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EXISTENCE OF RADIAL SOLUTIONS FOR A CLASS OF $p(x)$-LAPLACIAN EQUATIONS WITH CRITICAL GROWTH

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Abstract. Using variational methods we establish the existence of solutions for the following class of $p(x)$-Laplacian equations:

$$-div(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\lambda|u|^{q(x)-2}u+|u|^{p^*(x)-2}u, \quad \mathbb{R}^N, \qquad (P)$$

where $\lambda \in (0,\infty)$ is a parameter and $p(x), q(x) : \mathbb{R}^N \to \mathbb{R}$ are radial continuous functions satisfying $1 < p(x) < N$ and $p(x) << q(x) << p^*(x) = \frac{p(x)N}{N-p(x)}$.

1. Introduction

In this paper, we investigate the existence of solutions for the following class of $p(x)$-Laplacian equations:

$$(P) \quad \begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\lambda|u|^{q(x)-2}u+|u|^{p^*(x)-2}u, \ \mathbb{R}^N \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

where $\lambda \in (0,\infty)$ is a parameter and $p(x), q(x) : \mathbb{R}^N \to \mathbb{R}$ are continuous functions satisfying the following:

($\mathbf{H_1}$) $p(x)$ and $q(x)$ are radially symmetric; that is,

$$p(x) = p(r) \ \text{ and } \ q(x) = q(r) \ \text{ with } r = \|x\|,$$

where $\| \ \|$ denotes the usual norm in $\mathbb{R}^N$.

($\mathbf{H_2}$) $1 < p_- = \inf_{\mathbb{R}^N} p(x) \le p(x) \le p^+ = \sup_{\mathbb{R}^N} p(x) < N$

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($\mathbf{H_3}$) $1 < q_- = \inf_{\mathbb{R}^N} q(x) \leq q(x) \leq q^+ = \sup_{\mathbb{R}^N} q(x), p^+ < q_-$ and

$$p(x) << q(x) << p^*(x) = \frac{Np(x)}{N - p(x)}.$$

The notation $h << g$ means that $\inf_{\mathbb{R}^N}\{g(x) - h(x)\} > 0$.

Equations with variable exponents appear in various mathematical models, for example:

Electrorheological fluids: see Acerbi & Mingione [1, 2], Antontsev & Rodrigues [7] and Ruzicka [25],

Nonlinear Darcy law in porous medium: see Antontev & Shmarev [8, 10],

Image Processing: see Chambolle & Lions [11] and Chen, Levine & Rao [13].

As a consequence of the presence of variable exponents in the applications above and after Kovacik & Rákosník discussed some properties of the $L^{p(x)}$ spaces and $W^{1,p(x)}$ spaces in [22], a lot of research has been done concerning these kinds of problems, see, for example, Alves [4], Alves & Souto [3], Fan [18], Fu [20], El Hamidi [16], Chabrowski & Fu [12], Antontev, Chipot & Xie [9], Mihailescu, Pucci & Radulescu [23] and references therein.

In [20], Fu has established a principle of concentration compactness in $L^{p(x)}$ spaces. Using this principle, he proves the existence of solutions for a class of $p(x)$-Laplacian equations of the type

$$(P_1) \qquad \begin{cases} -div(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u & \\ \qquad = c(x)|u|^{p^*(x)-2}u + \lambda f(x,u), & \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary.

For the case that $p(x)$ and $q(x)$ are constants, problem (P) has seen considered by Alves, do Ó & Miyagaki [5]. In that paper, the authors used a result due to Lions [21] and showed the existence of solutions for all $\lambda > 0$. An important point in that work is the fact that the continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ has a best Sobolev constant, denoted by S, which is assumed by a special class of functions. When the functions $p(x)$ and $q(x)$ are not constant, we do not have this information, and thus new arguments and estimates are necessary.

The main proposal of this paper is to overcome these difficulties when $p(x)$ and $q(x)$ are variable. We show that, in some cases where the exponents are radial functions, it is not necessary to use the principle of concentration compactness of the type Lions found in [20]. Moreover, our main result enlarges the study made in [5], in the sense that we are working with variable exponents. The main tool used here is the variational method, more precisely, the mountain pass theorem due to Ambrosetti & Rabibowitz [6] and a result of Strauss-Lions type for the space $W_r^{1,p(x)}(\mathbb{R}^N)$ due to X. Fan, Y. Zhao & D. Zhao [18].

In all this work, K denotes a positive constant associated to the continuous embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$, in the sense that $K > 1$ and

$$\|u\|_{p^*(x)} \leq K\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} \ \ \forall u \in W^{1,p(x)}(\mathbb{R}^N). \tag{1.1}$$

Our main result is the following.

Theorem 1.1. *There exists $\lambda^* > 0$ such that (P) has a solution for all $\lambda > \lambda^*$.*

This paper is organized in the following way: In Section 2, we introduce the variational formulation associated to the problem. In Section 3, we show the existence of solutions to (P), and in Section 4 we observe that the arguments developed in this paper can be used to get solutions for a class of $p(x)$-Laplacian equations with critical growth and a coercive potential, completing this way the study done by Miyagaki [24].

2. Variational Framework

In all this paper, the Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)},$$

where

$$\|\nabla u\|_{p(x)} = \inf\Big\{\alpha > 0 : \int_{\mathbb{R}^N} \Big|\frac{\nabla u}{\alpha}\Big|^{p(x)} dx \leq 1\Big\},$$

and

$$\|u\|_{p(x)} = \inf\Big\{\alpha > 0 : \int_{\mathbb{R}^N} \Big|\frac{u}{\alpha}\Big|^{p(x)} dx \leq 1\Big\}.$$

Moreover, let us denote by $I : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ the energy functional related to (P) given by

$$I(u) = \int_{\mathbb{R}^N} \Big[\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|u|^{p(x)}}{p(x)}\Big] dx - \lambda \int_{\mathbb{R}^N} \frac{|u|^{q(x)}}{q(x)} dx - \int_{\mathbb{R}^N} \frac{|u|^{p^*(x)}}{p^*(x)} dx.$$

Since we do not have a compact embedding involving $W^{1,p(x)}(\mathbb{R}^N)$ and $L^{q(x)}(\mathbb{R}^N)$, we shall restrict I to the subspace $W_r^{1,p(x)}(\mathbb{R}^N)$ (subspace of the radial functions), which has a compact embedding in $L^{q(x)}(\mathbb{R}^N)$ (see [18]), therefore, in the next section, we shall work with $I : W_r^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$.

Using standard arguments, it is possible to prove that the functional I satisfies the well-known mountain pass geometry (see [6]), and thus, there exists a Palais-Smale sequence $\{u_n\} \subset W_r^{1,p(x)}(\mathbb{R}^N)$ satisfying

$$I(u_n) \to d_\lambda \ \text{ and } \ I'(u_n) \to 0, \tag{2.1}$$

where $d_\lambda = \inf_{g\in\Gamma} \max_{t\in[0,1]} I(g(t)) > 0$ and

$$\Gamma = \left\{ g \in C([0,1], W_r^{1,p(x)}(\mathbb{R}^N)) : g(0) = 0 \text{ and } I(g(1)) \le 0 \right\}.$$

Since I is invariant under rotation, using a symmetric critical principle of Palais type in Banach spaces developed by de Morais Filho, do Ó & Souto [15], we have that a critical point of I restricted to $W_r^{1,p(x)}(\mathbb{R}^N)$ is a critical point in $W^{1,p(x)}(\mathbb{R}^N)$.

3. Existence of Solution

In order to show the existence of a solution to (P), we commence studying the almost everywhere convergence of $\{\nabla u_n\}$ in $\mathbb{R}^N$. Using well-known arguments, the Palais-Smale sequence $\{u_n\}$ given by mountain pass geometry is bounded in $W_r^{1,p(x)}(\mathbb{R}^N)$ (see [3]). Thus, using [18], for some subsequence, still denoted by $\{u_n\}$, there exists $u \in W_r^{1,p(x)}(\mathbb{R}^N)$ such that

- $u_n \rightharpoonup u$ in $W_r^{1,p(x)}(\mathbb{R}^N)$
- $u_n(x) \to u(x)$ a.e in $\mathbb{R}^N$
- $u_n \to u$ in $L^{q(x)}(\mathbb{R}^N)$
- $u_n \to u$ in $L_{loc}^{p^*(x)}(\mathbb{R}^N \setminus \{0\})$.

The above limits imply that $u_n \to u$ in $W^{1,p(x)}(1/R < |x| < R)$ for all $R > 0$. Hence, for some subsequence $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in $A_R = \{x \in \mathbb{R}^N : 1/R < |x| < R\}$ from which it follows, up to a subsequence if necessary, that $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in $\mathbb{R}^N$.

From the above commentaries, it is a routine calculation to show the next lemma.

Lemma 3.1. *The weak limit u of the sequence $\{u_n\}$ is a critical point of I; that is, $I'(u) = 0$ and thus u is a solution to (P).*

Now, our goal is to prove that the weak limit u is not trivial; to this end, firstly, we study the behavior of d_λ in relation to λ.

Lemma 3.2. *The level d_λ goes to zero as λ goes to infinity; that is,*

$$\lim_{\lambda\to\infty} d_\lambda = 0.$$

Proof. For fixed $\Psi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, there exists $t_\lambda > 0$ satisfying

$$I(t_\lambda \Psi) = \max_{t\geq 0} I(t\Psi),$$

and

$$\int_{\mathbb{R}^N} t_\lambda^{p(x)}[|\nabla\Psi|^{p(x)} + |\Psi|^{p(x)}]dx = \lambda \int_{\mathbb{R}^N} t_\lambda^{q(x)}|\Psi|^{q(x)}dx + \int_{\mathbb{R}^N} t_\lambda^{p^*(x)}|\Psi|^{p^*(x)}dx.$$

This equality in conjunction with $(H_1) - (H_3)$ yields $t_\lambda \to 0$ as $\lambda \to \infty$. Since

$$0 \leq d_\lambda \leq I(t_\lambda\Psi) \leq t_\lambda^{p_-} \int_{\mathbb{R}^N} [|\nabla\Psi|^{p(x)} + |\Psi|^{p(x)}]dx,$$

it follows that $d_\lambda \to 0$ as $\lambda \to \infty$. □

Corollary 3.1. *There exists $\lambda^* > 0$ such that*

$$d_\lambda < \Theta \left(\tfrac{1}{K}\right)^{1/\Theta} \quad \forall \lambda \geq \lambda^*,$$

where $\Theta = \frac{1}{p^+} - \frac{1}{p^*_-}$.

Proof. This is an immediate consequence of Lemma 3.2.

Lemma 3.3. *There are $\lambda^* > 0$ and $n_0 = n_0(\lambda) \in \mathbb{N}$ such that*

$$\|u_n\|_{p^*(x)}, \|u_n\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq 1 \;\; \forall \lambda \geq \lambda^* \;\; \textit{and} \;\; n \geq n_0.$$

Proof. From Lemma 3.2, there exists $\lambda^* > 0$ such that

$$d_\lambda \leq \frac{1}{4K^{p^+}}\left(\frac{q_- - p^+}{q_- p^+}\right) \;\; \forall \lambda \geq \lambda^*.$$

Fixing $\lambda > \lambda^*$, the equality

$$I(u_n) - \frac{1}{q_-} I'(u_n)u_n = d_\lambda + o_n(1)$$

implies

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \left[|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right] dx \leq \left(\frac{p^+ q_-}{q_- - p^+}\right) d_\lambda \leq \frac{1}{4K^{p^+}}.$$

Therefore, there exists $n_0 = n_0(\lambda) \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} \left[|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right] dx \leq \frac{1}{K^{p^+}} \;\; \forall n \geq n_0,$$

showing that

$$\|u_n\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq \frac{1}{K} < 1 \quad n \geq n_0. \tag{3.1}$$

Combining (1.1) and (3.1), we get

$$\|u_n\|_{p^*(x)} \leq 1 \quad n \geq n_0, \tag{3.2}$$

finishing the proof. □

Corollary 3.2. *There are* $\lambda^* > 0$ *and* $n_0 = n_0(\lambda) \in \mathbb{N}$ *such that*

$$\Big(\int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx\Big)^{\frac{1}{p^*_-}} \leq K\Big(\int_{\mathbb{R}^N} \Big[|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big]\, dx\Big)^{\frac{1}{p^+}} \quad \forall n \geq n_0.$$

Proof. From Lemma 3.3, it follows that

$$\Big(\int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx\Big)^{\frac{1}{p^*_-}} \leq \|u_n\|_{p^*(x)} \quad \forall n \geq n_0, \tag{3.3}$$

and

$$\|u_n\|_{W^{1,p(x)}(\mathbb{R}^N)} \leq \Big(\int_{\mathbb{R}^N} \Big[|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big]\, dx\Big)^{\frac{1}{p^+}} \quad \forall n \geq n_0. \tag{3.4}$$

Thus, the corollary follows combining (1.1), (3.3) and (3.4). □

Proposition 3.1. *For* $\lambda > \lambda^*$*, the weak limit* u *of the sequence* $\{u_n\}$ *is not trivial.*

Proof. Arguing by contradiction, we shall assume $u = 0$. Once $\{u_n\}$ is bounded, there are $L > 0$ and some subsequence, still denoted by $\{u_n\}$, such that

$$\int_{\mathbb{R}^N} \Big[|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big]\, dx \to L. \tag{3.5}$$

On the other hand, using the compact embedding $W_r^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ (see [18]) it follows that $\|u_n\|_{q(x)} \to 0$ which is equivalent to

$$\int_{\mathbb{R}^N} |u_n|^{q(x)} dx \to 0. \tag{3.6}$$

From (2.1), (3.5) and (3.6), we have that

$$\int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx \to L. \tag{3.7}$$

Since, for each $\lambda \geq \lambda^*$ fixed, $d_\lambda + o_n(1) = I(u_n)$, using (3.6) we obtain

$$d_\lambda + o_n(1) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \Big[|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big]\, dx - \int_{\mathbb{R}^N} \frac{1}{p^*(x)} |u_n|^{p^*(x)} dx,$$

and thus

$$d_\lambda + o_n(1) \geq \frac{1}{p^+} \int_{\mathbb{R}^N} \left[|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right] dx - \frac{1}{p_-^*} \int_{\mathbb{R}^N} |u_n|^{p^*(x)} dx.$$

Taking $n \to +\infty$ in the preceding inequality, using (3.5), (3.7) and the definition of the number Θ given in Corollary 3.1, we get

$$d_\lambda \geq \Theta L. \tag{3.8}$$

On the other hand, (3.5), (3.7) and Corollary 3.2 yield

$$L^{\frac{1}{p_-^*}} \leq K L^{\frac{1}{p^+}},$$

which is equivalent to

$$L \geq \left(\tfrac{1}{K}\right)^{\frac{1}{\Theta}}. \tag{3.9}$$

From (3.8) and (3.9), we have $d_\lambda \geq \Theta\left(\frac{1}{K}\right)^{\frac{1}{\Theta}}$, $\lambda \geq \lambda^*$ which is absurd when combined with Corollary 3.1. Therefore, u is not trivial and the proof is finished. □

Proof of Theorem 1.1. This is an immediate consequence of Lemma 3.1 and Proposition 3.1. □

4. The Coercive Potential

The same type of arguments developed in this paper can be used to prove the existence of solutions for the following class of $p(x)-$Laplacian equations:

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + V(x)|u|^{p(x)-2}u = \lambda |u|^{q(x)-2}u + |u|^{p^*(x)-2}u, \quad \mathbb{R}^N \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

assuming that $\inf_{\mathbb{R}^N} V(x) > 0$ and that V is a coercive function; that is,

$$V(x) \to +\infty \ \text{ as } \ |x| \to +\infty. \tag{V_1}$$

For this class of problem, the appropriate Sobolev space is $W^{1,p(x)}_{V(x)}(\mathbb{R}^N)$, defined as a completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_* = \|\nabla u\|_{p(x)} + \|u\|_{p(x),V(x)},$$

where

$$\|\nabla u\|_{p(x)} = \inf\left\{\alpha > 0 : \int_{\mathbb{R}^N} \left|\frac{\nabla u}{\alpha}\right|^{p(x)} dx \leq 1\right\},$$

and

$$\|u\|_{p(x),V(x)} = \inf\left\{\alpha > 0 : \int_{\mathbb{R}^N} V(x)\left|\frac{u}{\alpha}\right|^{p(x)} dx \leq 1\right\}.$$

To prove an important embedding involving the space $W^{1,p(x)}_{V(x)}(\mathbb{R}^N)$, we shall need one more result, which is an interpolation type result for bounded sequences in spaces of variable exponents.

Lemma 4.1. *Let $p(x), r(x), q(x) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ with*

$$p(x) << q(x) << r(x), \quad \inf_{\mathbb{R}^N} p(x) > 1,$$

and $\{u_n\} \subset L^{p(x)}(\mathbb{R}^N) \cap L^{r(x)}(\mathbb{R}^N)$ satisfying $\|u_n\|_{p(x)}, \|u_n\|_{r(x)} \leq \eta$ for all $n \in \mathbb{N}$ for some $\eta > 0$. Then, $\{u_n\} \subset L^{q(x)}(\mathbb{R}^N)$ and there are $C > 0$ and $\alpha(x) \in L^\infty(\mathbb{R}^N)$ satisfying

$$0 << \alpha(x) << 1 \ \text{and} \ \frac{1}{q(x)} = \frac{\alpha(x)}{p(x)} + \frac{(1-\alpha(x))}{r(x)} \ \ \forall x \in \mathbb{R}^N,$$

such that

$$\|u_n\|_{q(x)} \leq C\|u_n\|^{\alpha_-}_{p(x)}\|u_n\|^{1-\alpha^+}_{r(x)} \ \ \forall n \in \mathbb{N}.$$

Proof. Defining the sequence $f_n = \frac{u_n}{\eta}$, it follows that $\|f_n\|_{p(x)}, \|f_n\|_{r(x)} \leq 1$, and so

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big|\frac{f_n}{\|f_n\|^{\alpha_-}_{p(x)}\|f_n\|^{(1-\alpha^+)}_{r(x)}}\Big|^{q(x)}dx\\ &\leq \int_{\mathbb{R}^N}\Big|\frac{f_n}{\|f_n\|_{p(x)}}\Big|^{q(x)\alpha(x)}\Big|\frac{f_n}{\|f_n\|_{r(x)}}\Big|^{q(x)(1-\alpha(x))}dx.\end{aligned}$$

Applying Holder's inequality for the exponents $\frac{p(x)}{q(x)\alpha(x)}$ and $\frac{r(x)}{q(x)(1-\alpha(x))}$ (see [17]), there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^N}\Big|\frac{f_n}{\|f_n\|^{\alpha_-}_{p(x)}\|f_n\|^{(1-\alpha^+)}_{p^*(x)}}\Big|^{q(x)}dx \leq C_1 \ \ \forall n \in \mathbb{N},$$

from which it follows that

$$\|f_n\|_{q(x)} \leq C_2\|f_n\|^{\alpha_-}_{p(x)}\|f_n\|^{(1-\alpha^+)}_{p^*(x)} \ \ \forall n \in \mathbb{N}, \tag{4.1}$$

where $C_2 = (1+C_1)^{q^+}$. Therefore, using the definition of f_n, we get

$$\|u_n\|_{q(x)} \leq C\|u_n\|^{\alpha_-}_{p(x)}\|u_n\|^{(1-\alpha^+)}_{p^*(x)},$$

where $C = \frac{C_2\eta}{\eta^{\alpha_-+1-\alpha^+}}$. □

The next lemma establishes a compact embedding, which is used in the proof of Proposition 3.1 when the potential V satisfies condition (V_1).

Lemma 4.2. *Assume that* $q(x) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ *satisfies*

$$p(x) << q(x) << p^*(x).$$

Then, the Sobolev embedding

$$W^{1,p(x)}_{V(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N) \tag{4.2}$$

is compact.

Proof. The proof follows by adapting some arguments developed by Costa [14], which has considered the case $p = 2$ and $q \in (2, 2^*)$. Without loss of generality, let $\{u_n\}$ be a sequence in $W^{1,p(x)}_{V(x)}(\mathbb{R}^N)$ with $u_n \rightharpoonup 0$ in $W^{1,p(x)}_{V(x)}(\mathbb{R}^N)$. For each $\epsilon \in (0,1)$ fixed, by (V_1), there exists $R > 0$ such that

$$\frac{1}{V(x)} \leq \epsilon^{p_-} \ \ \forall x \in B^c_R(0),$$

where $B_R(0) = \{x \in \mathbb{R}^N : \|x\| < R\}$. The last inequality implies

$$\|u_n\|_{L^{p(x)}(B^c_R(0))} \leq \epsilon \|u_n\|_* \ \ \forall n \in \mathbb{N}. \tag{4.3}$$

On the other hand, it is easy to check that the Sobolev embedding

$$W^{1,p(x)}_{V(x)}(B_R(0)) \hookrightarrow L^{p(x)}(B_R(0))$$

is compact, hence there exists $n_0 \in \mathbb{N}$ satisfying

$$\|u_n\|_{L^{p(x)}(B_R(0))} \leq \epsilon \ \ \forall n \geq n_0. \tag{4.4}$$

Using the boundedness of $\{u_n\}$ in $W^{1,p(x)}_{V(x)}(\mathbb{R}^N)$, (4.3) and (4.4), we get $\|u_n\|_{p(x)} \leq \epsilon C$ for all $n \geq n_0$ for some positive constant C. Therefore, the preceding inequality implies that

$$u_n \to 0 \ \text{ in } \ L^{p(x)}(\mathbb{R}^N). \tag{4.5}$$

Now, using again the fact that $\{u_n\}$ is bounded in $W^{1,p(x)}_{V(x)}(\mathbb{R}^N)$, there exists $\eta > 0$ satisfying

$$\|u_n\|_{p(x)}, \|u_n\|_{p^*(x)} \leq \eta \ \ \forall n \in \mathbb{N}. \tag{4.6}$$

This, in conjunction with Lemma 4.1, yields

$$\|u_n\|_{q(x)} \leq C \|u_n\|^{\alpha_-}_{p(x)} \|u_n\|^{1-\alpha^+}_{p^*(x)} \ \ \forall n \in \mathbb{N}, \tag{4.7}$$

for some positive constant C. From (4.5)-(4.7), we see that $\|u_n\|_{q(x)} \to 0$ from which follows the lemma. $\square$

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