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FULLY NONLINEAR PHASE TRANSITION PROBLEMS WITH FLAT FREE BOUNDARIES

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Abstract. In this paper we continue our study, started in [9], on the regularity theory of Stefan-like free boundary problems for a special class of fully nonlinear equations of parabolic type. We prove that degenerate Lipschitz free boundaries, with small Lipschitz constant in space, are C^1.

1. Introduction

In the present paper, we continue the study of the regularity properties of the free boundary for fully nonlinear parabolic phase transition problems.

These well-known problems arise when a state variable, u, diffuses nonlinearly in any of two given states but has a discontinuity in its behavior across the surface where u vanishes. On the other hand, across that zero level surface there is a balance condition between the speed of the interphase and the jump on the normal derivative of the variable u. In the linear case, where phenomena are governed by the heat equation, well-known examples are Stefan-like or melting-solidification problems; that is, problems where energy changes discontinuously across the melting temperature, creating in that way a solid-liquid interface. In the nonlinear case, examples appear in financial mathematics where the strategy in stock market changes when the present value of an option passes through a certain threshold (see [7] and references therein).

In general, as we will see below, a Lipschitz free boundary could have hyperbolic behavior and a corner could persist for an amount of time without any regularization effect. On the other hand, if one assumes a nondegeneracy condition (see [1] for the linear case and [9] for fully nonlinear equations), which is able to prevent simultaneous vanishing of the flows from the two sides of the zero level surface, then Lipschitz free boundaries are actually C^1 graphs in space and time. The purpose of this paper is to examine the case

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when this nondegeneracy condition is not satisfied. More precisely, we are able to prove C^1 regularity of the free boundary if the Lipschitz constant with respect to the space variable is small enough.

2. DEFINITIONS AND MAIN RESULT.

In this section, we introduce the two phase free boundary problems we are dealing with and we give the basic definitions concerning our solutions. At the end of the section we state our main regularity result.

We denote a point in $\mathbb{R}^{n+1}$ by $(x,t) = (x', x_n, t)$ and by $\mathcal{S}$ the space of $n \times n$ symmetric matrices. In the present paper, the operator $F : \mathcal{S} \subseteq \mathbb{R}^{n\times n} \to \mathbb{R}$, whenever it appears, is assumed to be smooth, concave, fully nonlinear, homogeneous of degree 1, with $F(0) = 0$, and uniformly elliptic with ellipticity constants λ_1 and λ_2 (i.e. $\lambda_1|\xi|^2 \le F_{ij}(M) \le \lambda_2|\xi|^2$ for all $M \in \mathcal{S}\setminus\{0\}$ and $\xi \in \mathbb{R}^n$ where $F_{ij}(M) = \frac{\partial F}{\partial M_{ij}}(M)$). Denote by $x_n = f(x', t)$ a Lipschitz function with Lipschitz constant L. We give the definition of F-solutions.

Definition 2.1. *Let Ω be one side of a Lipschitz graph; i.e., $\Omega = \{x_n > f(x', t)\}$ and D is the intersection of some $(n+1)$-dimensional cube Q with Ω. By an F-solution we mean a nonnegative solution, u, of the equation*

$$F(D^2u) - u_t = 0 \text{ in } D \subset \mathbb{R}^{n+1},$$

in the Crandall - Lions viscosity sense that vanishes locally on $(\partial\Omega) \cap Q$.

Next, we introduce a class of free boundary problems and define for them a notion of viscosity solutions.

Definition 2.2. *Let v be a continuous function in $D_1 := B_1(0) \times (-1, 1)$. Then v is called a subsolution (supersolution) to a free boundary problem if*
(i) $F(D^2v^+) - v_t^+ \ge 0(\le)$ in $\Omega^+ := D_1 \cap \{v > 0\}$;
(ii) $F(D^2v^-) - v_t^- \le 0(\ge)$ in $\Omega^- := D_1 \cap \{v \le 0\}^o$;
(iii) $v \in C^1(\bar{\Omega}^+) \cap C^1(\bar{\Omega}^-)$;
(iv) for any $(x,t) \in \partial\Omega^+ \cap D_1$, $\nabla_x v^+(x,t) \ne 0$, and

$$V_\nu \ge -G((x,t), \nu, v_\nu^+, v_\nu^-) \quad (\le),$$

where (i) and (ii) are satisfied in the viscosity sense and V_ν is the speed of the surface $\mathcal{F}_t := \partial\Omega^+ \cap \{t\}$ in the direction $\nu := \frac{\nabla_x v^+}{|\nabla_x v^+|}$.

We say that v is a solution to a free boundary problem if it is both a subsolution and a supersolution. To give a precise description of the speed of the interphase we can replace condition (iv) in the definition above by

$$\frac{v_t^+}{v_\nu^+} \leq G((x,t),\nu,v_\nu^+,v_\nu^-) \qquad (\geq).$$

We assume that the function G is given a priori, it is continuous in all of its arguments, is increasing in v_ν^+, decreasing in v_ν^- and that $G \to +\infty$ when $v_\nu^+ - v_\nu^- \to +\infty$. In the case of the Stefan problem we have

$$G((x,t),\nu,v_\nu^+,v_\nu^-) = v_\nu^+ - v_\nu^-.$$

Definition 2.3. *Assume that u is a continuous function in D_1; then u is a viscosity subsolution (supersolution) to a free boundary problem if, for any subcylinder Q of D_1 and for every supersolution (subsolution), according to Definition 2.2, v in Q, $u \leq v$ $(u \geq v)$ on $\partial_p Q$ implies that $u \leq v$ $(u \geq v)$ in Q.*

In this paper we consider viscosity solutions whose free boundary is given locally by a Lipschitz graph. In this case, using the special structure of the fully nonlinear operator we can prove the optimal regularity of the viscosity solution, along with several other properties concerning our free boundary problem. For the proofs of the theorems below we refer the reader to [9] (see also [10], [11]). Introduce, for $(\xi,\tau) \in \{x_n = f(x',t)\}$, $r > 0$:

$$A_r(\xi,\tau) := (\xi',\xi_n + b_0 r,\tau), \ \ \overline{A}_r(\xi,\tau) := (\xi',\xi_n + b_0 r,\tau + \tfrac{3}{2}r^2),$$
$$\underline{A}_r(\xi,\tau) := (\xi',\xi_n + b_0 r,\tau - \tfrac{3}{2}r^2),$$

and

$$Q_r(\xi,\tau) := \{(x,t) \in \mathbb{R}^{n+1} : |x'-\xi'| < r, |x_n - \xi_n| < b_0 r, |t-\tau| < r^2\},$$
$$K_r(\xi,\tau) := \{(x,t) \in \mathbb{R}^{n+1} : |x'-\xi'| < t, |x_n - \xi_n| < b_0 r, |t-\tau| < r\}.$$

Finally, we assume $d_{x,t} := inf\{dist((x,t),(y,t)) : y_n = f(y',t)\}$ and $(0,0) \in \{x_n = f(x',t)\}$. First, we give a theorem which is known as a boundary backward Harnack principle.

Theorem 2.4 ([9])**.** *Let u be an F-solution in $Q_1 \cap \Omega$ and $u(\underline{A}_{3/4}(0,0)) = m > 0$. Then there exists a constant $C = C(n,L,m/M,\lambda_1,\lambda_2)$ such that*

$$u(x,t+\rho^2) \leq Cu(x,t-\rho^2),$$

for all $(x,t) \in Q_{1/2} \cap \Omega$ and for all $\rho : 0 < \rho < d_{x,t}/b_0$, $b_0 = \max(4L,1)$.

Theorem 2.5 ([9])**.** *Let u be as in Theorem 2.4; then we have the following.*

(1) *There exist positive constants C_1, C_2 such that*

$$C_1 \frac{u(x,t)}{d_{x,t}} \leq |\nabla u(x,t)| \leq C_2 \frac{u(x,t)}{d_{x,t}},$$

for all $(x,t) \in K_r \cap \Omega$ *where* $\nabla u = (\nabla_x u, D_t u)$.

(2) *There exist* $\vartheta > 0$ *and* $\delta > 0$ *depending on* $n, L, \lambda_1, \lambda_2$ *such that the functions* $w_+ := u + |u|^{1+\vartheta}$ *and* $w_- := u - |u|^{1+\vartheta}$ *are subsolutions and supersolutions respectively of the equation* $F(D^2 v) = 0$ *in the viscosity sense in* $Q_\delta \cap \Omega \cap \{t = 0\}$.

We can also prove the optimal regularity of viscosity solutions as well as the existence of a whole cone where our solution is monotone. This is precisely the content of the following theorem.

Theorem 2.6 ([9])**.** *Let* u *be a viscosity solution of a free boundary problem in* $B_2(0) \times (-2,2)$ *and suppose that the free boundary of* u *is given by the graph* $\{x_n = f(x',t)\}$, $(x_n, x', t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, *which is Lipschitz in some space direction* ν *with Lipschitz constant* L *and passes through the origin. Assume that* $u(\underline{A}_{3/4}(0,0)) = m > 0$ *where* $b_0 = max(4L, 1)$ *and* $M = supu$. *Then, in* $B_2(0) \times (-2,2)$*:*

(1) u *is Lipschitz continuous across the free boundary.*

(2) *There exists a cone* $\Gamma(\theta, e_n)$, *with axis* e_n *and opening* $\theta = \theta(\lambda_1, \lambda_2, n, L, M)$, *such that* u *is monotone increasing along any direction lying on the cone.*

(3) *Let* (x_0, t_0) *be a free boundary point and suppose that there exists an* $(n+1)$*-dimensional ball* $B \subset \Omega^+$ *such that* $\overline{B} \cap \mathcal{F} = \{(x_0, t_0)\}$*; i.e.,* (x_0, t_0) *is a regular point from the right. If* ν *is the inward spatial normal vector at* (x_0, t_0) *of* $B \cap \{t = t_0\}$ *and* $d_{x,t}$ *is the distance between* (x,t) *and* (x_0, t_0), *then there exist numbers* $\alpha_+, \alpha_-, \beta_+, \beta_-$ *such that, near* (x_0, t_0),

$$u(x,t) \geq (\alpha_+ \langle \nu, x - x_0 \rangle + \beta_+ (t - t_0))^+ - (\alpha_- \langle \nu, x - x_0 \rangle + \beta_- (t - t_0))^- + o(d_{x,t}),$$

with $\alpha_+ > 0, \alpha_- \geq 0$, *equality holds on the hyperplane* $t = t_0$,

$$\beta_+ \geq \alpha_+ G(\nu, \alpha_+, \alpha_-), \quad \beta_- \geq \alpha_- G(\nu, \alpha_+, \alpha_-).$$

If (x_0, t_0) *is a regular point from the left, the previous inequalities are reversed,* $\alpha_+ \geq 0, \alpha_- > 0$ *and* ν *is the outward spatial normal vector.*

Counterexamples (see [1]) show that, in general, a Lipschitz free boundary could exhibit a hyperbolic behavior; i.e., a corner can persist for any amount of time. In this case, a nondegeneracy condition guarantees the improvement of the regularity and Lipschitz free boundaries are actually C^1 graphs. A suitable nondegeneracy condition for our problem could be the following (see [9], [10]).

Nondegeneracy condition: *There exists $k_0 > 0$ such that, if $(x_0, t_0) \in \mathcal{F}$ is a regular point from the right or from the left, then, for any small r,*

$$\fint_{B_r(x_0)} u^+ \geq k_0 r,$$

where we say that a point on the free boundary is a regular point from the left (right), if there exists a ball in the negative (positive) domain of the solution, touching the free boundary only at that point.

As we see in [1] (see also [2]), an examination of the above mentioned counterexamples shows that in the lack of nondegeneracy the achievement of further regularity of the free boundary depends on its Lipschitz constant. This is also the case in our fully nonlinear problem. More precisely, in our main theorem we show that if the Lipschitz constant (with respect to the space variable) is small enough then one is actually in a "nondegenerate situation" and C^1 regularity holds. The main result of this work can be stated in the following way.

Theorem 2.7. *Let u be a viscosity solution of a free boundary problem in $Q_2 = B_2 \times (-2, 2)$ whose free boundary, $\mathcal{F}$, contains the origin $(0, 0)$ and is given by the graph of a Lipschitz function $x_n = f(x', t)$ with Lipschitz constant L_1 in space and L_2 in time. Suppose that $u(e_n, -\frac{3}{2}) = 1$, where e_n is the unit vector in the x_n direction, $G(\nu, a, b)$ is a Lipschitz function in all of its arguments with Lipschitz constant L_G, and for some positive number γ_0, $D_aG \geq \gamma_0$ and $D_bG \leq -\gamma_0$. Then there exists a constant L_0 such that, if the Lipschitz constant in space of f, $L_1 \leq L_0$ the free boundary is a C^1 graph in space and time. Moreover, there exists a positive constant $C_1 = C_1(n, L_0, L_2, M, \lambda_1, \lambda_2, \gamma_0)$ such that,*

$$|\nabla_{x'} f(x', t) - \nabla_{x'} f(y', t)| \leq C_1(-\log|x' - y'|)^{-4/3}$$
$$|D_t f(x', t) - D_t f(x', s)| \leq C_1(-\log|t - s|)^{-1/3},$$

for every (x', x_n, t), $(y', y_n, t) \in \mathcal{F}$ where $M = sup_{Q_2} u$.

To achieve further regularity, we follow the method described in [3], [2] for flat free boundaries, with certain modifications in order to overcome the difficulties due to the nonlinear nature of our operator. We need to make use of the structure of viscosity solutions along with the concavity of our equation. Although we are in position to prove the majority of the steps for more general operators we are still in need of this special structure in certain crucial points, as we may see in the following sections. To describe the strategy of the proof, a first observation is that in order to get a solution

u monotone along any direction $\tau \in \Gamma(\theta_0, \nu)$, where by $\Gamma(\theta_0, \nu)$ we denote a cone with axis ν and opening θ_0, it enough to prove that for any $\varepsilon > 0$

$$u_\varepsilon(p) := \sup_{q \in B_{\varepsilon \sin(\theta_0)}(p)} u(q - \varepsilon\tau) \leq u(p),$$

where the function u_ε actually measures the opening of the monotonicity cone. Therefore, we need to get this inequality in a smaller region for τ belonging to a larger cone. This can be done away from the free boundary for any $\varepsilon > 0$. Next, we need to carry this information via a family of perturbations to the free boundary. In order to counter-balance the lack of nondegeneracy we need, as in the linear case, a control of u_ν at regular points among the free boundary points. This will allow us to achieve in a larger cone only ε-monotonicity rather than full monotonicity. On the other hand, since ε-monotonicity implies full monotonicity $\sqrt{\varepsilon}$ away from the free boundary, it is possible to improve ε-monotonicity itself. Finally, we need to perform a double iteration procedure which consists at each step of a cone enlargement and an ε-monotonicity improvement in a sequence of contracting domains. As in [3], [2] the double homogeneity of the problem gives logarithmic opening speeds of the monotonicity cones. We would like to emphasize the fact that the method of Athanasopoulos, Caffarelli and Salsa ([3], [2]) is quite general and in the present paper we follow closely their arguments.

To end this introductory section, let us give some notation to be used in the upcoming sections. For $0 < \lambda_1 \leq \lambda_2$ and $M \in \mathcal{S}$ we define Pucci's extremal operators (see also [6]) as

$$\mathcal{M}^-(M, \lambda_1, \lambda_2) = \lambda_1 \sum_{e_i > 0} e_i + \lambda_2 \sum_{e_i < 0} e_i,$$
$$\mathcal{M}^+(M, \lambda_1, \lambda_2) = \lambda_1 \sum_{e_i < 0} e_i + \lambda_2 \sum_{e_i > 0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of M. The various constants that will appear in the sequel may vary from formula to formula. If we do not give any explicit dependence for a constant, we mean that it depends only on the usual parameters such as ellipticity constants, dimension and the maximum value of our solution.

The regularity of flat free boundaries of two phase problems for fully nonlinear elliptic equations has been treated by P. Wang in [14].

3. From ε-monotonicity to full monotonicity.

The lack of a nondegeneracy condition restricts the arbitrariness of ε at each step of the iterative method we perform in the proof of the main theorem (see also [7]). In this section we show that at a distance of order $\sqrt{\varepsilon}$ away from the free boundary ε-monotonicity implies full monotonicity. In our approach we follow the method of Athanasopoulos, Caffarelli and Salsa (see [3], [2]).

Definition 3.1. *Given $\varepsilon \geq 0$, a function u is called ε-monotone in the direction τ if $u(p + h\tau) \geq u(p)$ for every $h \geq \varepsilon$.*

The following lemma can be considered as the nonlinear analog of Lemma 3.3 in [2].

Lemma 3.2. *Let u be a nonnegative $F-$solution in the cylinder $Q_{\varepsilon M} := B_{\varepsilon M} \times (-\varepsilon^2 M^2, \varepsilon^2 M^2)$. Suppose that u is monotone in $\Gamma_t(e_n, \theta_0)$ for some θ_0 and ε-monotone along a space direction e. Then, if $M = M(n, \theta_0)$ is large enough and $\varepsilon M < 1$ there exists a constant $c = c(n, \theta_0, \lambda_1, \lambda_2)$ such that*

$$D_{\bar{e}} u(0,0) \geq 0,$$

where $\bar{e} = e + c\varepsilon M e_n$.

Proof. For any $1 \leq \lambda < \frac{1}{2}M$ define $w_\lambda(x,t) := u(x + \lambda\varepsilon e, t) - u(x,t)$ for $(x,t) \in Q_{\varepsilon M/2}$. Then w_λ is nonnegative and $w_\lambda \in \mathcal{S}(\lambda_1, \lambda_2)$ in $Q_{\varepsilon M/4}$, where $\mathcal{S}(\lambda_1, \lambda_2)$ is the class of solutions corresponding to Pucci's extremal operators. Therefore, by Harnack's inequality we have

$$w_\lambda(x,t) \leq w_{[\lambda]+2}(x,t) \leq C\lambda w_1(x, t + rM^2),$$

for some fixed $r > 0$. Next, we linearize the equation to get an operator

$$\begin{aligned} Lw_\lambda :&= \alpha_{i,j}(x,t) D_{ij} w_\lambda - D_t w_\lambda \\ &= \int_0^1 F_{ij}[sD^2 u(x + \lambda e, t) + (1-s) D^2 u(x,t)] ds D_{ij} w_\lambda - D_t w_\lambda = 0. \end{aligned}$$

Let w^ϵ be the classical solution of the ϵ-regularized equation

$$L_\epsilon w^\epsilon = \alpha^\epsilon_{i,j}(x,t) D_{ij} w^\epsilon - D_t w^\epsilon,$$

with the same boundary data as w_λ, for which one can have $C^{1,\alpha}$ estimates. Extracting a subsequence, if necessary, we can find a limit w^0, which is an L^{n+1}-viscosity solution of $Lw = 0$ with the same boundary data as w_λ. By the uniqueness of the L^{n+1}-viscosity solution of the linear operator, $w_\lambda = w^0$ and in particular

$$|D_e w_\lambda(0,0)| \leq \frac{C}{M} w_\lambda(0, r\varepsilon^2 M^2).$$

Since

$$D_e w_\lambda(0,0) = D_e u(\lambda\varepsilon e, 0) - D_e u(0,0),$$

we obtain

$$\begin{aligned} w_1(\varepsilon e, 0) = u(2\varepsilon e, 0) - u(\varepsilon e, 0) &= \int_1^2 \varepsilon D_e w_\lambda(0,0) d\lambda + \varepsilon D_e u(0,0) \\ &\leq \frac{C}{M} \int_1^2 w_\lambda(0, r\varepsilon^2 M^2) d\lambda + \varepsilon D_e u(0,0) \\ &\leq \frac{c}{M} w_1(\varepsilon e, 2\varepsilon^2 M^2) + \varepsilon D_e u(0,0). \end{aligned}$$

On the other hand,

$$\begin{aligned} w_1(\varepsilon e, 2\varepsilon^2 M^2) &= w_1(\varepsilon e, 2\varepsilon^2 M^2) - w_1(\varepsilon e, 0) + w_1(\varepsilon e, 0) \\ &\leq \max |D_t w_1| 2\varepsilon^2 M^2 + w_1(\varepsilon e, 0) \\ &\leq C D_e u(0,0) 2\varepsilon^2 M^2 + w_1(\varepsilon e, 0). \end{aligned}$$

Therefore, if we choose M large enough,

$$D_e u(0,0) + C_0 \varepsilon M D_{e_n} u(0,0) \geq w_1(\varepsilon e, 0)(1 - \frac{C_0}{M})\frac{1}{\varepsilon} \geq 0. \qquad \square$$

Lemma 3.3. *Suppose that u is a nonnegative F-solution in the cylinder $Q_{\sqrt{\varepsilon M}} := B_{\sqrt{\varepsilon M}} \times (-\varepsilon M, \varepsilon M)$ which is monotone along every direction in a cone $\Gamma(e_n, \theta_0)$ and ε-monotone along a direction $\tau = \alpha e + \beta e_t$ where $\alpha^2 + \beta^2 = 1$, $\beta \neq 0$ and e is a spatial direction. Then if M is large enough and ε small,*

$$D_{\tau_\varepsilon} u(0,0) \geq 0,$$

where $\tau_\varepsilon = \tau + c|\alpha|\sqrt{\varepsilon M} e_n$, for some constant $c = c(n, \theta, \lambda_1, \lambda_2)$.

Proof. Suppose $\beta > 0$, $1 \leq \lambda \leq M/2$ and set $w_\lambda(p) := u(p) - u(p - \lambda\varepsilon\tau)$. Then

$$\begin{aligned} w_1(-\varepsilon\tau) = u(-\varepsilon\tau) - u(-2\varepsilon\tau) &= -\int_1^2 D_\lambda u(-\lambda\varepsilon\tau) d\lambda \\ &= \int_1^2 \varepsilon D_\tau u(\lambda\varepsilon\tau) d\lambda = \varepsilon D_\tau u(0) - \varepsilon \int_1^2 D_\tau w_\lambda(0) d\lambda \\ &= \varepsilon D_\tau u(0) - \varepsilon \int_1^2 \alpha D_e w_\lambda(0) d\lambda - \varepsilon \int_1^2 \beta D_t w_\lambda(0) d\lambda, \end{aligned}$$

since $D_\lambda u(-\lambda\varepsilon\tau) = -\varepsilon D_\tau u(-\lambda\varepsilon\tau)$. In addition in $B_{\sqrt{\varepsilon M}/4} \times (-\varepsilon M, 0)$ using the Harnack inequality (see [12], [13]) we have

$$D_e w_\lambda(0) \leq \frac{C}{\sqrt{\varepsilon M}} w_\lambda(\varepsilon M e_t) \leq \frac{C}{\sqrt{\varepsilon M}} w_1(-\alpha\varepsilon e + 2\varepsilon M e_t),$$

and

$$w_1(-\alpha\varepsilon e + 2\varepsilon M e_t) \leq C\varepsilon M D_{e_n} u(0) + w_1(-\varepsilon\tau).$$

On the other hand, since F is concave (see [8]) using the Harnack inequality again for M large enough,

$$D_t w_\lambda(0) \geq -\frac{C}{M\varepsilon} w_1(-\varepsilon\tau).$$

Thus,

$$w_1(-\varepsilon\tau) \leq \varepsilon D_\tau u(0) + \frac{C}{M}\beta w_1(-\varepsilon\tau) + c|\alpha|\frac{\sqrt{\varepsilon}}{\sqrt{M}} w_1(-\varepsilon\tau) + c|\alpha|\varepsilon\sqrt{\varepsilon M} D_{e_n} u(0);$$

that is,

$$D_{\tau_\varepsilon} u(0) \geq w_1(-\varepsilon\tau)\Big[1 - \frac{c\beta}{M} - c|\alpha|\frac{\sqrt{\varepsilon}}{\sqrt{M}}\Big]\frac{1}{\varepsilon} \geq 0,$$

for M large enough. The case $\beta < 0$ is treated similarly. $\square$

Before we present the basic monotonicity result of the section, let us give an alternative definition of ε-monotonicity.

Definition 3.4. *A function u is said to be ε-monotone in $\Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$, $\nu \in Sp(\{e_n, e_t\})$ if for small $\delta > 0$ and any unit vector $\tau \in \Gamma_x(\theta^x - \delta, e_n) \cup \Gamma_t(\theta^t - \delta, \nu)$,*

$$\sup_{B_{\varepsilon \sin\delta}(p)} u(q - \varepsilon\tau) \leq u(p).$$

Corollary 3.5. *Let u be an ε-monotone in $\Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$ with $\nu \in Sp(\{e_n, e_t\})$ viscosity solution to the free boundary problem. Then there exist positive constants c_1, c_2 such that u is fully monotone in*

$$\Gamma_x(\theta^x - c_1\varepsilon, e_n) \cup \Gamma_t(\theta^t - c_1\sqrt{\varepsilon}, \nu),$$

at a distance greater than $c_2\sqrt{\varepsilon}$ from the free boundary.

4. Interior gain.

According to Corollary 3.5 u is fully monotone away from the free boundary, so we can use the techniques of [1], [9] to enlarge the cone in space. More precisely we can prove the following lemma.

Lemma 4.1. *Let u be a viscosity solution to the free boundary problem in $Q_1 = B_1 \times (-1,1)$, which is fully monotone in $\Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$, $\nu \in Sp(\{e_n, e_t\})$ for $(x,t) \in Q_1 \cap \{dist((x,t),\mathcal{F}) > C\varepsilon^\gamma\}$ where $0 < \gamma < \frac{1}{2}$ and $\varepsilon > 0$ is small. Then there exist θ_1^x and $\bar{e}$ with $\Gamma_x(\theta^x, e_n) \subset \Gamma_x(\theta_1^x, \bar{e})$ and $\frac{\pi}{2} - \theta_1^x = c(\frac{\pi}{2} - \theta^x)$ for some $0 < c < 1$ such that u is fully monotone in $\Gamma_x(\theta_1^x, \bar{e}) \cup \Gamma_t(\theta^t, \nu)$ in the domain $B_{1/8}(\frac{3}{4}e_n) \times (-\frac{C\delta}{\mu}, \frac{C\delta}{\mu})$ provided that $\delta \le \mu$, where δ and μ are the defect angles in space and time respectively.*

The enlargement of the cone of monotonicity in time away from the free boundary in [9] requires full monotonicity up to the free boundary, but this is not our case here. Instead we are able to prove that u is trapped between two approximations.

Lemma 4.2. *Let u be a viscosity solution which is monotone increasing along every direction $\tau \in \Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$, $\nu \in Sp(\{e_n, e_t\})$, outside an ε^γ-neighborhood of the free boundary of u. Define, for $\gamma' < \gamma < 1/2$,*

$$\overline{v}(p) := \inf_{B_{\varepsilon^{\gamma'}}(p)} u, \quad \underline{v}(p) := \sup_{B_{\varepsilon^{\gamma'}}(p)} u.$$

Then $\overline{v}$, $\underline{v}$ satisfy the following:

1. *They are Lipschitz functions with the same Lipschitz constant as u and*
$$|\overline{v} - u| \le C\varepsilon^{\gamma'}, \quad |\underline{v} - u| \le C\varepsilon^{\gamma'}.$$
2. *They are monotone increasing along every direction in*
$$\Gamma_x(\theta^x - c\varepsilon^{1-\gamma'}, e_n) \cup \Gamma_t(\theta^t - c\varepsilon^{1-\gamma'}, \nu),$$
where $c = c(n, L, \lambda_1, \lambda_2)$.
3. *$\overline{v}$ is an F-supersolution in $\Omega^+(\overline{v}) \cup \Omega^-(\overline{v})$ and $\underline{v}$ is an F-subsolution in $\Omega^+(\underline{v}) \cup \Omega^-(\underline{v})$.*
4. *Each point of $\mathcal{F}(\overline{v})$ (respectively $\mathcal{F}(\underline{v})$) is regular from the left (respectively right) and if $B(y_0, s_0) \subset \Omega^-(\overline{v})$ (respectively $B(y_0, s_0) \subset \Omega^+(\underline{v})$) is tangent to $\mathcal{F}(\overline{v})$ (respectively $\mathcal{F}(\underline{v})$) at (x_0, t_0), then near x_0 at t_0-level*

$$\overline{v}(x, t_0) \le \alpha_+ < x - x_0, \overline{v} >^+ - \alpha_- < x - x_0, \overline{v} >^- + o(|x - x_0|)$$

$$(respectively \underline{v}(x, t_0) \ge \alpha_+ < x - x_0, \overline{v} >^+ - \alpha_- < x - x_0, \overline{v} >^- + o(|x - x_0|))$$

with $\frac{\beta_+}{\alpha_+} \geq G(\overline{\nu}, \alpha_+, \alpha_-)$ *(respectively* $\leq$*), where* $\overline{\nu} = \frac{y_0 - x_0}{|y_0 - x_0|}$.

Proof. The first and the third properties are obvious and for the second we may use the argument of Lemma 2 of [5]. The last one can be considered as a particular case of Theorem 2.6 (see also [9]). □

Now, let $(0,0) \in \mathcal{F}(\overline{v})$ be the center of the strip

$$R_\eta := \{|x'| < \frac{2\eta}{3}, |x_n| < \frac{2\eta}{3}, |t| < \eta\},$$

and set $R^{\pm}_\eta := \Omega^{\pm}(\overline{v}) \cap R_\eta$. Denote by v the solution of the following problem:

$$\begin{cases} F(D^2 v) - v_t = 0, & \text{in} \quad R^{\pm}_1 \\ v = \overline{v}, & \text{on} \quad \partial_p R^{\pm}_1. \end{cases}$$

By the maximum principle and Lemma 4.2

$$|u - v| \leq c\varepsilon^{\gamma'} \quad \text{in} \quad \overline{R_1}.$$

Also since $\overline{v}$ is an F-supersolution in $R^+_1 \cup R^-_1$, $v^+ \leq \overline{v}^+$ and $v^- \geq \overline{v}^-$. Therefore, at each point $(x_0, t_0) \in \mathcal{F}(v) = \mathcal{F}(\overline{v})$ we have the following asymptotic development:

$$v(x, t_0) \leq \alpha^*_+ < x - x_0, \overline{\nu} >^+ - \alpha^*_- < x - x_0, \overline{\nu} >^- + o(|x - x_0|),$$

with $\alpha^*_+ \leq \alpha_+$, $\alpha^*_- \geq \alpha_-$ and by the monotonicity properties of G

$$\frac{\beta^*_+}{\alpha^*_+} = \frac{\beta_+}{\alpha_+} = G(\overline{\nu}, \alpha_+, \alpha_-) \geq G(\overline{\nu}, \alpha^*_+, \alpha^*_-).$$

Let $M_\pm := \max_{R^{\pm}_1} v^{\pm}$ and $m_\pm := \min_{\Sigma^{\pm}_1} v^{\pm}$ where $\Sigma^{\pm} : \{x_n = \pm\frac{2}{3}\} \cap \partial R^{\pm}_1$. Since on $\partial R^{\pm}_1$ v is equal to $\overline{v}$, $\frac{M_+}{m_+}$ and $\frac{M_-}{m_-}$ should be controlled above by a constant depending on n, L, M, λ_1 and λ_2. This means that after an appropriate rescaling $v^{\pm}$ satisfy the hypothesis of Theorem 2.4. In particular, there exists a δ-neighborhood of $\mathcal{F}(v)$ in $R_{7/8}$ where v is monotone increasing along the directions of a space-time cone $\Gamma(\overline{\theta}, e_n)$ for $\overline{\theta} = \overline{\theta}(n, L, \lambda_1, \lambda_2)$. Actually, u is monotone increasing in all of $R_{4/5}$ for some cone of directions.

Lemma 4.3. *Let* $\gamma'' < \gamma'/4$. *Then in* $R_{4/5}$ v *is monotone increasing along any direction* $\tau \in \Gamma_x(\theta^x - c\varepsilon^{\gamma''}, e_n) \cup \Gamma_t(\theta^t - c\varepsilon^{\gamma''}, \nu)$ *where* $c = c(n, L, \lambda_1, \lambda_2)$.

Proof. As before we have that v^+ is monotone increasing along any direction $\tau \in \Gamma_x(\theta^x - c\varepsilon^{1-\gamma'}, e_n) \cup \Gamma_t(\theta^t - c\varepsilon^{1-\gamma'}, \nu)$. Therefore, for any $p \in R^+_{7/8}$,

$$D_\tau v^+(p) \geq z(p),$$

where $z(p)$ is the solution of the problem

$$\begin{cases} \Delta z - z_t = 0, & \text{in } R^+_{7/8} \setminus \mathcal{F}(v) \\ z = -|D_\tau v^+|, & \text{on } \partial_p R^+_{7/8} \setminus \mathcal{F}(v). \end{cases}$$

If $d_p := dist(p, \mathcal{F}(v)) \leq \delta$ and since by Theorem 2.5 $v^+(p) \leq c d_p D_{e_n} u^+$, following the proof of Lemma 2.5 of [9] we have

$$D_\tau v^+(p) + \varepsilon^{\gamma''} D_{e_n} v^+(p) \geq (\varepsilon^{\gamma''} - c d_p) D_{e_n} v^+(p) \geq 0,$$

for $\gamma'' < \frac{\gamma'}{4}$ provided that $d_p \leq c^{-1} \varepsilon^{\gamma''}$.

Now, if $d_p \geq c^{-1}\varepsilon^{\gamma''}$ we consider $u_\varepsilon(p) := u(p + 4\varepsilon^{\gamma''} e_n)$. Then by construction, $\mathcal{F}(u_\varepsilon)$ is at a distance greater than $3\varepsilon^{\gamma'}$ from R^+_1.

On $\partial_p R^+_1$ we have

$$|\overline{v} - u_\varepsilon| \leq c(n, L, M, \lambda_1, \lambda_2)\varepsilon^{\gamma'} D_{e_n} u_\varepsilon,$$

and by the maximum principle

$$|v^+ - u_\varepsilon| \leq c\varepsilon^{\gamma'} D_{e_n} u_\varepsilon.$$

Furthermore, if $d_p \geq c^{-1}\varepsilon^{\gamma''}$, $p \in R^+_{4/5}$,

$$|\nabla v^+ - \nabla u_\varepsilon| \leq c\varepsilon^{\gamma' - 2\gamma''} D_{e_n} u_\varepsilon \leq c\varepsilon^{\gamma''} D_{e_n} u_\varepsilon,$$

since $\gamma'' < \gamma'/4$. Therefore,

$$D_\tau v^+ - D_\tau u_\varepsilon \geq -c\varepsilon^{\gamma''} D_{e_n} u_\varepsilon,$$

or $D_\tau v^+ \geq 0$ if $\tau \in \Gamma_x(\theta^x - c\varepsilon^{\gamma''}, e_n) \cup \Gamma_t(\theta^t - c\varepsilon^{\gamma''}, \nu)$. The proof for v^- is similar. □

At this point we are able to apply the method prescribed in [1] to get interior gain process in time. Let $e_t + Be_n$ and $-e_t - Ae_n$ be the vectors of the generatrix of the cone $\Gamma_t(\theta^t - c\varepsilon^{\gamma''}, \nu)$. If $\mu = \frac{\pi}{2} - \theta^t$ is the defect angle in time and $\varepsilon^{\gamma''} < \mu$ then $B - A \leq c(\mu + \varepsilon^{\gamma''}) \leq C\mu$. In every strip $R_{4/5}(p_0)$ centered at a point $p_0 \in \mathcal{F}(v)$ the cone of monotonicity of v can be increased in time if we stay away from the free boundary. On the other hand,

$$|D_t v^+ - D_t u^+| \leq c\varepsilon^{\gamma'} D_{e_n} u^+ \quad \textit{and} \quad |D_{e_n} v^+ - D_{e_n} u^+| \leq c\varepsilon^{\gamma'} D_{e_n} u^+.$$

More precisely, we have proved the following.

Lemma 4.4. *If $G(\alpha_+, \alpha_-, e_n) \geq -\frac{1}{2}A + B$ (or $G(\alpha_+, \alpha_-, e_n) \leq -\frac{1}{2}A + B$), where $\alpha_\pm = D_{e_n}u^\pm(\frac{3}{4}e_n)$, then there exist constants $c, C > 0$ such that for δ (the defect angle in space) small enough, $\delta \leq c\mu^3$,*

$$-D_t u^+ \leq (B - c\mu)D_{e_n}u^+, \quad \Big(D_t u^+ \leq -(A + c\mu)D_{e_n}u^+\Big),$$

for all $(x,t) \in B_{1/8}(\frac{3}{4}e_n) \times (-C\delta/\mu, C\delta/\mu)$.

5. Improvement of ε-monotonicity.

In this section, we are using the results we have so far to prove that full monotonicity away from the free boundary yields an improvement of the ε-monotonicity on the free boundary. But, first we need a control on the normal derivative at a contact point. The elliptic analog of the following lemma is due to P. Y. Wang (see [14]).

Lemma 5.1. *Let u be a viscosity solution to our free boundary problem and consider a subsolution w defined in Theorem 2.5 for ϑ small. Suppose that there exists an n-dimensional ball B, with radius of order ε, tangent to the Lipschitz free boundary at the origin. Then there exists a small positive constant k_0 such that $w_\nu(0) \leq c\varepsilon^{k_0}$ at the t_0-time level.*

Proof. Let H be the radial positive solution of

$$\begin{cases} F(D^2H) = 0 & \text{in } \; B^2 \setminus B \\ H = 0 & \text{on } \; \partial B \\ H = 1 & \text{on } \; \partial B^2, \end{cases}$$

where B^2 has radius twice the radius of B and the same center. Let w_0 be the positive solution which is above w in $B^2 \setminus B$ and attains the same boundary data. Then in $\Omega^-(u) \cap (B^2 \setminus B)$

$$w \leq (\sup_{B^2} w_0)H \leq C\frac{\sup_{B^2} w_0}{\varepsilon},$$

and since $w(0) = H(0) = 0$,

$$w_\nu(0) \leq (\sup_{B^2} w_0)H_\nu \leq C\frac{\sup_{B^2} w_0}{\varepsilon}.$$

Now, if we keep d away from the free boundary and the boundary of the domain, the Harnack inequality gives

$$w_0 \leq d^{-l}C(\sup_{\partial\Omega}|w_0|)\varepsilon^{\frac{1}{2}},$$

where $C > 1$ and l are large. Therefore, if $d > \varepsilon^{\frac{1}{4l}}$ we have

$$w_0 \leq C(\sup_{\partial\Omega} |w_0|)\varepsilon^{\frac{1}{4}}.$$

Now, take a cone $\Gamma = \Gamma(\theta_0, e_n)$ with vertex zero on $\mathcal{F}' = \mathcal{F}_{t_0} + 2\varepsilon$, where $\mathcal{F}_{t_0} := \{(x, t_0) \in \mathcal{F} : |x| > 1\}$ and define a function g with $F(D^2 g) = 0$ below the cone and $g = 0$ on $\partial\Gamma$. Then $g(x) \leq C(dist\{x, 0\})^{1-\delta}$ where δ depends only on the opening θ_0 and can be as small as we need.

Next, consider $\mathcal{F}_+ := \mathcal{F}_{t_0} + de_n$, $\mathcal{F}_- := \mathcal{F}_{t_0} - de_n$. Then

$$w_0 \leq C(\sup |w_0|)\varepsilon^{\frac{1}{4}}$$

on $\mathcal{F}_-$. Define $\tilde{w}$ to be the solution of $F(D^2\tilde{w}) = 0$ in $Q_{1-d} \cap D_\pm$, where $D_\pm$ is the domain between $\mathcal{F}_+$ and $\mathcal{F}_-$ and $\tilde{w} = 0$ on $\mathcal{F}_+ \cup \mathcal{F}_-$. Since $(w_0 - C\varepsilon^{1/4} g)^+ = 0$ on $\mathcal{F}_-$,

$$(w_0 - C\varepsilon^{1/4} g)^+ \leq (\sup |w_0|)\tilde{w}$$

in $D_\pm$. On the other hand, since the free boundary is Lipschitz, there exists a constant m such that $\tilde{w} \leq Cm^{(1-|x|)/d} \leq C\varepsilon^2$ for ε small and $|x| \leq 9/10$, therefore,

$$(w_0 - C\varepsilon^{1/4} g)^+ \leq C(\sup |w_0|)\varepsilon^2.$$

Collecting all the estimates we have for $|x| \leq 9/10$, if we stay 2ε-close to $\mathcal{F}_{t_0}$,

$$\begin{aligned} w_0 &\leq (w_0 - C\varepsilon^{1/4} g)^+ + C\varepsilon^{1/4} g \\ &\leq C(\sup |w_0|)\varepsilon^2 + C\varepsilon^{1/4} C(\sup |w_0|)\varepsilon^{1-\delta} \leq C(\sup |w_0|)\varepsilon^{\frac{5}{4}-\delta}. \end{aligned}$$

We conclude the proof by taking $k_0 = \frac{1}{4} - \delta > 0$. □

The full monotonicity away from the free boundary yields a gain in the ε-monotonicity on the free boundary. This can be achieved by means of continuous family of perturbations as in [9], [10]. First, we introduce the following notation:

$\mathcal{F} := \{x_n = f(x', t)\}$ is the free boundary

$\Upsilon_{b,r,T} := \{(x', x_n, t) : dist\{(x', x_n, t), \mathcal{F}\} < b\varepsilon^\gamma\} \cap \{|x'| < r\} \cap \{|t| < T\}$

Q is a smooth domain such that $\Upsilon_{b/2,r,T} \subset Q \subset \Upsilon_{b,r,T}$.

Lemma 5.2. *Let b_1, b_2, b_3, C_0 be positive constants; then there exists a family of C^2 functions $\psi_\eta := \psi_\eta(x, t)$, $\eta \in [0, 1]$ in Q such that*

(i) $0 < \psi_\eta \leq 1 + \eta$,

(ii) $\mathcal{M}^-(D^2\psi_\eta) - b_1 D_t\psi_\eta - b_2 \frac{|\nabla_x \psi_\eta|^2}{\psi_\eta} - b_3 |\nabla_x \psi_\eta| \geq 0$,

(iii) $|\nabla_x \psi_\eta| \leq CC_0\varepsilon^{\beta-\gamma}$, $0 < \beta < \gamma < 1$,

(iv) $D_t\psi_\eta \geq 0$,

(v) $\psi_\eta \leq 1$ *in* $Q \cap \Big(\{-T < t < -T + \varepsilon^\alpha\} \cup \{r - \frac{\varepsilon^{\alpha/4}}{2} < |x'| < r\}\Big)$,

(vi) $\psi_\eta \geq 1 + \eta(1 - C\varepsilon^\beta)$ *in* $Q \cap \Big(\{t > -T + 2\varepsilon^\alpha\} \cup \{|x'| < r - \frac{\varepsilon^{\alpha/4}}{2}\}\Big)$ *for* $0 < \alpha < \gamma - \beta$.

Lemma 5.3. *Let* u *be as before,* $\varepsilon > 0$, $\sigma << \lambda < 1$ *and let*

$$v_\eta(p) := \sup_{B_{\sigma\psi_\eta}(p)} u(q - \lambda\varepsilon\tau),$$

where ψ_η *are the auxiliary functions of the previous lemma. Then* v_η *satisfies:*

(i) v_η *is* F*-subsolution in* $\Omega^+(v_\eta)$ *and in* $\Omega^-(v_\eta)$.

(ii) $\mathcal{F}(v_\eta)$ *is uniformly Lipschitz with Lipschitz constant* $L' \leq L + C\varepsilon^{1-\alpha}$.

(iii) Let $(x_0, t_0) \in \mathcal{F}(v_\eta)$ *and* $(y_0, s_0) \in \mathcal{F}(u) \cap \partial B_{\sigma\psi_\eta}(x_0, t_0)(x_0, t_0)$*; then* (x_0, t_0) *is a regular point from the right and if, near* (y_0, s_0) *at* s_0*-level,* u *has asymptotic development*

$$u(y, s_0) = \alpha_+ < y - y_0, \nu >^+ - \alpha_- < y - y_0, \nu >^- + o(|y - y_0|)$$

where $\nu = (y_0 - x_0)/|y_0 - x_0|$*, then near* (x_0, t_0) *at* t_0*-level*

$$v_\eta(x, t_0) \geq \alpha_+ \langle x - x_0, \nu + \frac{\sigma\psi_\eta(x_0, t_0)}{|y_0 - x_0|}\nabla_x(\sigma\psi_\eta)\rangle^+$$
$$-\alpha_- \langle x - x_0, \nu + \frac{\sigma\psi_\eta(x_0, t_0)}{|y_0 - x_0|}\nabla_x(\sigma\psi_\eta)\rangle^- + o(|x - x_0|).$$

We refer the reader to [9] for the proofs of Lemma 5.2 and Lemma 5.3. Using now this family of subsolutions we obtain a gain in ε-monotonicity on the free boundary. Then ε can be decreased enough so that the hypotheses of the main inductive argument are fulfilled.

Let u be a viscosity solution to our free boundary problem which is ε-monotone along any direction in $\Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$. If the defect angle in space is very small (smaller than the defect angle in time) and $\varepsilon << \delta$ then we have

$$\sup_{B_{\varepsilon \sin\delta}(p)} u(q - \varepsilon\tau) \leq u(p),$$

for $p \in B_{1-\varepsilon} \times (-1 + \varepsilon, 1 - \varepsilon)$ and any $\tau \in \Gamma_x(\theta^x - \delta, e_n) \cup \Gamma_t(\theta^t - \mu, \nu)$, $|\tau| = 1$. Since for any $\lambda < 1$ close enough to 1

$$B_\tau(p - \lambda\varepsilon\tau) \subset B_{\varepsilon\sin\delta}(p - \varepsilon\tau),$$

it follows that

$$\sup_{B_\sigma(p)} u(q - \lambda\varepsilon\tau) \leq u(p),$$

for $\sigma = \varepsilon(\sin\delta - 1 + \lambda)$. On the other hand for $0 < \lambda \leq 1$

$$\sup_{B_{\lambda\varepsilon\sin\delta}(p)} u(q - \lambda\varepsilon\tau) \leq u(p),$$

since u is fully monotone in $\Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$, not in the ε^γ-neighborhood $(0 < \gamma \leq 1/2)$ of the free boundary, but for a small error of order ε in space and $\sqrt{\varepsilon}$ in time, which we are allowed to omit since $\varepsilon << \delta$.

Lemma 5.4. *Let Q be a smooth domain and α, β, γ be such that $\Upsilon_{b/2,r,T} \subset Q \subset \Upsilon_{b,r,T}$ with*

$$\Upsilon_{b,r,T} := \{(x', x_n, t) : dist\{(x', x_n, t), \mathcal{F}\} < b\varepsilon^\gamma\}$$
$$\cap\{|x'| < r - c_1\varepsilon^\alpha\} \cap \{|t| < T - c_2\varepsilon^\alpha\}.$$

Suppose that

$$w_1(p) := \sup_{B_{l_1}(p)} u(q - \lambda\varepsilon\tau) \leq u(p),$$

for $p \in Q$, $l_1 := \varepsilon(\lambda\sin\delta - c\varepsilon^\beta)$ and $\tau \in \Gamma_x(\theta^x - \delta, e_n) \cup \Gamma_t(\theta^t - \mu, \nu)$. Then on ∂Q ε^γ-away from the free boundary

$$w_1(p) \leq (1 - c\varepsilon^{1+\beta-\gamma})u(p).$$

Proof. Consider the function $w := \sup_B u(q - \lambda\varepsilon\tau)$ where B is a ball with radius $\varepsilon\lambda\sin\delta$ and center at p. Then using Theorem 2.5 we have

$$w_1(p) \leq w(p) - C\varepsilon^{1+\beta}|\nabla u(p)| \leq u(p) - c\varepsilon^{1+\beta-\gamma}u(p),$$

for $p \in Q$ and $dist\{p, \mathcal{F}\} \geq \varepsilon^\gamma$. □

Now, in order to gain in ε-monotonicity on the free boundary we only need to find an intermediate radius $\sigma\psi_\eta$ where ψ_η is the auxiliary function of Lemma 5.2. More precisely, we have the following.

Lemma 5.5. *Let u be a viscosity solution to the free boundary problem in $D_{r,T} := B_r(0) \times (-T, T)$ which is ε-monotone along any direction in the cone $\Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$. Suppose that δ, μ, the defect angles in space and time, satisfy $\delta << \mu^3$, δ very small. Then there exists an $\varepsilon_0 > 0$, $\varepsilon_0 << \delta$ and $0 < \lambda < 1$ such that if $\varepsilon < \varepsilon_0$, u is $\lambda\varepsilon$-monotone in $\Gamma_x(\theta^x - c\varepsilon^\beta, e_n) \cup \Gamma_t(\theta^t - c\varepsilon^\beta, \nu)$, in the domain $D_{r-\varepsilon^\alpha, T-\varepsilon^\alpha} \cap Q$ where Q is as in Lemma 5.4 and $0 < \alpha < \beta < 1/2$.*

Proof. Choose $\overline{\eta}$ such that $\sigma(1 + \overline{\eta}) = \lambda\varepsilon\sin\delta - c\varepsilon^{1+\beta}$, where $\sigma = \varepsilon(\sin\delta + \lambda - 1)$ and for simplicity let $\gamma = \max\{\gamma, k_0\}$. If $1 - \lambda = \frac{1}{2}\sin\delta$ then $\frac{1}{3} < \overline{\eta} < 1$ since $0 < \varepsilon << \delta << 1$. Set

$$\overline{v}_\eta := v_\eta + c\varepsilon^{1+\beta-\gamma}v,$$

where v_η is the function defined in Lemma 5.3 and v solves

$$\begin{cases} F(D^2v) - v_t = 0 & in \;\; \Omega^+(u) \cap Q \\ v = u & \partial_p Q \cap \{p : dist(p, F(u)) > \varepsilon^\gamma\} \\ v = 0 & everywhere \;\; else \;\; in \;\; Q. \end{cases}$$

It is enough to show that, for every $\eta \in [0, \overline{\eta}]$, $\overline{v}_\eta \leq u$ in $D_{r-\varepsilon^\alpha, T-\varepsilon^\alpha} \cap Q$.

We make use of the maximum principle and the previous lemma to get that the closed set $\{\eta \in [0, \overline{\eta}] : \overline{v}_\eta \leq u\}$ is nonempty. To show that it is open it is enough to prove that $Q \cap \{\overline{v}_{\eta_0} > 0\}$ is compactly contained in $Q \cap \{u > 0\}$ if $\overline{v}_{\eta_0} \leq u$ for $\eta_0 \in [0, \overline{\eta})$. Suppose not; then there exists a regular point $(x_0, t_0) \in \mathcal{F}(\overline{v}_{\eta_0}) \cap \mathcal{F}(u)$ with distance from the parabolic boundary of Q greater than $c\varepsilon^\gamma$.

It is not hard to show that $\overline{v}_{\eta_0}$ has asymptotic behavior

$$\overline{v}_\eta(x, t_0) \geq \overline{\alpha}_+ < x - x_0, \nu >^+ - \alpha_- < x - x_0, \nu >^- + o(|x - x_0|),$$

where

$$\overline{\alpha}_+ \geq \alpha_+(1 - \sigma|\nabla_x(\psi_{\eta_0}|) + c\varepsilon^{1+\beta-\gamma},$$

with

$$\nu = \frac{(y_0 - x_0 + \sigma^2\psi_\eta \nabla_x \psi_\eta)/|y_0 - x_0|}{|(y_0 - x_0 + \sigma^2\psi_\eta \nabla_x \psi_\eta)/|y_0 - x_0||},$$

and

$$u(x, t_0) = \alpha'_+ < x - x_0, \nu >^+ - \alpha'_- < x - x_0, \nu >^- + o(|x - x_0|),$$

near the contact point (x_0, t_0) at t_0-time level with

$$G(\nu, \alpha'_+, \alpha'_-) \leq G(\frac{y_0 - x_0}{|y_0 - x_0|}, \alpha_+, \alpha_-) + C\delta\varepsilon^{1-\alpha}.$$

Using the Lipschitz continuity of G and since D_aG, D_bG are bounded below and above respectively we have

$$G(\nu, \alpha'_+, \alpha'_-) \leq G(\nu, \overline{\alpha}_+, \alpha_-) - C'\varepsilon^{1+\beta} \leq G(\nu, \overline{\alpha}_+, \alpha_-),$$

which gives us a contradiction since by Hopf's principle it should be $\alpha'_+ > \overline{\alpha}_+$ and $\alpha'_- < \alpha_-$. □

Now take $\varepsilon = \varepsilon_k = 2^{-k}$ with k large, $\delta = \delta = ck^{-1-\alpha}$ for some $\alpha > 0$ and $\lambda_k = 1 - \frac{1}{2}\sin(\delta_k)$. Iterate Lemma 5.5 m-times to get $\lambda_k^{-m} \leq \frac{1}{2}$ and choose new α_1, β_1 such that

$$\sum_m c\varepsilon_k^\beta \lambda_k^{m\beta} \leq c\varepsilon_k^\beta \frac{1}{1 - \lambda^{\beta_k}} \leq c\varepsilon_{k+1}^{\beta_1},$$

and

$$\sum_m c\varepsilon_k^{\alpha}\lambda_k^{m\alpha} \leq c\varepsilon_k^{\alpha}\frac{1}{1-\lambda^{\alpha_k}} \leq c\varepsilon_{k+1}^{\alpha_1}.$$

More precisely, we have proved the following.

Corollary 5.6. *There exist* α_1, β_1 *such that* u *is* ε_{k+1}*-monotone in the domain* $D_{r-\varepsilon_{k+1}^{\alpha_1},T-\varepsilon_{k+1}^{\alpha_1}}$ *for every direction in* $\Gamma_x(\theta^x - c\varepsilon_{k+1}^{\beta_1}, e_n) \cup \Gamma_t(\theta^t - c\varepsilon_{k+1}^{\beta_1}, \nu)$.

6. Final iteration

As in [2], [7] the gain of full monotonicity yields only an enlarged cone of ε-monotonicity on the free boundary. First, we need to propagate our estimates to the free boundary. This can be achieved by the use of a powerful topological method first introduced by Caffarelli (see [4]).

Lemma 6.1. *Let* $u_1 \leq u_2$ *be two viscosity solutions in* Q_2 *and* $\mathcal{F}(u_2)$ *be Lipschitz through the origin. Assume that*

$$v_\varepsilon(x,t) := \sup_{B_\varepsilon(x,t)} u_1 \leq u_2(x,t)$$

in $B_1 \times (-T,T)$*, and for* h *small,*

$$u_2(x,t) - v_{(1+h\sigma)\varepsilon}(x,t) \geq C\sigma\varepsilon u_2(\tfrac{3}{4}e_n, 0),$$

in $B_{1/8}(\frac{3}{4}e_n) \times (-T,T) \subset \{u_2 > 0\}$*. Then for* $\varepsilon > 0$*,* $h > 0$ *small there exists a constant* c*,* $0 < c < 1$ *such that* $v_{(1+ch\sigma)\varepsilon}(x,t) \leq u_2(x,t)$ *in* $B_{1/2} \times (-\frac{1}{2}T, \frac{1}{2}T)$*.*

Proof. Let $0 < m_0 \leq T$ and C be large enough. Then as in Lemma 3.5 of [9], we can find constants $\bar{C}$, k, $h_0 > 0$ depending on m_0 and C, such that, for any $h \in (0, h_0)$ there exists a family of functions $\phi_\eta \in C^2$, $0 \leq \eta \leq 1$, defined in $D := [B_1 \setminus \{\bar{B}_{1/8}(\frac{3}{4}e_n)] \times (-T,T)$ such that

(*i*) $1 \leq \phi_\eta \leq 1 + \eta h$ in D,

(*ii*) $\mathcal{M}^-(D^2\phi_\eta) - c_1 D_t\phi_\eta - C\frac{|\nabla_x\phi_\eta|^2}{\phi_\eta} - c_2|\nabla_x\phi_\eta| \geq 0$ in D,

(*iii*) $\phi_\eta \equiv 1$ outside $B_{8/9} \times (-\frac{7}{8}T, T)$,

(*iv*) $\phi_\eta \geq 1 + k\eta h$, for $(x,t) \in B_{1/2} \times (-\frac{T}{2}, \frac{T}{2})$,

(*v*) $|\nabla\phi_\eta| \leq \bar{C}\eta h$ and $D_t\phi_\eta \geq 0$ in D, provided that c_1, c_2 are sufficiently small.

Let

$$\overline{v}_\eta(x,t) := v'_\eta(x,t) + C\sigma\varepsilon w(x,t),$$

where $v'_\eta(p) = \sup_{B_{\varepsilon\phi_{\sigma\eta}}(p)} u_1$ and w is the solution of the problem

$$\begin{cases} F(D^2w) - D_t w = 0 & in \;\; D \cap \{u_2 > 0\}, \\ w = 0 & in \;\; D \cap \overline{\{u_2 \le 0\}}, \\ w = 0 & on \;\; \partial_p D \setminus \{\partial B_{1/8}(\frac{3}{4}e_n) \times (-\frac{9}{10}T, \frac{9}{10}T)\}, \\ w = u_2(\frac{3}{4}e_n, 0) & on \;\; \partial B_{1/8}(\frac{3}{4}e_n) \times (-\frac{9}{10}T, \frac{9}{10}T), \end{cases}$$

where $D := \left[B_{\frac{9}{10}}(0) \setminus B_{\frac{1}{8}}(\frac{3}{4}e_n)\right] \times \left(-\frac{9}{10}T, \frac{9}{10}T\right)$. On the other hand, our continuous family of functions $\overline{v}_\eta$ satisfies $\overline{v}_\eta \le u_2$ for $\eta \in [0,1]$ and $\overline{v}_1 \ge v_(1 + ch\sigma)\varepsilon$ in $B_{1/2} \times (-\frac{1}{2}T, \frac{1}{2}T)$ since, as in Lemma 5.5, the corresponding set of η's, H=$\{\eta \in [0,1] : \overline{v}_\eta \le u_2\}$ is both open and closed. □

Now, if we take any vector $\tau \in \Gamma_x(\theta^x - \delta, e_n)$ with $|\tau| << \delta$, $\varepsilon = |\tau| \sin\delta$ and define $u_1(x,t) := u(x - \tau, t)$ we enlarge the cone in space as in the nondegenerate case. In a similar fashion we achieve a gain in time by taking $\tau \in \Gamma_x(\theta^t - \delta, \nu)$. We state this observation as the basic iteration lemma.

Lemma 6.2. *Let u be a viscosity solution in $B_1 \times (-1,1)$, monotone in $\Gamma_x(\theta^x, e_n) \cup \Gamma_t(\theta^t, \nu)$. Suppose that $\delta := \frac{\pi}{2} - \theta^x << \mu := \frac{\pi}{2} - \theta^t$ and $\varepsilon << \delta$. Then there exist positive constants c', c'', C such that in $B_{1/2} \times (-C\delta/2\mu, C\delta/2\mu)$ u is ε-monotone in $\Gamma_x(\theta_1^x, e_n) \cup \Gamma_t(\theta_1^t, \nu_1)$ with*

$$\frac{\pi}{2} - \theta_1^x := \delta_1 \le \delta - c'\frac{\delta^2}{\mu}, \quad \frac{\pi}{2} - \theta_1^t := \mu_1 \le \mu - c''\delta.$$

Proof of Theorem 2.7. Let $\delta_k = \frac{1}{(k+c)^{4/3}}$ for c large enough, $\mu_k = \frac{1}{k^{1/3}}$ and $\varepsilon_k = 16^{-k}$. Therefore we get u to be ε_k-monotone in $\Gamma_x(\theta_k^x, e^k) \cup \Gamma_t(\theta_k^t, \nu_k)$ in the domain $B_{2^{-k}} \times (-\frac{C\delta_k}{2^k\mu_k}, \frac{C\delta_k}{2^k\mu_k}$. Consider the function

$$u_k(x,t) := 2^k u(2^{-k}x, 2^{-k}t),$$

which is now $2^k\varepsilon_k$-monotone in $\Gamma_x(\theta_k^x, e^k) \cup \Gamma_t(\theta_k^t, \nu_k)$ and fully monotone $2^{k\gamma}\varepsilon_k^\gamma$ away from the free boundary. Now, from Corollary 5.6 u_k is $2^{k+1}\varepsilon_{k+1}$-monotone in

$$\Gamma_x(\theta_k^x - c2^{\beta'(k+1)}\varepsilon_{k+1}^{\beta'}, e^k) \cup \Gamma_t(\theta_k^t - c2^{\beta'(k+1)}\varepsilon_{k+1}^{\beta'}, \nu_k).$$

Finally, if we apply the basic iteration lemma, Lemma 6.2, we have in $B_{1/2} \times (-\frac{C\delta_k}{2\mu_k}, \frac{C\delta_k}{2\mu_k})$ that u_{k+1} is $2^{k+1}\varepsilon_{k+1}$-monotone in

$$\Gamma_x(\theta_{k+1}^x, e^{k+1}) \cup \Gamma_t(\theta_{k+1}^t, \nu_{k+1}),$$

where

$$\frac{\pi}{2} - \theta^x_{k+1} := \delta_{k+1} \le \delta_k - c' \frac{\delta_k^2}{\mu_k} - c2^{\beta'(k+1)}\varepsilon_{k+1},$$
$$\frac{\pi}{2} - \theta^t_{k+1} := \mu_{k+1} \le \mu_k - c''\delta_k - c2^{\beta'(k+1)}\varepsilon_{k+1},$$

and the proof is completed.

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