')} e^{-(1-t^{\frac{1}{3}})(a_{\mathcal{I}}|\xi'|^{\frac{4}{3}} - i\,b_{\mathcal{I}}\xi'|\xi'|^{\frac{1}{3}})} d\xi'.\end{aligned}$$

For $t < 1$, define $G((1-t^{\frac{1}{3}}),\cdot) := \mathcal{F}^{-1}\big(\xi \to e^{-(1-t^{\frac{1}{3}})(a_{\mathcal{I}}|\xi|^{\frac{4}{3}} - i\, b_{\mathcal{I}}\xi|\xi|^{\frac{1}{3}})}\big)$. It is readily seen that G is L^1 as an inverse Fourier transform of a $W^{2,1}$ function. Moreover, for $t_0 \in (0,1)$ and all $t \in (0, t_0]$,

$$\|G((1-t^{\frac{1}{3}}),\cdot)\|_{L^1(\mathbb{R})} \leq C \Big\| e^{-(1-t^{\frac{1}{3}})(a_{\mathcal{I}}|\cdot|^{\frac{4}{3}} - i\, b_{\mathcal{I}}\cdot|\cdot|^{\frac{1}{3}})} \Big\|_{W^{2,1}(\mathbb{R},\mathbb{C})} \leq C(t_0),$$

where $C(t_0)$ only depends on t_0. Classical formulas on the Fourier transform then give

$$K(t,x) = t^{-\frac{1}{2}}\big(K(1,\cdot) * G((1-t^{\frac{1}{3}}),\cdot)\big)(t^{-\frac{1}{2}}x).$$

Observe now that $\partial_x K(1,\cdot) = \mathcal{F}^{-1}\big(\xi \to 2i\,\xi\pi e^{-\psi_{\mathcal{I}}(\xi)}\big)$ is L^1 as an inverse Fourier transform of a $W^{2,1}$ function. Then,

$$\partial_x K(t,x) = t^{-1}\big(\partial_x K(1,\cdot) * G((1-t^{\frac{1}{3}}),\cdot)\big)(t^{-\frac{1}{2}}x)$$

is L^1 and its L^1 norm can be computed by the change of variable $x' = t^{-\frac{1}{2}}x$ as follows:

$$\begin{aligned}\|\partial_x K(t,\cdot)\|_{L^1(\mathbb{R})} &= t^{-\frac{1}{2}}\|\partial_x K(1,\cdot) * G((1-t^{\frac{1}{3}}),\cdot)\|_{L^1(\mathbb{R})} \\ &\leq t^{-\frac{1}{2}}\|\partial_x K(1,\cdot)\|_{L^1(\mathbb{R})} C(t_0),\end{aligned}$$

for any $t \in (0, t_0]$. Since

$$\|\partial_x K(t,\cdot)\|_{L^1(\mathbb{R})} \leq C\|\xi \to 2i\,\xi\pi e^{-t\psi_{\mathcal{I}}(\xi)}\|_{W^{2,1}(\mathbb{R},\mathbb{C})} \leq C(t_0, T),$$

for all $t \in [t_0, T]$, the proof of (2.7) is now complete. □

Remark 4. For any $u_0 \in L^2(\mathbb{R})$ and $t > 0$,

$$\|K(t,\cdot) * u_0\|_{L^2(\mathbb{R})} \leq e^{\omega_0 t}\|u_0\|_{L^2(\mathbb{R})}, \tag{2.10}$$

where $\omega_0 = -\min Re(\psi_{\mathcal{I}}) > 0$.

Proof. This is readily established with Plancherel's formula, as in (2.9).

3. Duhamel's formula

Using the Fourier transform and Corollary 1, we formally see that any solution to (1.1) satisfies Duhamel's formula (3.1). This observation is the starting point of the definition of mild solution below.

Definition 1. *Let $T > 0$ and $u_0 \in L^2(\mathbb{R})$. We say that $u \in L^\infty((0,T); L^2(\mathbb{R}))$ is a mild solution to (1.1) if for a.e. $t \in (0,T)$,*

$$u(t,\cdot) = K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * u^2(s,\cdot)ds. \tag{3.1}$$

The following proposition shows that all the terms in (3.1) are well-defined and that Equation (1.1) generates a (non-linear) semi-group.

Proposition 5. *Let* $T > 0$, $u_0 \in L^2(\mathbb{R})$ *and* $v \in L^\infty((0,T); L^1(\mathbb{R}))$. *Then the function*

$$u : t \in (0,T] \to K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * v(s,\cdot)ds \in L^2(\mathbb{R}) \quad (3.2)$$

is well defined and belongs to $C([0,T]; L^2(\mathbb{R}))$ (*being extended at* $t = 0$ *by the value* $u(0,.) = u_0$*).*

(*Semi-group property*) *Moreover, for all* $t_0 \in (0,T)$ *and all* $t \in [0, T - t_0]$,

$$u(t_0 + t, .) = K(t,\cdot) * u(t_0,\cdot) - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * v(t_0 + s,\cdot)ds.$$

Proof. By (2.5), it is classical that the function $t \in (0,T] \to K(t,\cdot) * u_0 \in L^2(\mathbb{R})$ is continuous and can be continuously extended by the value $u(0,\cdot) = u_0$ at $t = 0$. What is left to prove is thus the continuity of the function

$$w : t \in [0,T] \to \int_0^t \partial_x K(t-s,\cdot) * v(s,\cdot)ds \in L^2(\mathbb{R}).$$

Let us extend $\partial_x K$ and v for all time in the following way:

$$\mathcal{H}(t,\cdot) := \begin{cases} \partial_x K(t,\cdot) & \text{if } t > 0, \\ 0 & \text{if not} \end{cases} \quad \text{and} \quad \mathcal{V}(t,\cdot) := \begin{cases} v(t,\cdot) & \text{if } t \in (0,T), \\ 0 & \text{if not.} \end{cases}$$

Then we have

$$w(t,\cdot) = \int_{\mathbb{R}} \mathcal{H}(t-s,\cdot) * \mathcal{V}(s,\cdot)ds.$$

It is immediate that $\mathcal{V} \in L^\infty(\mathbb{R}; L^1(\mathbb{R}))$. Moreover, (2.6) implies that

$$\|\mathcal{H}(t,\cdot)\|_{L^2(\mathbb{R})} \le \mathbf{1}_{\{0<t<T\}} \mathcal{K}_0 t^{-\frac{3}{4}}, \quad (3.3)$$

and it follows that $\mathcal{H} \in L^1(\mathbb{R}; L^2(\mathbb{R}))$. Young's inequalities imply that for all $t \in \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}} \|\mathcal{H}(t-s,\cdot) * \mathcal{V}(s,\cdot)\|_{L^2(\mathbb{R})} ds &\le \int_{\mathbb{R}} \|\mathcal{H}(t-s,\cdot)\|_{L^2(\mathbb{R})} \|\mathcal{V}(s,\cdot)\|_{L^1(\mathbb{R})} ds, \\ &\le \|\mathcal{H}\|_{L^1(\mathbb{R};L^2(\mathbb{R}))} \|\mathcal{V}\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}))}. \quad (3.4) \end{aligned}$$

This implies, in particular, that the function w is well defined. Let us now take $t, s \in \mathbb{R}$ and define

$$I := \left\| \int_{\mathbb{R}} \mathcal{H}(t-\tau,\cdot) * \mathcal{V}(\tau)d\tau - \int_{\mathbb{R}} \mathcal{H}(s-\tau,\cdot) * \mathcal{V}(\tau)d\tau \right\|_{L^2(\mathbb{R})}.$$

We have

$$I \leq \int_{\mathbb{R}} \left\| \Big(\mathcal{H}(t-\tau,\cdot) - \mathcal{H}(s-\tau,\cdot) \Big) * \mathcal{V}(\tau) \right\|_{L^2(\mathbb{R})} d\tau,$$
$$\leq \int_{\mathbb{R}} \|\mathcal{H}(t-\tau,\cdot) - \mathcal{H}(s-\tau,\cdot)\|_{L^2(\mathbb{R})} \|\mathcal{V}(\tau)\|_{L^1(\mathbb{R})} d\tau,$$

thanks to Young's inequalities. It follows that

$$I \leq \int_{\mathbb{R}} \|\mathcal{H}(t-\tau,\cdot) - \mathcal{H}(s-\tau,\cdot)\|_{L^2(\mathbb{R})} d\tau \; \|\mathcal{V}\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}))}.$$

Since the translations are continuous in $L^1(\mathbb{R}; L^2(\mathbb{R}))$, we see that $I \to 0$ as $|t-s| \to 0$. In particular, the function w is continuous and this completes the proof of the continuity of u.

Let us now prove the semi-group property. By (2.5) and (2.8), we infer that

$$u(t_0+t,\cdot) = K(t,\cdot) * K(t_0,\cdot) * u_0 - \frac{1}{2}\int_0^{t_0} \partial_x K(t+t_0-s,\cdot) * v(s,\cdot) ds$$
$$- \frac{1}{2}\int_{t_0}^{t+t_0} \partial_x K(t+t_0-s,\cdot) * v(s,\cdot) ds,$$
$$= K(t_0,\cdot) * K(t,\cdot) * u_0 - \frac{1}{2}\int_0^{t_0} K(t,\cdot) * \partial_x K(t_0-s,\cdot) * v(s,\cdot) ds$$
$$- \frac{1}{2}\int_0^{t} \partial_x K(t-s',\cdot) * v(t_0+s',\cdot) ds',$$

thanks to the change of variables $s' = s - t_0$ to compute the last integral term. Thus

$$u(t_0+t,\cdot) = K(t,\cdot) * K(t_0,\cdot) * u_0 - K(t,\cdot) * \frac{1}{2}\int_0^{t_0} \partial_x K(t_0-s,\cdot) * v(s,\cdot) ds$$
$$- \frac{1}{2}\int_0^{t} \partial_x K(t-s',\cdot) * v(t_0+s',\cdot) ds',$$
$$= K(t,\cdot) * \Big(K(t_0,\cdot) * u_0 - \frac{1}{2}\int_0^{t_0} \partial_x K(t_0-s,\cdot) * v(s,\cdot) ds \Big)$$
$$- \frac{1}{2}\int_0^{t} \partial_x K(t-s',\cdot) * v(t_0+s',\cdot) ds',$$
$$= K(t,\cdot) * u(t_0,\cdot) - \frac{1}{2}\int_0^{t} \partial_x K(t-s',\cdot) * v(t_0+s',\cdot) ds'.$$

The proof of the semi-group property is now complete. □

Remark 5. For $v \in L^\infty((0,T);L^1(\mathbb{R}))$, $u \in C([0,T];L^2(\mathbb{R}))$ defined in (3.2) satisfies

$$\|u\|_{C([0,T];L^2(\mathbb{R}))} \leq e^{\omega_0 T}\|u_0\|_{L^2(\mathbb{R})} + 2\mathcal{K}_0 T^{\frac{1}{4}}\|v\|_{L^\infty((0,T);L^1(\mathbb{R}))}. \tag{3.5}$$

Proof. Indeed, with (3.3) and (3.4), we estimate the integral term of (3.2) and, with (2.10), we estimate the L^2 norm of $K(t,\cdot) * u_0$. □

4. Uniqueness of a solution

Let us state a lemma that will be needed later.

Lemma 1. *Let $T > 0$, $u_0 \in L^2(\mathbb{R})$. For $i = 1, 2$, let $v_i \in L^\infty((0,T);L^1(\mathbb{R}))$ and define $u_i \in C([0,T];L^2(\mathbb{R}))$ as in Proposition 5 by*

$$u_i(t,\cdot) := K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * v_i(s,\cdot)ds.$$

Then we have the estimate

$$\|u_1 - u_2\|_{C([0,T];L^2(\mathbb{R}))} \leq 2\mathcal{K}_0 T^{\frac{1}{4}}\, \|v_1 - v_2\|_{L^\infty((0,T);L^1(\mathbb{R}))}. \tag{4.1}$$

Proof. For all $t \in [0,T]$, we have

$$u_1(t,\cdot) - u_2(t,\cdot) = -\frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * (v_1(s,\cdot) - v_2(s,\cdot))ds.$$

Hence,

$$\begin{aligned}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} &= \frac{1}{2}\Big|\Big|\int_0^t \partial_x K(t-s,\cdot) * (v_1(s,\cdot) - v_2(s,\cdot))ds\Big|\Big|_{L^2(\mathbb{R})}, \\ &\leq \frac{1}{2}\int_0^t \|\partial_x K(t-s,.) * (v_1(s,.) - v_2(s,.))\|_{L^2(\mathbb{R})}\, ds. \end{aligned} \tag{4.2}$$

By (2.6),

$$\begin{aligned}&\|\partial_x K(t-s,.) * (v_1(s,\cdot) - v_2(s,\cdot))\|_{L^2(\mathbb{R})} \\ &\leq \|\partial_x K(t-s,\cdot)\,\|_{L^2(\mathbb{R})}\|v_1(s,\cdot) - v_2(s,\cdot)\|_{L^1(\mathbb{R})} \\ &\leq \mathcal{K}_0(t-s)^{-\frac{3}{4}}\|v_1(s,\cdot) - v_2(s,\cdot)\|_{L^1(\mathbb{R})}.\end{aligned}$$

Inequality (4.2) then gives

$$\begin{aligned}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} &\leq \frac{\mathcal{K}_0}{2}\int_0^t (t-s)^{-\frac{3}{4}}ds\, \|v_1 - v_2\|_{L^\infty((0,t);L^1(\mathbb{R}))}, \\ &= 2\mathcal{K}_0 t^{\frac{1}{4}}\, \|v_1 - v_2\|_{L^\infty((0,t);L^1(\mathbb{R}))}.\end{aligned}$$

In particular, for all $s \in [0,t]$

$$\begin{aligned}\|u_1(s,\cdot)-u_2(s,\cdot)\|_{L^2(\mathbb{R})} &\leq 2\mathcal{K}_0 s^{\frac14}\ \|v_1-v_2\|_{L^\infty((0,s);L^1(\mathbb{R}))}\\ &\leq 2\mathcal{K}_0 t^{\frac14}\ \|v_1-v_2\|_{L^\infty((0,t);L^1(\mathbb{R}))},\end{aligned}$$

and we have proved that

$$\|u_1-u_2\|_{C([0,t];L^2(\mathbb{R}))} \leq 2\mathcal{K}_0 t^{\frac14}\ \|v_1-v_2\|_{L^\infty((0,t);L^1(\mathbb{R}))}. \tag{4.3}$$

Proposition 6. *Let $T>0$ and $u_0 \in L^2(\mathbb{R})$. There exists at most one $u \in L^\infty((0,T);L^2(\mathbb{R}))$ which is a mild solution to* (1.1).

Proof. Let $u,v \in L^\infty((0,T);L^2(\mathbb{R}))$ be two mild solutions. Let $t \in [0,T]$. With Lemma 1 applied to $v_1 = u^2$ and $v_2 = v^2$, we get

$$\|u-v\|_{C([0,t];L^2(\mathbb{R}))} \leq 2\mathcal{K}_0 t^{\frac14}\ \|u^2-v^2\|_{L^\infty((0,t);L^1(\mathbb{R}))}. \tag{4.4}$$

Since

$$\|u^2-v^2\|_{L^\infty((0,t);L^1(\mathbb{R}))} \leq M\|u-v\|_{C([0,t];L^2(\mathbb{R}))},$$

with $M = \|u\|_{C([0,T],L^2(\mathbb{R}))} + \|v\|_{C([0,T],L^2(\mathbb{R}))}$, we get

$$\|u-v\|_{C([0,t];L^2(\mathbb{R}))} \leq 2M\mathcal{K}_0 t^{\frac14}\ \|u-v\|_{C([0,t];L^2(\mathbb{R}))}.$$

We then have established that $u=v$ on $[0,t]$ for any $t \in (0,T]$ such that $t < (2M\mathcal{K}_0)^{-4}$. Notice that since u and v are continuous with values in L^2, $u=v$ on $[0,T_*]$ with $T_* = (2M\mathcal{K}_0)^{-4} > 0$. To prove that $u=v$ on $[0,T]$, let us define $t_0 := \sup\{t \in (0,T] \text{ s.t. } u=v \text{ on } [0,t]\}$ and let us assume that $t_0 < T$. The continuity of u and v implies that $u(t_0,\cdot) = v(t_0,\cdot)$. The semi-group property of Proposition 5, thus, implies that $u(t_0+\cdot,\cdot)$ and $v(t_0+\cdot,\cdot)$ are mild solutions of (1.1)with the same initial condition; that is to say $u(t_0+0,\cdot) = v(t_0+0,\cdot)$. The first step of the proof then implies that $u(t_0+\cdot,\cdot) = v(t_0+\cdot,\cdot)$ on $[0,T_*]$; hence, we get a contradiction with the definition of t_0 and we deduce that $t_0 = T$. The proof of the uniqueness is now complete. □

5. Existence of a regular solution

This section is devoted to the proof of the existence of a classical solution u to (1.1). As far as the regularity is concerned, we show that $u \in C^{1,2}((0,T]\times\mathbb{R})$; that is to say $\partial_t u \in C((0,T]\times\mathbb{R})$ and $\partial_x u,\ \partial^2_{xx}u \in C((0,T]\times\mathbb{R})$ and for the sake of simplicity we only give the main ideas to get further regularity.

We first need the following technical result.

Lemma 2. *Let* $u_0 \in L^2(\mathbb{R})$ *and* $T > 0$. *Let* $v \in C([0,T]; L^1(\mathbb{R})) \cap C((0,T]; W^{1,1}(\mathbb{R}))$ *that satisfies*

$$\sup_{t \in (0,T]} t^{\frac{1}{2}} \|\partial_x v(t,\cdot)\|_{L^1(\mathbb{R})} < +\infty. \tag{5.1}$$

Let $u \in C([0,T]; L^2(\mathbb{R}))$ *be the function defined in* (3.2). *Then,* $u \in C((0,T]; H^1(\mathbb{R}))$ *with*

$$\sup_{t \in (0,T]} t^{\frac{1}{2}} \|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \le \mathcal{K}_1 \|u_0\|_{L^2(\mathbb{R})} + \frac{\mathcal{K}_0 I}{2}\, T^{\frac{1}{4}} \sup_{t \in (0,T]} t^{\frac{1}{2}} \|\partial_x v(t,\cdot)\|_{L^1(\mathbb{R})}, \tag{5.2}$$

where I *is a constant equal to*

$$\int_0^1 (1-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds = B(1/2, 1/4),$$

B *being the beta function. Moreover, let* $v_i \in C([0,T]; L^1(\mathbb{R})) \cap C((0,T]; W^{1,1}(\mathbb{R}))$ *satisfy* (5.1) *and define* u_i *by* (3.2) *(with* u *and* v *replaced, respectively, by* u_i *and* v_i*) for* $i = 1, 2$. *Then,*

$$\sup_{t \in (0,T]} t^{\frac{1}{2}} \|\partial_x (u_1 - u_2)(t,\cdot)\|_{L^2(\mathbb{R})} \le \frac{\mathcal{K}_0 I}{2}\, T^{\frac{1}{4}} \sup_{t \in (0,T]} t^{\frac{1}{2}} \|\partial_x (v_1 - v_2)(t,\cdot)\|_{L^1(\mathbb{R})}. \tag{5.3}$$

Proof. Recall that Proposition 5 ensures that $u \in C([0,T]; L^2(\mathbb{R}))$. It is easy to check that the distribution derivative of u with respect to the space variable satisfies the following: for any $t \in (0,T]$,

$$\partial_x u(t,\cdot) = \partial_x K(t,\cdot) * u_0 - \frac{1}{2} \int_0^t \partial_x K(t-s,\cdot) * \partial_x v(s,\cdot) ds.$$

Let us verify that all the terms are well defined in L^2. Since $\partial_x K(t,\cdot) \in L^1(\mathbb{R})$, it is obvious that $\partial_x K(t,\cdot) * u_0 \in L^2(\mathbb{R})$. Moreover, define

$$w(t,\cdot) := \frac{1}{2} \int_0^t \partial_x K(t-s,\cdot) * \partial_x v(s,\cdot) ds.$$

Young's inequalities and (2.6) give

$$\begin{aligned} \|\partial_x K(t-s,\cdot) * \partial_x v(s,\cdot)\|_{L^2(\mathbb{R})} &\le \|\partial_x K(t-s,\cdot)\|_{L^2(\mathbb{R})} \|\partial_x v(s,\cdot)\|_{L^1(\mathbb{R})} \\ &= \|\partial_x K(t-s,\cdot)\|_{L^2(\mathbb{R})} s^{-\frac{1}{2}} s^{\frac{1}{2}} \|\partial_x v(s,\cdot)\|_{L^1(\mathbb{R})} \\ &\le \mathcal{K}_0 (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} \sup_{\tau \in (0,T]} \tau^{\frac{1}{2}} \|\partial_x v(\tau,\cdot)\|_{L^1(\mathbb{R})}. \end{aligned} \tag{5.4}$$

Since $\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}}\, ds < +\infty$, by (5.1) we deduce that $w(t,\cdot)$ is well defined in L^2 and thus, for all $t \in (0,T]$, $\partial_x u(t,\cdot) \in L^2(\mathbb{R})$. Let us now prove that $\partial_x u$ is continuous on $(0,T]$ with values in L^2. For $\delta > 0$ and $t \in (0,T]$, define

$$w_\delta(t,\cdot) := \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * \Big(\mathbf{1}_{\{s>\delta\}} \partial_x v(s,\cdot)\Big) ds.$$

Since $\mathbf{1}_{\{s>\delta\}} \partial_x v(s,\cdot) \in L^\infty([0,T];L^1(\mathbb{R}))$, Proposition 5 ensures that w_δ is continuous on $[0,T]$ with values in L^2. Moreover, for any $t_0 \in (0,T]$, $\delta \le t_0$ and $t \in [t_0,T]$,

$$\begin{aligned}
\|w(t,\cdot) - w_\delta(t,\cdot)\|_{L^2(\mathbb{R})} &\le \frac{1}{2}\int_0^\delta \|\partial_x K(t-s,\cdot) * \partial_x v(s,\cdot)\|_{L^2(\mathbb{R})} ds, \\
&\le \frac{\mathcal{K}_0}{2}\int_0^\delta (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds \sup_{s\in(0,T]} s^{\frac{1}{2}} \|\partial_x v(s,\cdot)\|_{L^1(\mathbb{R})} \quad \text{by (5.4)}, \\
&\le \frac{\mathcal{K}_0}{2}\int_0^\delta (t_0-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds \sup_{s\in(0,T]} s^{\frac{1}{2}} \|\partial_x v(s,\cdot)\|_{L^1(\mathbb{R})}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sup_{t\in[t_0,T]} \|w(t,\cdot) - w_\delta(t,\cdot)\|_{L^2(\mathbb{R})} \\
&\le \frac{\mathcal{K}_0}{2}\int_0^\delta (t_0-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds \sup_{s\in(0,T]} s^{\frac{1}{2}} \|\partial_x v(s,\cdot)\|_{L^1(\mathbb{R})} \to 0,
\end{aligned}$$

as $\delta \to 0$. We deduce that $w \in C((0,T];L^2(\mathbb{R}))$ as a local uniform limit of continuous functions. Moreover,

$$\partial_x K(t,\cdot) * u_0 = \mathcal{F}^{-1}\Big(\xi \to 2i\,\pi\xi e^{-t\psi_{\mathcal{I}}(\xi)} \mathcal{F}u_0(\xi)\Big).$$

The dominated convergence theorem immediately implies that, for any $t_0 >$ 0,

$$\int_{\mathbb{R}} 4\pi^2 |\xi|^2 \Big| e^{-t\psi_{\mathcal{I}}(\xi)} - e^{-t_0\psi_{\mathcal{I}}(\xi)} \Big|^2 |\mathcal{F}u_0(\xi)|^2 d\xi \to 0, \quad \text{as } t \to t_0.$$

This means that $t > 0 \to \big(\xi \to 2i\pi\xi e^{-t\psi_{\mathcal{I}}(\xi)}\mathcal{F}u_0\big) \in L^2(\mathbb{R})$ is continuous and, since $\mathcal{F}$ is an isometry of L^2, we deduce that $t > 0 \to \partial_x K(t,\cdot) * u_0 \in L^2(\mathbb{R})$ is continuous. We then have established that $\partial_x u \in C((0,T];L^2(\mathbb{R}))$. Let us now estimate how the L^2 norm of $\partial_x u$ can explode at $t = 0$. By (5.4),

$$\|w(t,\cdot)\|_{L^2(\mathbb{R})} \le \frac{\mathcal{K}_0}{2}\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds \sup_{\tau\in(0,T]} \tau^{\frac{1}{2}} \|\partial_x v(\tau,\cdot)\|_{L^1(\mathbb{R})}$$

$$= \frac{\mathcal{K}_0 I}{2} t^{-\frac{1}{4}} \sup_{\tau\in(0,T]} \tau^{\frac{1}{2}} \|\partial_x v(\tau,\cdot)\|_{L^1(\mathbb{R})},$$

where $I = \int_0^1 (1-s')^{-\frac{3}{4}} s'^{-\frac{1}{2}} ds' = B(1/2, 1/4)$; notice that the last integral term has been computed with the help of the change of variables $s' = \frac{s}{t}$. Moreover, (2.7) and Young's inequalities imply that

$$\|\partial_x K(t,\cdot) * u_0\|_{L^2(\mathbb{R})} \leq \mathcal{K}_1 t^{-\frac{1}{2}} \|u_0\|_{L^2(\mathbb{R})}.$$

We deduce that, for any $t \in (0,T]$,

$$\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \leq \mathcal{K}_1 t^{-\frac{1}{2}} \|u_0\|_{L^2(\mathbb{R})} + \frac{\mathcal{K}_0 I}{2} t^{-\frac{1}{4}} \sup_{s\in(0,T]} s^{\frac{1}{2}} \|\partial_x v(s,\cdot)\|_{L^1(\mathbb{R})},$$

which implies immediately (5.2).

Let us now prove (5.3). For any $t \in (0,T]$,

$$\begin{aligned}
&\|\partial_x (u_1-u_2)(t,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\int_0^t \|\partial_x K(t-s,.) * \partial_x (v_1-v_2)(s,\cdot)\|_{L^2(\mathbb{R})} ds,\\
&\leq \frac{\mathcal{K}_0}{2}\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds \sup_{s\in(0,T]} s^{\frac{1}{2}} \|\partial_x (v_1-v_2)(s,\cdot)\|_{L^1(\mathbb{R})},\\
&= \frac{\mathcal{K}_0 I}{2} t^{-\frac{1}{4}} \sup_{s\in(0,T]} s^{\frac{1}{2}} \|\partial_x (v_1-v_2)(s,\cdot)\|_{L^1(\mathbb{R})},
\end{aligned}$$

which implies immediately (5.3). □

Remark 6. Let u_0, T, v and u satisfy the assumptions of Lemma 2. Then, we have established that, for any $t \in (0,T]$,

$$\partial_x u(t,\cdot) = \partial_x K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * \partial_x v(s,\cdot) ds.$$

With Lemma 2 in hand, we can prove the local-in-time existence.

Proposition 7. *Let $u_0 \in L^2(\mathbb{R})$. There exists $T_* > 0$ that only depends on $\|u_0\|_{L^2(\mathbb{R})}$ such that* (1.1) *admits a (unique) mild solution $u \in C([0,T_*];L^2(\mathbb{R})) \cap C((0,T_*];H^2(\mathbb{R}))$ on $(0,T_*)$.*

Proof. We use a contracting fixed-point theorem. For $u \in C([0,T_*];L^2(\mathbb{R})) \cap C((0,T_*];H^1(\mathbb{R}))$, define the norm

$$|||u||| := \|u\|_{C([0,T_*];L^2(\mathbb{R}))} + \sup_{t\in(0,T_*]} t^{\frac{1}{2}} \|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}. \tag{5.5}$$

Define the space $X := \Big\{u \in C([0,T_*];L^2(\mathbb{R})) \cap C((0,T_*];H^1(\mathbb{R})) : u(0,\cdot) = u_0 \text{ and } |||u||| < +\infty\Big\}$. It is readily seen that X is a complete metric space endowed with the distance induced by the norm $|||\cdot|||$. For $u \in X$, define the function

$$\Theta u : t \in [0,T_*] \to K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * u^2(s,\cdot)ds \in L^2(\mathbb{R}). \quad (5.6)$$

By Proposition 5, $\Theta u \in C([0,T_*];L^2(\mathbb{R}))$ and satisfies $\Theta u(0,\cdot) = u_0$. Define $v := u^2$. We have $\partial_x v = 2u\partial_x u$. Therefore, $v \in C([0,T_*];L^1(\mathbb{R})) \cap C((0,T_*];W^{1,1}(\mathbb{R}))$ and (5.1) holds true. By Lemma 2, we deduce that $\Theta u \in X$. Let us take $R > \|u_0\|_{L^2(\mathbb{R})} + \mathcal{K}_1\|u_0\|_{L^2(\mathbb{R})}$ and assume that $|||u||| \le R$. Since $\|u^2\|_{L^\infty((0,T_*);L^1(\mathbb{R}))} = \|u\|^2_{C([0,T_*];L^2(\mathbb{R}))}$, estimate (3.5) of Remark 5 implies that

$$\begin{aligned}\|\Theta u\|_{C([0,T_*];L^2(\mathbb{R}))} &\le e^{\omega_0 T_*}\|u_0\|_{L^2(\mathbb{R})} + 2\mathcal{K}_0 T_*^{\frac{1}{4}}\ \|u\|^2_{C([0,T_*];L^2(\mathbb{R}))},\\ &\le e^{\omega_0 T_*}\|u_0\|_{L^2(\mathbb{R})} + 2\mathcal{K}_0 T_*^{\frac{1}{4}}\ R^2. \end{aligned} \quad (5.7)$$

Estimate (5.2) of Lemma 2, implies that

$$\begin{aligned}&\sup_{t\in(0,T_*]} t^{\frac{1}{2}}\|\partial_x(\Theta u(t,\cdot))\|_{L^2(\mathbb{R})}\\ &\le \mathcal{K}_1\|u_0\|_{L^2(\mathbb{R})} + \frac{\mathcal{K}_0 I}{2}\ T_*^{\frac{1}{4}} \sup_{t\in(0,T_*]} t^{\frac{1}{2}}\|\partial_x(u^2)(t,\cdot)\|_{L^1(\mathbb{R})},\\ &\le \mathcal{K}_1\|u_0\|_{L^2(\mathbb{R})} + \mathcal{K}_0 I\ T_*^{\frac{1}{4}} R^2,\end{aligned}$$

by the Cauchy-Schwarz inequality. Adding this inequality to (5.7), we get

$$|||\Theta u||| \le e^{\omega_0 T_*}\|u_0\|_{L^2(\mathbb{R})} + \mathcal{K}_1\|u_0\|_{L^2(\mathbb{R})} + (2+I)\mathcal{K}_0 T_*^{\frac{1}{4}} R^2.$$

For $T_* \in (0,T]$ sufficiently small such that

$$e^{\omega_0 T_*}\|u_0\|_{L^2(\mathbb{R})} + \mathcal{K}_1\|u_0\|_{L^2(\mathbb{R})} + (2+I)\mathcal{K}_0 T_*^{\frac{1}{4}} R^2 \le R, \quad (5.8)$$

we deduce that $|||\Theta u||| \le R$. To sum up, we have established that for any $T_* \in (0,T]$ such that (5.8) holds true, Θ (defined by (5.6)) maps $\overline{B}_R$ into itself, where $\overline{B}_R$ denotes the ball of X (endowed with the $|||\cdot|||$ norm) centered at the origin and of radius R. Let us now prove that Θ is a contraction. For $u, v \in \overline{B}_R$, Estimate (4.1) of Lemma 1 implies that

$$\|\Theta u - \Theta v\|_{C([0,T_*];L^2(\mathbb{R}))} \le 4R\mathcal{K}_0 T_*^{\frac{1}{4}}\ \|u-v\|_{C([0,T_*];L^2(\mathbb{R}))}, \quad (5.9)$$

where we again used

$$\|u^2 - v^2\|_{C([0,T_*];L^1(\mathbb{R}))} \\ \leq (\|u\|_{C([0,T_*],L^2(\mathbb{R}))} + \|v\|_{C([0,T_*],L^2(\mathbb{R}))})\|u - v\|_{C([0,T_*];L^2(\mathbb{R}))}.$$

Moreover, Estimate (5.3) of Lemma 2 implies that

$$\sup_{t\in(0,T_*]} t^{\frac{1}{2}}\|\partial_x(\Theta u - \Theta v)(t,\cdot)\|_{L^2(\mathbb{R})} \\ \leq \mathcal{K}_0 I T_*^{\frac{1}{4}} \sup_{t\in(0,T_*]} t^{\frac{1}{2}}\|(u\partial_x u - v\partial_x v)(t,\cdot)\|_{L^1(\mathbb{R})}.$$

Since

$$t^{\frac{1}{2}}\|(u\partial_x u - v\partial_x v)(t,\cdot)\|_{L^1(\mathbb{R})} \leq t^{\frac{1}{2}}\|\partial_x v(t,\cdot)\|_{L^2(\mathbb{R})}\|(u-v)(t,\cdot)\|_{L^2(\mathbb{R})} \\ + t^{\frac{1}{2}}\|u(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x(u-v)(t,\cdot)\|_{L^2(\mathbb{R})} \\ \leq |||v|||\ \|(u-v)(t,\cdot)\|_{L^2(\mathbb{R})} + |||u|||\ t^{\frac{1}{2}}\|\partial_x(u-v)(t,\cdot)\|_{L^2(\mathbb{R})} \leq R|||u-v|||,$$

we get

$$\sup_{t\in(0,T_*]} t^{\frac{1}{2}}\|\partial_x(\Theta u - \Theta v)(t,\cdot)\|_{L^2(\mathbb{R})} \leq R\mathcal{K}_0 I T_*^{\frac{1}{4}}|||u-v|||.$$

Adding this inequality to (5.9), we find that

$$|||\Theta u - \Theta v||| \leq (4+I)R\mathcal{K}_0 T_*^{\frac{1}{4}}|||u-v|||.$$

Consequently, for any $T_* > 0$ sufficiently small such that (5.8) holds true and $(4+I)R\mathcal{K}_0 T_*^{\frac{1}{4}} < 1$, Θ is a contraction from $\overline{B}_R$ into itself. The Banach fixed-point theorem then implies that Θ admits a (unique) fixed point $u \in C([0,T_*];L^2(\mathbb{R})) \cap C((0,T_*];H^1(\mathbb{R}))$ which is, of course, a mild solution to (1.1).

To prove the H^2 regularity of u, we have to use again a contracting fixed-point theorem. But, this is now the gradient of the solution which is searched for as a fixed point. Let $t_0 \in (0,T_*)$ and, for $t \in (0, T_* - t_0]$, define

$$\overline{u}(t,\cdot) := u(t_0 + t, \cdot).$$

Let $T'_* \in (0, T_* - t_0]$ and define the complete metric space $X' := \Big\{v \in C([0,T'_*];L^2(\mathbb{R})) \cap C((0,T'_*];H^1(\mathbb{R})) : v(0,\cdot) = v_0$ and $|||v||| < +\infty\Big\}$, where

$v_0 := \partial_x \overline{u}(0,\cdot)$ and $|||\cdot|||$ is defined in (5.5) with T_* replaced by T'_*. For $v \in X'$, define the function

$$\Theta' v : t \in [0,T'_*] \to K(t,\cdot)*v_0 - \int_0^t \partial_x K(t-s,\cdot)*(\overline{u}v)(s,\cdot)ds \in L^2(\mathbb{R}). \quad (5.10)$$

Arguing as in the first step of the proof, we claim that Proposition 5, Remark 5, Lemmas 1 and 2 imply that Θ' maps X' into itself with, for any $u,v \in X'$,

$$|||\Theta' v||| \leq e^{\omega_0 T'_*}\|v_0\|_{L^2(\mathbb{R})} + \mathcal{K}_1\|v_0\|_{L^2(\mathbb{R})} + C{T'_*}^{\frac{1}{4}}|||v|||, \quad (5.11)$$

$$|||\Theta' v - \Theta' w||| \leq C{T'_*}^{\frac{1}{4}}|||v-w|||,$$

for some non-negative constant C that only depends on $\mathcal{K}_0$ and $\|\overline{u}\|_{C([t_0,T_*];H^1(\mathbb{R}))}$. Let us take R' such that

$$R' > e^{\omega_0 T'_*}\|v_0\|_{L^2(\mathbb{R})} + \mathcal{K}_1\|v_0\|_{L^2(\mathbb{R})}.$$

If $T'_* > 0$ satisfies

$$e^{\omega_0 T'_*}\|v_0\|_{L^2(\mathbb{R})} + \mathcal{K}_1\|v_0\|_{L^2(\mathbb{R})} + C{T'_*}^{\frac{1}{4}}R' \leq R' \quad \text{and} \quad C{T'_*}^{\frac{1}{4}} < 1,$$

then Θ' maps $\overline{B}_{R'}(X')$ into itself and is a contraction. Let v denote its unique fixed point. Observe now that $\Theta'\partial_x\overline{u} = \partial_x\overline{u}$, thanks to Remark 6. But, using the same argument as in Proposition 6, one can easily prove that there is at most one function $w \in L^\infty((0,T'_*);L^2(\mathbb{R}))$ which is a fixed point of Θ'. It follows that $\partial_x\overline{u} = v \in X'$ on $(0,T'_*)$ and thus that $u \in C((t_0,t_0+T'_*];H^2(\mathbb{R}))$. Recalling that t_0 is an arbitrary time in $(0,T_*]$, we see that u is continuous on $(0,T_*]$ with values in H^2. The proof of Proposition 7 is complete. □

Let us now prove the regularity of the solution. We also establish the regularity up to time $t=0$ when u_0 is $C^2 \cap H^2$, since this will be needed in the proof of the maximum principle failure.

Proposition 8. *Let* $u_0 \in H^2(\mathbb{R})$ *and* $T > 0$. *Assume that* u *is a mild solution to* (1.1) *that satisfies the regularity of Proposition* 7 *up to time* T. *Then,* u *belongs in fact to* $C([0,T];H^2(\mathbb{R})) \cap C^{1,2}((0,T]\times\mathbb{R})$ *and satisfies* (1.1) *in the classical sense. Moreover, if* $u_0 \in C^2(\mathbb{R})$, *then* $u \in C^{1,2}([0,T]\times\mathbb{R})$ *and satisfies the PDE up to time* $t=0$.

Proof. First, we leave it to the reader to verify that the continuity with values in H^2 up to the time $t=0$ can be proved again by the use of a contracting fixed-point theorem. Note that the regularity of u_0 allows us to work in a space of continuous functions with values in H^2 up to time $t=0$; more precisely, we argue as in the proof of Proposition 7, but we can directly use the $C([0,T_*];H^2)$ norm instead of the $|||\cdot|||$ norm defined in (5.5). Let us now give the proof of the $C^{1,2}$-regularity in three steps: first, we prove that u satisfies the PDE (1.1) in the distribution sense; next, we prove the C^2-regularity in space that implies the $C^{1,2}$-regularity, thanks to the equation.

First step: u is a distribution solution. Taking the Fourier transform with respect to the space variable in (3.1), we get, for all $t \in [0,T]$,

$$\mathcal{F}(u(t,\cdot)) = e^{-t\psi_{\mathcal{I}}}\mathcal{F}u_0 - \int_0^t i\,\pi \cdot e^{-(t-s)\psi_{\mathcal{I}}}\mathcal{F}(u^2(s,\cdot))ds. \tag{5.12}$$

Since $u^2 \in C([0,T];L^1(\mathbb{R}))$, we know that $\mathcal{F}(u^2) \in C([0,T];C_b(\mathbb{R},\mathbb{C})) \subset C([0,T]\times\mathbb{R},\mathbb{C})$. The function

$$w(t,\xi) := -\int_0^t i\,\pi\xi e^{-(t-s)\psi_{\mathcal{I}}(\xi)}\mathcal{F}(u^2(s,\cdot))(\xi)ds$$

is, thus, continuous on $[0,T]\times\mathbb{R}$; moreover, classical results on ODEs imply that w is differentiable with respect to the time variable with $\partial_t w \in C([0,T]\times\mathbb{R},\mathbb{C})$ and

$$\partial_t w(t,\xi) + \psi_{\mathcal{I}}(\xi)w(t,\xi) = -i\,\pi\xi\mathcal{F}(u^2(t,\cdot))(\xi) = -\mathcal{F}\Big(\partial_x\big(\frac{u^2}{2}\big)(t,\cdot)\Big)(\xi). \tag{5.13}$$

Let us prove that all these terms are continuous with values in L^2. First, $u \in C([0,T];H^1(\mathbb{R}))$, therefore $\partial_x(u^2) \in C([0,T];L^2(\mathbb{R}))$, and we deduce that $\mathcal{F}(\partial_x(\frac{u^2}{2}))$ is continuous with values in L^2. Moreover, Equation (5.12) implies that

$$\psi_{\mathcal{I}}\,w(t,\cdot) = \psi_{\mathcal{I}}\big(\mathcal{F}(u(t,\cdot)) - e^{-t\psi_{\mathcal{I}}}\mathcal{F}u_0\big).$$

Since $u \in C([0,T];H^2(\mathbb{R}))$ and $\psi_{\mathcal{I}}$ behaves at infinity as $|\cdot|^2$, we clearly have that the function $t \in [0,T] \to \psi_{\mathcal{I}}\,w(t,\cdot) \in L^2(\mathbb{R},\mathbb{C})$ is continuous. Finally, we have proved that all the terms in (5.13) are continuous with values in L^2. In particular, $\partial_t w \in C([0,T];L^2(\mathbb{R},\mathbb{C}))\cap C([0,T]\times\mathbb{R},\mathbb{C})$ and it follows that $w \in C^1([0,T];L^2(\mathbb{R},\mathbb{C}))$ with

$$\frac{d}{dt}(w(t,\cdot)) + \psi_{\mathcal{I}}\,w(t,\cdot) = -\mathcal{F}\Big(\partial_x\big(\frac{u^2}{2}\big)(t,\cdot)\Big).$$

Moreover, it is easy to see that $t \in [0,T] \to e^{-t\psi_{\mathcal{I}}}\mathcal{F}u_0 \in L^2(\mathbb{R},\mathbb{C})$ is C^1 with

$$\frac{d}{dt}\big(e^{-t\psi_{\mathcal{I}}}\mathcal{F}u_0\big) + \psi_{\mathcal{I}}e^{-t\psi_{\mathcal{I}}}\mathcal{F}u_0 = 0.$$

From (5.12), we infer that $\mathcal{F}u$ is C^1 on $[0,T]$ with values in L^2 with

$$\begin{aligned}\frac{d}{dt}(\mathcal{F}(u(t,\cdot))) &= -\psi_{\mathcal{I}}w(t,\cdot) - \psi_{\mathcal{I}}e^{-t\psi_{\mathcal{I}}}\mathcal{F}u_0 - \mathcal{F}\Big(\partial_x\Big(\frac{u^2}{2}\Big)(t,\cdot)\Big)\\ &= -\psi_{\mathcal{I}}\mathcal{F}(u(t,\cdot)) - \mathcal{F}\Big(\partial_x\big(\frac{u^2}{2}\big)(t,\cdot)\Big).\end{aligned}$$

Since $\mathcal{F}$ is an isometry of L^2, we deduce that $u \in C^1([0,T];L^2(\mathbb{R}))$ and that

$$\begin{aligned}\frac{d}{dt}(u(t,.)) &= -\partial_x\Big(\frac{u^2}{2}\Big)(t,\cdot) - \mathcal{F}^{-1}\Big(\psi_{\mathcal{I}}\mathcal{F}(u(t,\cdot))\Big)\\ &= -\partial_x\Big(\frac{u^2}{2}\Big)(t,\cdot) - \mathcal{I}[u(t,\cdot)] + \partial^2_{xx}u(t,\cdot),\end{aligned}$$

where we used Corollary 1 to compute the pseudo-differential term. In particular, u satisfies the PDE (1.1) in the distribution sense.

Second step: C^2-regularity in space. Differentiating (3.1) two times with respect to the space variable, we get, for any $t \in [0,T]$,

$$\partial^2_{xx}u(t,\cdot) = K(t,\cdot) * u_0'' - \int_0^t \partial_x K(t-s,\cdot) * v(s,\cdot)ds, \tag{5.14}$$

where $v = (\partial_x u)^2 + u\partial^2_{xx}u$. By the Sobolev imbedding $H^2(\mathbb{R}) \hookrightarrow C^1_b(\mathbb{R})$, we know that $v \in C([0,T];L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$. By Lemma 4 in Appendix, we know that, for all $x,y \in \mathbb{R}$,

$$\begin{aligned}&|\partial_x K(t-s,\cdot) * v(s,\cdot)(x) - \partial_x K(t-s,\cdot) * v(s,\cdot)(y)|\\ &\le \|\partial_x K(t-s)\|_{L^2(\mathbb{R})}\|\mathcal{T}_{(x-y)}(v(s,\cdot)) - v(s,\cdot)\|_{L^2(\mathbb{R})}.\end{aligned}$$

By (2.6), we deduce that, for all $t \in [0,T]$ and all $x,y \in \mathbb{R}$,

$$\begin{aligned}&\Big|\int_0^t \partial_x K(t-s,\cdot) * v(s,\cdot)(x)ds - \int_0^t \partial_x K(t-s,\cdot) * v(s,\cdot)(y)ds\Big|\\ &\le \int_0^t \mathcal{K}_0(t-s)^{-\frac{3}{4}}\|\mathcal{T}_{(x-y)}(v(s,\cdot)) - v(s,\cdot)\|_{L^2(\mathbb{R})}ds\\ &\le 4T^{\frac{1}{4}}\sup_{s\in[0,T]}\|\mathcal{T}_{(x-y)}(v(s,\cdot)) - v(s,\cdot)\|_{L^2(\mathbb{R})}.\end{aligned}$$

By Lemma 5 in Appendix, we deduce that the second term of (5.14) is continuous with respect to the space variable uniformly in the time variable (equicontinuity with respect to the time variable). Moreover, we already know that this term is continuous on $[0,T]$ with values in L^2 (by Proposition 5) and Lemma 6 in Appendix implies that it is continuous with respect to the couple (t,x) on $[0,T] \times \mathbb{R}$. We now leave it to the reader to verify that

$(t,x) \to K(t,\cdot) * u_0''(x)$ is continuous on $(0,T] \times \mathbb{R}$ when $u_0 \in H^2(\mathbb{R})$. By Formula (5.14), we then have proved that $\partial^2_{xx} u \in C((0,T] \times \mathbb{R})$.

Conclusion: $C^{1,2}$-regularity. By the first step, the terms $\partial_t u$ and $-\partial_x(\frac{u^2}{2})$ $-\mathcal{I}[u] + \partial^2_{xx} u$ have the same regularity; by the second step, it is enough to establish the continuity of $\mathcal{I}[u]$ with respect to the space-time variable. To do this, we recall that $u \in C([0,T]; H^2(\mathbb{R}))$ and we use the second item of Remark 3. We conclude that $\mathcal{I}[u] \in C([0,T] \times \mathbb{R})$ and thus, $u \in C^{1,2}((0,T] \times \mathbb{R})$. This completes the proof when $u_0 \in H^2(\mathbb{R})$.

To finish, we prove the regularity up to time $t = 0$ when $u_0 \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$. We leave it to the reader to verify that $(t,x) \to K(t,\cdot) * u_0''(x)$ is continuous up to time $t = 0$. Moreover, we already have seen in the second step that the integral term of (5.14) is continuous on $[0,T] \times \mathbb{R}$. We deduce that $\partial^2_{xx} u \in C([0,T] \times \mathbb{R})$. But, we also have seen above that $\mathcal{I}[u] \in C([0,T] \times \mathbb{R})$; hence, Equation (1.1) implies that $\partial_t u \in C([0,T] \times \mathbb{R})$ and thus $u \in C^{1,2}([0,T] \times \mathbb{R})$. The proof of Proposition 8 is complete. □

We can finally prove the global-in-time existence.

Proposition 9. *Let $u_0 \in L^2(\mathbb{R})$ and $T > 0$. There exists a (unique) mild solution $u \in C([0,T]; L^2(\mathbb{R})) \cap C((0,T]; H^2(\mathbb{R}))$ to (1.1). Moreover, u belongs to $C^{1,2}((0,T] \times \mathbb{R})$ and satisfies the PDE (1.1) in the classical sense.*

Proof. First step: *a priori* estimate. We begin by deriving an L^2 estimate on an *a priori* solution of (1.1) with regularity: $u \in C([0,T]; L^2(\mathbb{R})) \cap C((0,T]; H^2(\mathbb{R})) \cap C^{1,2}((0,T] \times \mathbb{R})$. Multiplying (1.1) by u and integrating with respect to the space variable, we get

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u^2 dx + \int_{\mathbb{R}} (\mathcal{I}[u] - \partial^2_{xx} u) u dx = 0. \tag{5.15}$$

Indeed, the following computations show that the nonlinear term equals 0:

$$\int_{\mathbb{R}} \partial_x\Big(\frac{u^2}{2}\Big) u dx = -\int_{\mathbb{R}} \frac{u^2}{2} \partial_x u dx = -\frac{1}{2}\int_{\mathbb{R}} u(u\partial_x u) dx = -\frac{1}{2}\int_{\mathbb{R}} u\, \partial_x\Big(\frac{u^2}{2}\Big) dx;$$

notice that "there is no boundary term from the infinity," because $u(t,\cdot)$ and $u^2(t,\cdot)$ belong to $H^1(\mathbb{R})$, for all $t \in (0,T]$, since $u(t,\cdot) \in H^2(\mathbb{R})$. But, Corollary 1 implies that

$$\begin{aligned}\int_{\mathbb{R}} (\mathcal{I}[u] - \partial^2_{xx} u) u dx &= \int_{\mathbb{R}} \mathcal{F}^{-1}(\psi_{\mathcal{I}} \mathcal{F} u) u dx = \int_{\mathbb{R}} \psi_{\mathcal{I}} |\mathcal{F} u|^2 d\xi \\ &= \int_{\mathbb{R}} \mathrm{Re}(\psi_{\mathcal{I}}) |\mathcal{F} u|^2 d\xi,\end{aligned}$$

since $\int_{\mathbb{R}}(\mathcal{I}[u]-\partial_{xx}^2 u)udx$ is real. It follows that

$$\int_{\mathbb{R}}(\mathcal{I}[u]-\partial_{xx}^2 u)udx \geq \min \operatorname{Re}(\psi_{\mathcal{I}}) \int_{\mathbb{R}} |\mathcal{F}u|^2 d\xi = \min \operatorname{Re}(\psi_{\mathcal{I}}) \int_{\mathbb{R}} u^2 dx,$$

thanks to Plancherel's equality. Equation (5.15) then implies that

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u^2 dx \leq \omega_0 \int_{\mathbb{R}} u^2 dx,$$

and by Gronwall's lemma, we deduce that, for all $t \in [0,T]$,

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} \leq e^{\omega_0 t}\|u_0\|_{L^2(\mathbb{R})}. \tag{5.16}$$

Conclusion. With this estimate, we can prove the global-in-time existence. Define

$$t_0 := \sup\{t>0 : \text{ there exists a (unique) mild solution } u \text{ to (1.1) on } (0,t) \text{ that satisfies the regularity of Proposition 9}\}.$$

We have to prove that $t_0 \geq T$. To do so, we assume the contrary $t_0 < T$ and we seek a contradiction. Let us first verify that $t_0 > 0$. By Proposition 7, Equation (1.1) admits a local-in-time mild solution $u \in C([0,T_*];L^2(\mathbb{R})) \cap C((0,T_*];H^2(\mathbb{R}))$ on $(0,T_*)$. Applying Proposition 8 with $t_1 \in (0,T_*)$ as the initial time, we see that $u \in C^{1,2}((t_1,T_*]\times\mathbb{R})$ for all $t_1 \in (0,T_*)$. Finally, we infer that u has the following regularity: $u \in C([0,T_*];L^2(\mathbb{R})) \cap C((0,T_*];H^2(\mathbb{R})) \cap C^{1,2}((0,T_*]\times\mathbb{R})$, which implies that $t_0 \geq T_* > 0$. Using again Propositions 7 and 8, there exists $T'_* > 0$ such that, for any initial data v_0 that satisfy

$$\|v_0\|_{L^2(\mathbb{R})} \leq e^{\omega_0 t_0}\|u_0\|_{L^2(\mathbb{R})}, \tag{5.17}$$

Equation (1.1) admits a regular mild solution v on $(0,T'_*)$ with initial datum v_0. Since (5.16) implies that $v_0 := u(t_0 - T'_*/2)$ satisfies (5.17), we infer that (1.1) admits such a solution v. Using the uniqueness and the semigroup property, we see that $u(t_0 - T'_*/2 + t,\cdot) = v(t,\cdot)$ for all $t \in [0,T'_*/2)$. Finally, define $\widetilde{u}$ by $\widetilde{u} = u$ on $[0,t_0)$ and $\widetilde{u}(t_0 - T'_*/2 + t,\cdot) = v(t,\cdot)$ for $t \in [T'_*/2, T'_*]$. Then, $\widetilde{u}$ is still a mild solution to (1.1) that satisfies the regularity of Proposition 9. Since the solution $\widetilde{u}$ lives on $[0, t_0 + T'_*/2]$, this gives us a contradiction. We conclude that $t_0 \geq T$ and this completes the proof of the global existence of a $C^{1,2}$-solution. □

For the sake of completeness, we give the main ideas to get further regularity. The proof follows closely the one given in [4].

Sketch of the proof of the C^∞-regularity. We use a bootstrap method. Let us first introduce some notation. For $n \in \mathbb{N}$, $\partial^n_{x^n} u$ denotes the spatial partial derivatives of u of order n. We also say that $u \in C^{n,\infty}((0,T]\times\mathbb{R})$ if all the mixed derivatives of u up to order n in time are continuous on $(0,T]\times\mathbb{R}$. The proof follows in three steps. First, we prove the spatial regularity at any order and the degree of regularity in time in order to differentiate the equations with respect to t. Finally, we conclude by induction.

First step: C^1-regularity of the spatial derivatives. Up to this point, we know that

$$u \in C((0,T]; H^2(\mathbb{R})) \cap C^{1,2}((0,T] \times \mathbb{R}).$$

Differentiating Duhamel's formula with respect to x, we proved in Proposition 7 that $\partial_x u$ is a mild solution of Equation (1.1) differentiated with respect to to x:

$$\partial_x u(t_0+t,\cdot) = K(t,\cdot) * \partial_x u(t_0,\cdot) - \int_0^t \partial_x K(t-s,\cdot) * (u\,\partial_x u)(t_0+s,\cdot)ds.$$

Moreover, taking any $t_0 \in (0,T]$ as the initial time, we know that the initial datum $\partial_x u(t_0,\cdot) \in L^2(\mathbb{R})$; hence, we argue as in Propositions 7, 8 and 9 to prove that $\partial_x u \in C((t_0,T]; H^2(\mathbb{R})) \cap C^{1,2}((t_0,T]\times\mathbb{R})$. Since t_0 is arbitrary in $(0,T]$, we deduce that $\partial_x u \in C((0,T]; H^2(\mathbb{R})) \cap C^{1,2}((0,T] \times \mathbb{R})$. We can do the same for the second derivative $\partial^2_{xx} u$ which is a mild solution of Eq. (1.1) differentiated twice, see Equation (5.14). By induction, we extend this to all the spatial derivatives of u: for all $n \in \mathbb{N}$, $\quad \partial^n_{x^n} u \in C((0,T]; L^2(\mathbb{R})) \cap C^1((0,T] \times \mathbb{R})$.

Second step: $C^{2,\infty}$-regularity. Let us pay attention to the regularity of the non-local term. Let us recall that the second item of Remark 3 implies that $\mathcal{I}[u] \in C((0,T]; L^2(\mathbb{R})) \cap C((0,T] \times \mathbb{R})$. Moreover, we have

$$\partial_x \mathcal{I}[u] = \mathcal{I}[\partial_x u]. \tag{5.18}$$

Indeed, observe that for $\varphi \in H^3(\mathbb{R})$, (2.2) and a classical formula on Fourier transforms give

$$\mathcal{F}(\partial_x \mathcal{I}[\varphi]) = 2i\pi \cdot \mathcal{F}(\mathcal{I}[\varphi]) = 2i\pi \cdot \psi\mathcal{F}\varphi = \psi\mathcal{F}(\partial_x\varphi),$$

where ψ is given by (2.3). This implies that $\partial_x\mathcal{I}[\varphi] = \mathcal{I}[\partial_x\varphi]$. Since the first step implies, in particular, that u is continuous in time with values in H^3, we deduce that (5.18) holds true. Using again the second item of Remark 3, it follows that $\partial_x\mathcal{I}[u] \in C((0,T]; L^2(\mathbb{R})) \cap C((0,T] \times \mathbb{R})$. Since, from the first step, $u \in C((0,T];H^n(\mathbb{R}))$, for all $n \in \mathbb{N}$, by induction we get the same result for the spatial derivatives of $\mathcal{I}[u]$ of any order:

$$\forall n \in \mathbb{N}, \quad \partial^n_{x^n}\mathcal{I}[u] \in C((0,T];L^2(\mathbb{R})) \cap C((0,T] \times \mathbb{R}).$$

Since $\partial_t u = -\partial_x\big(\frac{u^2}{2}\big) - \mathcal{I}[u] + \partial^2_{xx}u$ holds in the classical sense by Proposition 8 we deduce that, for all $n \in \mathbb{N}$, $\quad \partial^n_{x^n}\partial_t u \in C((0,T];L^2(\mathbb{R})) \cap C((0,T] \times \mathbb{R})$. Now, we show that

$$\partial_t\mathcal{I}[u] = \mathcal{I}[\partial_t u] \in C((0,T] \times \mathbb{R});$$

indeed, $\partial_t u$ is sufficiently regular in order to apply the theorems of continuity and differentiation under the integral sign to Formula (2.1). Arguing by induction on the the spatial derivatives of u, we see that

$$\forall n \in \mathbb{N}, \quad \partial_t\partial^n_{x^n}\mathcal{I}[u] = \partial_t\mathcal{I}[\partial^n_{x^n}u] = \mathcal{I}[\partial_t\partial^n_{x^n}u] \in C((0,T] \times \mathbb{R}).$$

We deduce that

$$\forall n \in \mathbb{N}, \quad \partial^n_{x^n}\mathcal{I}[u] \in C((0,T];L^2(\mathbb{R})) \cap C^1((0,T] \times \mathbb{R}).$$

Using again the fact that Equation (1.1) holds in the classical sense, we get

$$\forall n \in \mathbb{N}, \quad \partial^n_{x^n}\partial_t u \in C((0,T];L^2(\mathbb{R})) \cap C^1((0,T] \times \mathbb{R}).$$

Moreover, we have seen in the first step that $\partial_x u \in C^{1,2}((0,T] \times \mathbb{R})$; hence, the following equation holds true in the classical sense:

$$\partial_t\partial_x u + (\partial_x u)^2 + u\partial^2_{xx}u + \mathcal{I}[\partial_x u] - \partial^3_{xxx}u = 0.$$

Performing the second step with u replaced by $\partial_x u$, we obtain

$$\forall n \in \mathbb{N}, \quad \partial^n_{x^n}\partial_t\partial_x u \in C((0,T];L^2(\mathbb{R})) \cap C^1((0,T] \times \mathbb{R}).$$

We then can iterate this for all the spatial derivatives of u and we get finally the following regularity:

$$\forall n,m \in \mathbb{N}, \quad \partial^n_{x^n}\partial_t\partial^m_{x^m}u \in C((0,T];L^2(\mathbb{R})) \cap C^1((0,T] \times \mathbb{R}).$$

Hence, $u \in C^{2,\infty}((0,T] \times \mathbb{R})$ with all mixed derivatives up to order 1 in time continuous with values in L^2.

Conclusion. Now, we can differentiate Equation (1.1) with respect to t and argue as previously to prove that $u \in C^{3,\infty}((0,T] \times \mathbb{R})$ with all mixed derivatives up to order 2 in time continuous with values in L^2. Arguing by induction, we conclude that u is $C^\infty((0,T] \times \mathbb{R})$ with all derivatives continuous with values in L^2. This is the desired regularity in Theorem 1. □

To finish, we prove the L^2-stability stated in Proposition 1.

Proof of Proposition 1 . Let (u,v) be solutions to (1.1) with respective L^2 initial data (u_0,v_0), let $T>0$ and $t\in[0,T]$. Subtracting

$$u(t,\cdot)=K(t,\cdot)*u_0-\frac{1}{2}\int_0^t \partial_x K(t-s,\cdot)*u^2(s,\cdot)ds,$$

and

$$v(t,.)=K(t,\cdot)*v_0-\frac{1}{2}\int_0^t \partial_x K(t-s,\cdot)*v^2(s,\cdot)ds,$$

we get

$$u(t,\cdot)-v(t,\cdot)=K(t,\cdot)*(u_0-v_0)-\frac{1}{2}\int_0^t \partial_x K(t-s,\cdot)*(u^2(s,\cdot)-v^2(s,\cdot))ds. \tag{5.19}$$

Hence, by (2.10) of Remark 4 and Young's inequality

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^2(\mathbb{R})}$$
$$\leq e^{\omega_0 T}\|u_0-v_0\|_{L^2(\mathbb{R})}+\frac{1}{2}\int_0^t \|\partial_x K(t-s,\cdot)\|_{L^2(\mathbb{R})}\|u^2(s,\cdot)-v^2(s,\cdot)\|_{L^1(\mathbb{R})}ds.$$

Taking

$$\begin{aligned} M&=\max\Big(\|u\|_{C([0,T];L^2(\mathbb{R}))},\|v\|_{C([0,T];L^2(\mathbb{R}))}\Big)\\ &\leq e^{\omega_0 T}\max\Big(\|u_0\|_{L^2(\mathbb{R})},\|v_0\|_{L^2(\mathbb{R})}\Big),\ \text{by (5.16)},\end{aligned}$$

we can bound

$$\begin{aligned}&\|u(t,\cdot)-v(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\leq e^{\omega_0 T}\|u_0-v_0\|_{L^2(\mathbb{R})}+M\int_0^t \|\partial_x K(t-s,\cdot)\|_{L^2(\mathbb{R})}\|u(s,\cdot)-v(s,\cdot)\|_{L^2(\mathbb{R})}\,ds\\ &\leq e^{\omega_0 T}\|u_0-v_0\|_{L^2(\mathbb{R})}+M\mathcal{K}_0\int_0^t (t-s)^{-\frac{3}{4}}\|u(s,\cdot)-v(s,\cdot)\|_{L^2(\mathbb{R})}\,ds,\end{aligned}$$

thanks to (2.6). With Lemma 3 in Appendix, the proof is finished. □

6. Failure of the maximum principle

We now investigate the proof of Theorem 2, which is an immediate consequence of the integral formula (2.1).

Proof of Theorem 2. Propositions 8 and 2 imply that the solution u to (1.1) is $C^{1,2}$ up to time $t=0$ and that

$$u_t(0,x_*)+u_0(x_*)u_0'(x_*)+C_{\mathcal{I}}\int_{-\infty}^0 \frac{u_0(x_*+z)-u_0(x_*)-u_0'(x_*)z}{|z|^{7/3}}dz-u_0''(x_*)=0.$$

It follows that

$$u_t(0,x_*) = -C_{\mathcal{I}} \int_{-\infty}^{0} \frac{u_0(x_*+z)}{|z|^{7/3}}\, dz < 0.$$

There then exists $t_* > 0$ such that $u(t_*, x_*) < 0$. The proof of Theorem 2 is now complete. □

7. Numerical simulations

The aim of this part is to show some numerical simulations for (1.1). An explicit discretization gives results in line with the theoretical study (see Remark 2).

We write (1.1) with a viscous coefficient $\varepsilon > 0$ as follows:

$$\partial_t u + \partial_x\big(\tfrac{1}{2}u^2 + \mathcal{L}[u]\big) - \varepsilon \partial^2_{xx} u = 0, \tag{7.1}$$

where, for any $\varphi \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\mathcal{L}[\varphi](x) := \int_0^{+\infty} |\zeta|^{-\frac{1}{3}} \varphi'(x-\zeta) d\zeta.$$

The viscous coefficient is taken sufficiently small, in order to magnify the erosive effect of the non-local term. The new definition of the non-local term ($\mathcal{I}[u] = \partial_x \mathcal{L}[u]$) follows [6], which interprets $\mathcal{L}[u]$ as a flow. Notice that in [6, 7], the bottom is, in fact, $s(t,x) = u(t, x + q'(1)t)$, where q is the bedload transport of sediments; for the sake of simplicity, we continue to work with u.

To shed light on the effect of the nonlocal term, we compare the evolution of the solution of (7.1) with the solution of the viscous Burgers equation:

$$\partial_t u + \partial_x\big(\tfrac{1}{2}u^2\big) - \varepsilon \partial^2_{xx} u = 0. \tag{7.2}$$

7.1. **Maximum principle for the viscous Burgers equation.** It is well known that (7.2) satisfies the maximum principle: for any initial data $u_0 \in L^\infty(\mathbb{R})$, ess-inf $u_0 \le u \le$ ess-sup u_0. As a consequence, (7.2) cannot take into account erosion phenomena. To simulate the evolution of u, we define a regular discretization of $[0, L]$ with a spatial step Δx such that $L = M\Delta x$, and a discretization of $[0,T]$ with a time step Δt such that $T = N\Delta t$. We let x_i, t_n and u_i^n respectively denote the point $i\Delta x$, the time $n\Delta t$ and the computed solution at the point $(n\Delta t, i\Delta x)$. We use the following explicit centered scheme:

$$u_i^{n+1} = u_i^n + \Delta t\Big[-\frac{1}{2}\frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{2\Delta x} + \varepsilon \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\Big]. \tag{7.3}$$

It is well known that this scheme is stable under the CFL-Peclet condition

$$\Delta t \leq \min\Big(\frac{\Delta x}{\max_i |u_i^n|}, \frac{\Delta x^2}{2\varepsilon}\Big). \tag{7.4}$$

To convince the reader, let us simulate the evolution of the well-known following travelling waves of (7.2) for $\varepsilon = 1$

$$u(t,x) := \tfrac{1}{2}\big[1 - \tanh\big(\tfrac{1}{4}(x - \tfrac{1}{2}t)\big)\big].$$

Remark 7. Equation (1.1) also admits travelling wave solutions, see [1].

We expose in Figure 3 both analytic and numerical solutions. We observe an error of the order of 10^{-4} between these solutions.

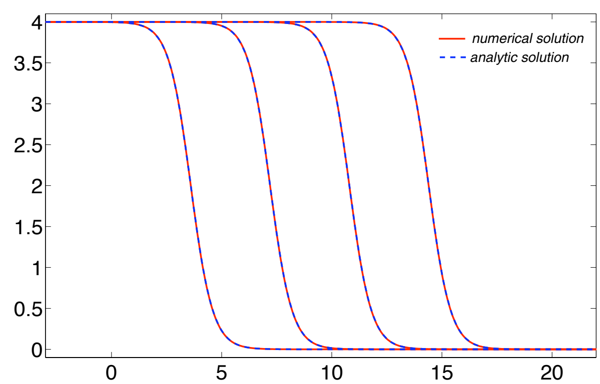

FIGURE 3. Numerical and analytic travelling waves of the viscous Burgers equation.

Let us now take, as an initial dune, the following small regular perturbation on the bottom:

$$u_0(x) = \begin{cases} e^{\frac{-1}{1-(x-\frac{L}{2})^2}} & \text{if } \frac{L}{2} - 1 < x < \frac{L}{2} + 1. \\ 0 & \text{otherwise.} \end{cases} \tag{7.5}$$

We describe its evolution in Figure 4. The dune propagates, but as mentioned above the erosion phenomena are not taken into account since u remains positive (because of the maximum principle).

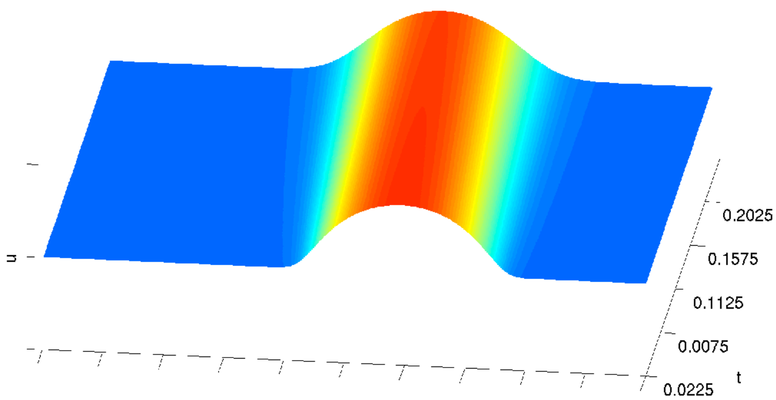

FIGURE 4. Evolution of the solution of (7.2) with u_0 defined in (7.5) ($L = 30$, $M = 4001$ and $\varepsilon = 0.1$).

7.2. **Erosive effect of the nonlocal term.** Let us return to the study of (7.1). We add the discretization of the non-local operator $\mathcal{L}$ to the explicit centered scheme (7.3). It is natural to consider the following discretization:

$$\mathcal{L}[u_i^n] \approx \sum_{j=1}^{+\infty} |j\Delta x|^{-\frac{1}{3}} \frac{u_{i-j+1}^n - u_{i-j-1}^n}{2\Delta x}. \tag{7.6}$$

Then we obtain the discretization for (7.1)

$$u_i^{n+1} = u_i^n + \Delta t \Big[-\frac{1}{2}\frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{2\Delta x} - \frac{\mathcal{L}[u_{i+1}^n] - \mathcal{L}[u_{i-1}^n]}{2\Delta x} + \varepsilon \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \Big]. \tag{7.7}$$

Remark 8. Taking into account the explicit discretization (7.6), we see that $\mathcal{L}[u_i^n]$ depends only on u_j^n for $j \le i$, therefore, in (7.7), u_i^{n+1} does not depend on u_k^n for $k \ge i+2$:

$$u_i^{n+1} = u_i^n + \Delta t\, F(u_{i+1}^n, u_i^n, u_{i-1}^n, \dots, u_{i-j}^n, \dots).$$

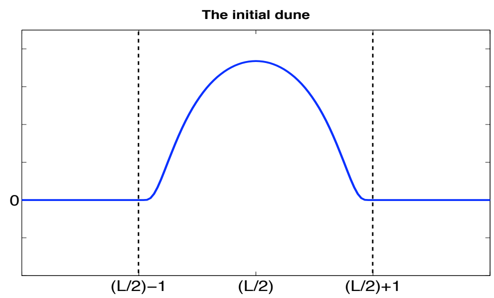

FIGURE 5. The initial dune defined in (7.5).

Now, take any integer $A \geq 1$ and assume that $u_0(x) = 0$, for all $x \leq A\Delta x$; then by induction we have $u_i^n = 0$, for all $i \leq (A - n)$, therefore $\mathcal{L}$ can be computed *using only a finite number of terms*:

$$\mathcal{L}[u_i^n] = \begin{cases} 0 & \text{if } i < A - n \\ \sum_{j=1}^{i+n-A} |j\Delta x|^{-\frac{1}{3}} \frac{u_{i-j+1}^n - u_{i-j-1}^n}{2} & \text{otherwise.} \end{cases} \tag{7.8}$$

To take advantage of the previous remark, we take again the initial datum u_0 defined in (7.5), which satisfies $\mathrm{supp}(u_0) \subset\subset (0, L)$ (see Figure 5), and we use the explicit scheme (7.7) and (7.8). Concerning the stability condition, one can numerically see that (7.4) is still ensuring stability for small Δx. The evolution of the initial dune (7.5) is given in Picture 6. Like the solutions of the viscous Burgers equation, the dune is propagated downstream but we now observe an erosive process behind the dune: the bottom is eroded downstream from the dune, as shown in Remark 2.

Let us make a final remark. We are aware of the fact that these numerical simulations are a first crude attempt. To tackle rigorously the non-local term would need further study, which will be reported elsewhere; see also the recent work [3] for a rigorous numerical study in the monotone case.

APPENDIX A. SOME TECHNICAL LEMMAS.

We first recall a classical generalization of Gronwall's lemma proved e.g. in [5].

Lemma 3. *Let $g : [0, T] \to \mathbb{R}_+$ be a measurable function and suppose that there are positive constants C, A and $\theta > 0$ such that, for all $t \leq T$,*

$$g(t) \leq A + C \int_0^t (t - s)^{\theta - 1} g(s)\, ds.$$

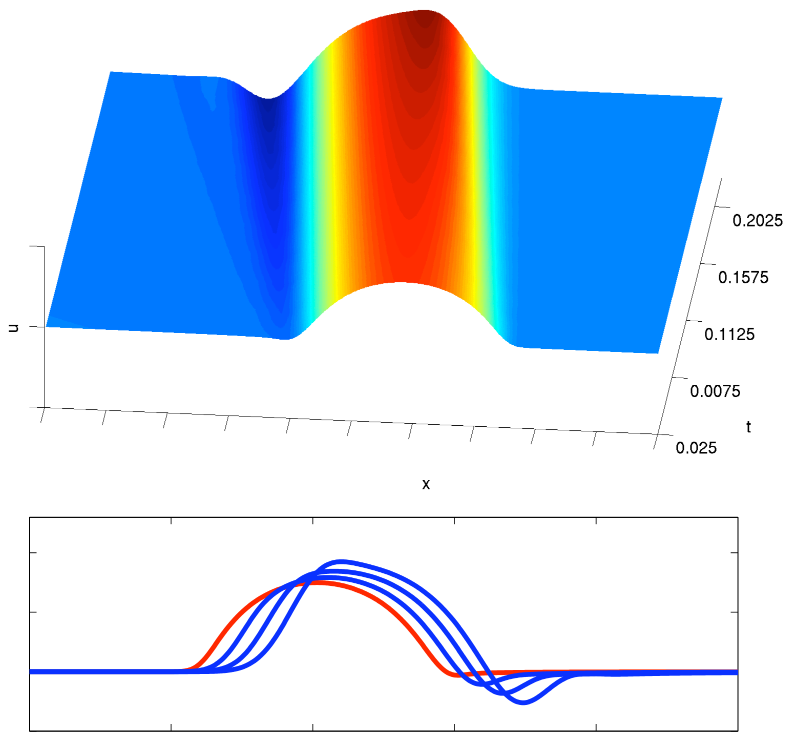

FIGURE 6. Evolution of an initial dune, by using the non-local model (7.1) ($L = 30$, $M = 4001$ and $\varepsilon = 0.1$).

Then,

$$\sup_{0 \leq t \leq T} g(t) \leq C_T A,$$

where the constant C_T does not depend on A.

Lemma 4. *Let $f, g \in L^2(\mathbb{R})$. Then, $f * g \in C(\mathbb{R})$ and, for all $x, y \in \mathbb{R}$,*

$$|f * g(x) - f * g(y)| \leq \|\mathcal{T}_{(x-y)} f - f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}.$$

Proof. The result is immediate if f and g are smooth; indeed,

$$|f * g(x) - f * g(y)| = \left| \int_{\mathbb{R}} f(x-z) g(z) dz - \int_{\mathbb{R}} f(y-z) g(z) dz \right|$$

$$\leq \int_{\mathbb{R}} |f(x-z) - f(y-z)g(z)|dz$$
$$\leq \|\mathcal{T}_{(x-y)}f - f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}.$$

The result for general f and g only L^2 is then obtained by density. □

Lemma 5. *Let $u \in C([0,T]; L^2(\mathbb{R}))$. Then,*

$$\sup_{t\in[0,T-h]} \|\mathcal{T}_h(u(t,\cdot)) - u(t,\cdot)\|_{L^2(\mathbb{R})} \to 0, \quad \text{as } h \to 0.$$

Proof. The function u is uniformly continuous with values in L^2 as a continuous function on a compact set $[0,T]$. For any $\varepsilon > 0$, there then exists a finite sequence $0 = t_0 < t_1 < \ldots < t_N = T$ such that, for any $t \in [0,T]$, there exists $j \in \{0, \ldots, N-1\}$ with

$$\|u(t,\cdot) - u(t_j,\cdot)\|_{L^2(\mathbb{R})} \leq \varepsilon.$$

Moreover,

$$\|\mathcal{T}_h(u(t,\cdot)) - u(t,\cdot)\|_{L^2(\mathbb{R})} \leq \|\mathcal{T}_h(u(t,\cdot)) - \mathcal{T}_h(u(t_j,\cdot))\|_{L^2(\mathbb{R})}$$
$$+ \|\mathcal{T}_h(u(t_j,\cdot)) - u(t_j,\cdot)\|_{L^2(\mathbb{R})} + \|u(t_j,\cdot) - u(t,\cdot)\|_{L^2(\mathbb{R})}.$$

Since $\|\mathcal{T}_h(u(t,\cdot)) - \mathcal{T}_h(u(t_j,\cdot))\|_{L^2(\mathbb{R})} = \|u(t,\cdot) - u(t_j,\cdot)\|_{L^2(\mathbb{R})}$, we get

$$\|\mathcal{T}_h(u(t,\cdot)) - u(t,\cdot)\|_{L^2(\mathbb{R})}$$
$$\leq \|\mathcal{T}_h(u(t_j,\cdot)) - u(t_j,\cdot)\|_{L^2(\mathbb{R})} + 2\|u(t_j,\cdot) - u(t,\cdot)\|_{L^2(\mathbb{R})}$$
$$\leq \|\mathcal{T}_h(u(t_j,\cdot)) - u(t_j,\cdot)\|_{L^2(\mathbb{R})} + 2\varepsilon.$$

By the continuity of translation in $L^2(\mathbb{R})$, $\|\mathcal{T}_h(u(t_j,\cdot)) - u(t_j,\cdot)\|_{L^2(\mathbb{R})} \to 0$, as $h \to 0$. Thus,

$$\limsup_{h\to 0} \|\mathcal{T}_h(u(t,\cdot)) - u(t,\cdot)\|_{L^2(\mathbb{R})} \leq 2\varepsilon.$$

Taking the infimum with respect to $\varepsilon > 0$ implies the result. □

Lemma 6. *Let $u \in C([0,T]; L^2(\mathbb{R}))$ such that u is continuous with respect to the variable x uniformly in t. Then, $u \in C([0,T] \times \mathbb{R})$.*

Proof. Let $(t_0, x_0) \in [0,T] \times \mathbb{R}$. Let $\varepsilon > 0$. By the regularity of u with respect to the space variable, we know that there exists $\eta > 0$ such that, for any $t \in [0,T]$ and all $x, y \in [x_0 - \eta, x_0 + \eta]$,

$$|u(t_0, x_0) - u(t,x)|$$
$$\leq |u(t_0,x_0) - u(t_0,y)| + |u(t_0,y) - u(t,y)| + |u(t,y) - u(t,x)|$$
$$\leq \varepsilon + |u(t_0,y) - u(t,y)| + \varepsilon.$$

If we integrate with respect to $y \in [x_0 - \eta, x_0 + \eta]$, then we get

$$2\eta|u(t_0, x_0) - u(t, x)| \leq 4\varepsilon\eta + \int_{x_0-\eta}^{x_0+\eta} |u(t_0, y) - u(t, y)|dy$$
$$\leq 4\varepsilon\eta(2\eta)^{\frac{1}{2}} \|u(t_0, \cdot) - u(t, \cdot)\|_{L^2(\mathbb{R})}.$$

By the continuity of u with values in L^2,

$$\limsup_{(t,x)\to(t_0,x_0)} |u(t_0, x_0) - u(t, x)| \leq 2\varepsilon.$$

Taking the infimum with respect to $\varepsilon > 0$ completes the proof. □

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