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GLOBAL NONEXISTENCE OF POSITIVE INITIAL-ENERGY SOLUTIONS OF A SYSTEM OF NONLINEAR WAVE EQUATIONS WITH DAMPING AND SOURCE TERMS

Belkacem Said-Houari
Laboratoire de Mathématiques Appliquées, Université Badji Mokhtar
B.P. 12 Annaba 23000, Algérie

(Submitted by: Reza Aftabizadeh)

Abstract. This work is concerned with a system of nonlinear wave equations with nonlinear damping and source terms acting in both equations. We will prove that the solution of our considered system blows up in finite time provided that the initial data are large enough. This result extends a previous result in [1] to a large class of initial data. The key ingredient in the proof is a method used in [25] with necessary modifications imposed by the nature of our problem.

1. Introduction

A single wave equation similar to the following form:

$$u_{tt} - \Delta u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ and a and b are positive constants, together with initial and boundary conditions of Dirichlet type, has been extensively studied and results concerning existence, blow up and asymptotic behavior of smooth, as well as weak solutions have been established by several authors over the past three decades. See in this regard [4, 26, 25, 17, 15, 16, 8, 7, 3] and references therein.

For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial data (see [6], [11]).
For $a = 0$, the source term causes finite-time blow-up of solutions with a large initial data (negative initial energy) (see [2] and [10]). The interaction between the damping term $a\,|u_t|^{m-2}\,u_t$ and the source term $b\,|u|^{p-2}\,u$ makes

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Current Address: Institut de Mathématiques de Toulouse, MIP , Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex 09, France.

the problem more interesting. This situation was first considered by Levine [15, 16] in the linear damping case ($m = 2$), where he showed that solutions with negative initial energy blow up in finite time. The main tool used in [15] and [16] is the "concavity method" where the basic idea of this method is to construct a positive defined functional $\theta(t)$ of the solution and show that $\theta^{-\alpha}(t)$ is a concave function of t, for some $\alpha > 0$. Unfortunately, this method fails in the case of a nonlinear damping term ($m > 2$).

Georgiev and Todorova, in [5], extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (1.1) and introduced a method different from the one known as the concavity method. They showed that solutions continue to exist globally "in time" if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting and to unbounded domains by Levine and Serrin [13] and Levine and Park [12]. In [13], the authors showed that no solution with negative energy can be extended on $[0, \infty)$ if $p > m$ and proved several noncontinuation theorems. This generalization allowed them also to apply their result to quasi-linear situations, of which the problem in [5] is a particular case. Vitillaro in [25] combined the arguments in [5] and [13] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy. Similar results have also been established by Todorova [22], [23] and Levine and Park [12] for different Cauchy problems. We recall here that the lack of the Poincaré inequality in $\mathbb{R}^n$ renders the problem more difficult; for this reason people usually used the finite speed of propagation property of solutions. See [21] for more details. We mention also the work by Levine and Todorova [14] in which the authors considered the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, & x \in \mathbb{R}^n,\ t > 0 \\ u(0,x) = u_0(x)\,,\ u_t(0,x) = u_1(x)\,, & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

and showed that, for any $\lambda \geq 0$, there are compactly supported data u_0, u_1 such that $E(0) = \lambda$ and the solution of (1.2) with these choices of initial data blows-up in finite time. We refer also to the work in [24].

Concerning the system of wave equations, Milla Miranda and Medeiros [18] considered the following system:

$$\begin{cases} u_{tt} - \Delta u + u - |v|^{\rho+2}|u|^{\rho}u = f_1(x) \\ v_{tt} - \Delta v + v - |u|^{\rho+2}|v|^{\rho}v = f_2(x), \end{cases} \tag{1.3}$$

in $\Omega \times (0,T)$. By using the method of potential well, the authors determined the existence of weak solutions of system (1.3). Some special cases of system (1.3) arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field. See [20] and [9].

In [1], Agre and Rammaha studied the following system of wave equations:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u,v) \\ v_{tt} - \Delta v + |v_t|^{r-1} u_t = f_2(u,v) \end{cases} \tag{1.4}$$

in $\Omega \times (0,T)$ with initial and boundary conditions of Dirichlet type and the nonlinear functions $f_1(u,v)$ and $f_2(u,v)$ satisfying appropriate conditions. They proved under some restrictions on the parameters and the initial data several results on local existence and global existence of a weak solution. They also showed that any weak solution with negative initial energy blows up in finite time. To prove this later result the authors used the same techniques as in [5]. In this paper, we consider the same problem treated by Agre and Rammaha [1], and we improve the blow up result obtained in [1], for a large class of initial data in which our initial energy can take positive values. The main tool of the proof is a technique introduced by Georgiev and Todorova [5] combined with a potential well method of Payne and Sattinger [19] developed by Vitillaro [25].

2. Preliminaries

In this paper, we consider the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r-1} u_t = f_2(u,v), \end{cases} \tag{2.1}$$

where $u = u\,(t,x)\,,\ v = v\,(t,x)\,,\ x \in \Omega$, Δ denotes the Laplacian operator with respect to the x variable, and Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$.

First, let us introduce some notation used throughout this paper. By $\|.\|_q$ we denote the usual $L^q(\Omega)$-norm and for $\varphi \in H_0^1(\Omega)$, we denote by $\|\nabla\varphi\|_2$ the norm of φ in $H_0^1(\Omega)$ which is equivalent to the $H^1(\Omega)$-norm. Furthermore, we set

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx,$$

as the usual $L^2(\Omega)$-inner product.

Problem (2.1) may be completed by the following initial and boundary conditions:

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), \; x \in \Omega \tag{2.2}$$

$$u(x) = v(x) = 0, \; x \in \partial\Omega. \tag{2.3}$$

Concerning the functions $f_1(u, v)$ and $f_2(u, v)$ we assume that

$$\begin{aligned} f_1(u, v) &= [a|u + v|^{2(\rho+1)}(u + v) + b|u|^\rho u|v|^{(\rho+2)}] \\ f_2(u, v) &= [a|u + v|^{2(\rho+1)}(u + v) + b|u|^{(\rho+2)}|v|^\rho v], \end{aligned} \tag{2.4}$$

and

$$uf_1(u, v) + vf_2(u, v) = 2(\rho + 2)F(u, v), \quad \forall (u, v) \in \mathbb{R}, \tag{2.5}$$

where $F(u, v) = \frac{1}{2(\rho+2)}[a|u + v|^{2(\rho+2)} + 2b|uv|^{\rho+2}]$. Suppose that there exist two positive constants c_0 and c_1 such that

$$\frac{c_0}{2(\rho + 2)}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}) \leq F(u, v) \leq \frac{c_1}{2(\rho + 2)}(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}). \tag{2.6}$$

In this work, we will deal with the weak solution of the problem (2.1)-(2.3); consequently, we use the same definition as in [1].

Definition 2.1. *A pair of functions* (u, v) *is said to be a weak solution of* (2.1)-(2.3) *on* $[0, T]$ *if* $u, v \in C_w([0, T], H_0^1(\Omega))$, $u_t, v_t \in C_w([0, T], L^2(\Omega))$, $u_t \in L^{m+1}(\Omega \times (0, T))$, $v_t \in L^{r+1}(\Omega \times (0, T))$, $(u(0), v(0)) = (u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $(u_t(0), v_t(0)) = (u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$, *and* (u, v) *satisfies*

$$\begin{aligned} &\int_\Omega u'(t)\phi - \int_\Omega u_1\phi + \int_0^t \int_\Omega \nabla u(\tau)\nabla\phi d\tau dx + \int_0^t \int_\Omega |u^{'}|^{m-1}u^{'}\phi d\tau dx \\ &= \int_0^t \int_\Omega f_1(u(\tau), v(\tau))\phi d\tau dx, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\int_\Omega v'(t)\psi - \int_\Omega v_1\psi + \int_0^t \int_\Omega \nabla v(\tau)\nabla\psi d\tau dx + \int_0^t \int_\Omega |v^{'}|^{r-1}v^{'}\psi d\tau dx \\ &= \int_0^t \int_\Omega f_2(u(\tau), v(\tau))\psi d\tau dx, \end{aligned} \tag{2.8}$$

for all test functions $\phi \in H_0^1(\Omega) \cap L^{m+1}(\Omega), \psi \in H_0^1(\Omega) \cap L^{r+1}(\Omega)$ *and for almost all* $t \in [0, T]$.

For the sake of completeness, we state here the local existence result of [1].

Theorem 2.1. *Assume that the assumptions* (2.4)-(2.5) *hold. Suppose further that* $m, r \geq 1$, $\rho > 0$ *if* $n = 1, 2$, $\rho = 0$ *if* $n = 3$. *Then for any initial data* $u_0, v_0 \in H_0^1(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$, *there exists a unique local weak solution* (u, v) *of* (2.1)-(2.3) *defined in* $[0, T_0]$ *for some* $T_0 > 0$. *In addition, the solution satisfies the following energy identity:*

$$E(t) + \int_0^t (\|u_t(s)\|_{m+1}^{m+1} + \|v_t(s)\|_{r+1}^{r+1}) ds = E(0), \tag{2.9}$$

where

$$E(t) = \frac{1}{2}(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \int_\Omega F(u, v) dx. \tag{2.10}$$

3. Main Result

In this section, we will prove a global nonexistence result of the system (2.1)-(2.3), provided that our initial data are large enough. Our result restates [1], Theorem 1.6, for a wider class of initial data. Our techniques of proof follow very carefully the techniques used in [25]. To the author's knowledge, this is the first blow up result obtained for systems of wave equations and positive initial energy.

Let us now state our main result.

Theorem 3.1. *Suppose that* $-1 < \rho$ *if* $n = 1, 2$ *and* $-1 < \rho \leq (4-n)/(n-2)$ *if* $n \geq 3$. *Assume further that* $2(\rho+2) > \max(m+1, r+1)$. *Then any solution of* (2.1)-(2.3) *with initial data satisfying*

$$(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^{1/2} > \alpha_1 \quad \textit{and} \quad E(0) < E_1$$

cannot exist for all time, where the constants α_1 *and* E_1 *are defined in* (3.3).

Remark 3.1. The advantage of our Theorem 3.1 is in the fact that we can not apply Theorem 2 in [25], since our restriction on the initial data,

$$(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^{1/2} > \alpha_1,$$

in Theorem 3.1, is weaker than the conditions

$$\|u_0\|_{2(\rho+2)} + \|v_0\|_{2(\rho+2)} > \overline{\alpha}_1,$$

if we apply Theorem 2 of [25]. By the way, by setting

$$D = [H_0^1(\Omega)]^2, \qquad W = [L^{2(\rho+2)}(\Omega)]^2 \qquad V = [L^2(\Omega)]^2,$$

and

$$A(u, v) = (-\Delta u, -\Delta v), \quad F(u, v) = (f_1(u, v), f_2(u, v)), \quad P = \text{Identity}$$

in [25], Theorem 3.1 completely contains Theorem 2 in the reference [25].

The basic tool in our proof of Theorem 3.1 is the following lemma.

Lemma 3.1. *Let $L(t)$ be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t), \tag{3.1}$$

defined in $[0,\infty)$, where $\nu > 0$. If $L(0) > 0$, then the solution ceases to exist for $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$.

Now, we have the following result.

Lemma 3.2. *Assume that $-1 < \rho \leq (4-n)/(n-2)$ if $n \geq 3$. Then there exists $\eta > 0$ such that for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \leq \eta(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{\rho+2} \tag{3.2}$$

holds.

Proof. It is clear that by using the Minkowski inequality we get

$$\|u+v\|_{2(\rho+2)}^2 \leq 2(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2).$$

Also, Holder's and Young's inequalities give us

$$\|uv\|_{(\rho+2)} \leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)} \leq c(\|\nabla u\|_2^2 + \|\nabla v\|_2^2).$$

A combination of the last two inequalities and the embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ give us (3.2).

In order to state and prove our result, and for the sake of simplicity, we set $a = b = 1$. We introduce the following: Let

$$B = \eta^{1/2(\rho+2)}, \ \alpha_1 = B^{-(\rho+2)/(\rho+1)}, \ \ E_1 = \left(\frac{1}{2} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \tag{3.3}$$

where η is the optimal constant in (3.2).

The following lemma will play an essential role in the proof of our main result, and it is similar to a lemma used firstly by Vitillaro [25], in order to study some classes of a single wave equation.

Lemma 3.3. *Let (u,v) be a solution of (2.1)-(2.3). Suppose that $-1 < \rho$ if $n = 1,2$ and $-1 < \rho \leq (4-n)/(n-2)$ if $n \geq 3$. Assume further that $E(0) < E_1$ and*

$$(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)^{1/2} > \alpha_1. \tag{3.4}$$

Then there exists a constant $\alpha_2 > \alpha_1$ such that

$$(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)^{1/2} > \alpha_2, \tag{3.5}$$

and

$$[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}]^{1/(2(\rho+2))}\geq B\alpha_2, \quad \forall t\in[0,T). \tag{3.6}$$

Proof. We first note that, by (2.10) and the the definition of B, we have

$$\begin{aligned}E(t) &\geq \frac{1}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)-\frac{1}{2(\rho+2)}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2})\\ &\geq \frac{1}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)-\frac{B^{2(\rho+2)}}{2(\rho+2)}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)^{\rho+2}\\ &= \frac{1}{2}\alpha^2-\frac{B^{2(\rho+2)}}{2(\rho+2)}\alpha^{2(\rho+2)}=g(\alpha), \end{aligned}\tag{3.7}$$

where $\alpha=\left[\|\nabla u\|_2^2+\|\nabla v\|_2^2\right]^{1/2}$. It is not hard to verify that g is increasing for $0<\alpha<\alpha_1$, decreasing for $\alpha>\alpha_1$, $g(\alpha)\to-\infty$ as $\alpha\to+\infty$, and

$$g(\alpha_1)=\frac{1}{2}\alpha_1^2-\frac{B^{2(\rho+2)}}{2(\rho+2)}\alpha_1^{2(\rho+2)}=E_1,$$

where α_1 is given in (3.3). Therefore, since $E(0)<E_1$, there exists $\alpha_2>\alpha_1$ such that $g(\alpha_2)=E(0)$.

If we set $\alpha_0=\left[\|\nabla u(0)\|_2^2+\|\nabla v(0)\|_2^2\right]^{1/2}$, then by (3.7) we have $g(\alpha_0)\leq E(0)=g(\alpha_2)$, which implies that $\alpha_0\geq\alpha_2$.

Now, to establish (3.5), we suppose by contradiction that

$$\left[\|\nabla u(t_0)\|_2^2+\|\nabla v(t_0)\|_2^2\right]^{1/2}<\alpha_2,$$

for some $t_0>0$; by the continuity of $\|\nabla u(.)\|_2^2+\|\nabla v(.)\|_2^2$ we can choose t_0 such that

$$\left[\|\nabla u(t_0)\|_2^2+\|\nabla v(t_0)\|_2^2\right]^{1/2}>\alpha_1.$$

Again, the use of (3.7) leads to

$$E(t_0) \quad\geq\quad g\left(\|\nabla u(t_0)\|_2^2+\|\nabla v(t_0)\|_2^2\right)>g(\alpha_2)=E(0).$$

This is impossible since $E(t)\leq E(0)$, for all $t\in[0,T)$. Hence, (3.5) is established.

To prove (3.6), we make use of (2.10) to get

$$\frac{1}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)\leq E(0)+\frac{1}{2(\rho+2)}[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}].$$

Consequently, (3.5) yields

$$\frac{1}{2(\rho+2)}[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}] \geq \frac{1}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)-E(0)$$
$$\geq \frac{1}{2}\alpha_2^2 - E(0) \geq \frac{1}{2}\alpha_2^2 - g(\alpha_2) = \frac{B^{2(\rho+2)}}{2(\rho+2)}\alpha_2^{2(\rho+2)}. \tag{3.8}$$

Therefore, (3.8) and (3.3) yield the desired result.

Proof of Theorem 3.1. We suppose that the solution exists for all time and we reach a contradiction. For this purpose, a multiplication of the first equation in (2.1) by u_t, the second by v_t, integrating over Ω, using integration by parts, and addition of equalities yield

$$E'(t) = -\|u_t\|_{m+1}^{m+1} - \|v_t\|_{r+1}^{r+1} \leq 0, \quad \forall t > 0. \tag{3.9}$$

We then set

$$H(t) = E_1 - E(t). \tag{3.10}$$

By using (2.10) and (3.10) we get

$$0 < H(0) \leq H(t) = E_1 - \frac{1}{2}(\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2)$$
$$+\frac{1}{2(\rho+2)}[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}]. \tag{3.11}$$

From (2.10) and (3.5), we obtain, for all $t \geq 0$,

$$E_1 - \frac{1}{2}(\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2) < E_1 - \frac{1}{2}\alpha_1^2 = -\frac{1}{2(\rho+2)}\alpha_1^2 < 0.$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}[\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}], \quad \forall t \geq 0.$$

Then, by (2.6), we have

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(\rho+2)}(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}), \quad \forall t \geq 0. \tag{3.12}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega (u.u_t+v.v_t)(x,t)\,dx, \tag{3.13}$$

for ε small to be chosen later and

$$0 < \sigma \leq \min\Big(\frac{1}{2}, \frac{\rho+1}{2(\rho+2)}, \frac{2(\rho+2)-(m+1)}{2m(\rho+2)}, \frac{2(\rho+2)-(r+1)}{2r(\rho+2)}\Big). \tag{3.14}$$

Our goal is to show that $L(t)$ satisfies the differential inequality (3.1). This, of course, will lead to a blow up in finite time. By taking a derivative of (3.13) and using the equations (2.1) we obtain

$$L^{'}(t) = (1-\sigma)\, H^{-\sigma}(t)\, H^{'}(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) - \varepsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \varepsilon \int_{\Omega} (u|u_t|^{m-1}u_t dx + v|v_t|^{r-1}v_t)dx + \varepsilon \int_{\Omega} (uf_1(u,v) + vf_2(u,v))dx. \tag{3.15}$$

By exploiting (2.10) and (3.10), the equation (3.15) takes the form

$$\begin{aligned} L^{'}(t) = {} & (1-\sigma)\, H^{-\sigma}(t)\, H^{'}(t) + 2\varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) \\ & - \varepsilon \int_{\Omega} (u|u_t|^{m-1}u_t dx + v|v_t|^{r-1}v_t)dx \\ & + \varepsilon(1 - \frac{1}{\rho+2})(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) + 2\varepsilon H(t) - 2\varepsilon E_1. \end{aligned} \tag{3.16}$$

Then using (3.6) we obtain

$$\begin{aligned} L^{'}(t) \geq {} & (1-\sigma)\, H^{-\sigma}(t)\, H^{'}(t) + 2\varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) \\ & + \varepsilon\bar{c}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}) + 2\varepsilon H(t) \\ & - \varepsilon \int_{\Omega} (u|u_t|^{m-1}u_t dx + v|v_t|^{r-1}v_t)dx, \end{aligned} \tag{3.17}$$

where $\bar{c} = 1 - \frac{1}{\rho+2} - 2E_1(B\alpha_2)^{-2(\rho+2)}$. It is clear that $\bar{c} > 0$, since $\alpha_2 > B_2^{-(\rho+2)/(\rho+1)}$. In order to estimate the last two terms in (3.17) we make use of the following Young's inequality:

$$XY \leq \frac{\delta^{\alpha} X^{\alpha}}{\alpha} + \frac{\delta^{-\beta} Y^{\beta}}{\beta},$$

$X, Y \geq 0,\ \delta > 0,\ \alpha, \beta \in \mathbb{R}^+$ such that $1/\alpha + 1/\beta = 1$. Consequently, we get

$$\Big| \int_{\Omega} u|u_t|^{m-1}u_t dx \Big| \leq \frac{\delta_1^{m+1}}{m+1} \|u\|_{m+1}^{m+1} + \frac{m}{m+1} \delta_1^{-(m+1)/m} \|u_t\|_{m+1}^{m+1},\ \forall \delta_1 > 0,$$

and

$$\Big| \int_{\Omega} v|v_t|^{r-1}v_t dx \Big| \leq \frac{\delta_2^{r+1}}{r+1} \|v\|_{r+1}^{r+1} + \frac{r}{r+1} \delta_2^{-(r+1)/r} \|v_t\|_{r+1}^{r+1},\ \forall \delta_2 > 0.$$

Inserting the last two estimates into (3.17), we have

$$L^{'}(t) \geq (1-\sigma)\, H^{-\sigma}(t)\, H^{'}(t) + 2\varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2)$$

$$
\begin{aligned}
&+\varepsilon\bar{c}(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2})+2\varepsilon H(t)\\
&-\varepsilon\frac{\delta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1}-\varepsilon\frac{m}{m+1}\delta_1^{-(m+1)/m}\|u_t\|_{m+1}^{m+1}\\
&-\varepsilon\frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1}-\varepsilon\frac{r}{r+1}\delta_2^{-(r+1)/r}\|v_t\|_{r+1}^{r+1}.
\end{aligned}
\tag{3.18}
$$

Let us choose δ_1 and δ_2 such that

$$
\delta_1^{-(m+1)/m}=M_1H^{-\sigma}(t),\qquad \delta_2^{-(r+1)/r}=M_2H^{-\sigma}(t),\tag{3.19}
$$

for M_1 and M_2 large constants to be fixed later. Thus, by using (2.6) and (3.19), the inequality (3.18) then takes the form

$$
\begin{aligned}
L^{'}(t)\geq{}&((1-\sigma)-M\varepsilon)\,H^{-\sigma}(t)\,H^{'}(t)+2\varepsilon(\|u_t\|_2^2+\|v_t\|_2^2)\\
&+\varepsilon c_2(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)})+2\varepsilon H(t)\\
&-\varepsilon M_1^{-m}H^{\sigma m}(t)\,\|u\|_{m+1}^{m+1}-\varepsilon M_2^{-r}H^{\sigma r}(t)\,\|v\|_{r+1}^{r+1},
\end{aligned}
\tag{3.20}
$$

where $M=m/(m+1)M_1+r/(r+1)M_2$ and c_2 is a positive constant. Since $2(\rho+2)>max(m+1,r+1)$, we obtain by using (2.6) and (3.12)

$$
H^{\sigma m}(t)\,\|u\|_{m+1}^{m+1}\leq c_3(\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+(m+1)}+\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{m+1}^{m+1}),\tag{3.21}
$$

and

$$
H^{\sigma m}(t)\,\|v\|_{r+1}^{r+1}\leq c_4(\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+(r+1)}+\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{r+1}^{r+1}),\tag{3.22}
$$

for some positive constants c_3 and c_4. By using (3.14) and the algebraic inequality

$$
z^{\nu}\leq(z+1)\leq\left(1+\frac{1}{a}\right)(z+a)\,,\quad\forall z\geq0,\quad 0<\nu\leq1,a\geq0,\tag{3.23}
$$

we have, for all $t\geq0$,

$$
\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+(m+1)}\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H\left(0\right)\right)\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H\left(t\right)\right),\tag{3.24}
$$

where $d=1+1/H\left(0\right)$. Similarly,

$$
\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+(r+1)}\quad\leq\quad d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H\left(t\right)\right),\ \forall t\geq0.\tag{3.25}
$$

Also, since

$$
(X+Y)^s\leq C(X^s+Y^s),\qquad X,Y\geq0,s>0,
$$

by using (3.14) and (3.23) we conclude

$$\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{m+1}^{m+1} \leq C(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{m+1)}^{2(\rho+2)}) \tag{3.26}$$
$$\leq C(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{2(\rho+2)}^{2(\rho+2)}),$$
$$\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{r+1}^{r+1} \leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{r+1)}^{2(\rho+2)}) \tag{3.27}$$
$$\leq C(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{2(\rho+2)}^{2(\rho+2)}),$$

where C is a generic positive constant. Taking into account (3.21)-(3.27), (3.20) therefore takes the form

$$\begin{aligned} L^{'}(t) &\geq ((1-\sigma) - M\varepsilon)\, H^{-\sigma}(t)\, H^{'}(t) + 2\varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) \\ &+ \varepsilon(c_2 - CM_1^{-m} - CM_2^{-r})(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}) \\ &+ \varepsilon(2 - CM_1^{-m} - CM_2^{-r})H(t). \end{aligned} \tag{3.28}$$

At this point, and for large values of M_1 and M_2, we can find positive constants Λ_1 and Λ_2 such that (3.28) becomes

$$\begin{aligned} L^{'}(t) &\geq ((1-\sigma) - M\varepsilon)\, H^{-\sigma}(t)\, H^{'}(t) + 2\varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) \\ &+ \varepsilon\Lambda_1\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) + \varepsilon\Lambda_2 H(t). \end{aligned} \tag{3.29}$$

Once M_1 and M_2 are fixed (hence, Λ_1 and Λ_2), we pick ε small enough so that $(1-\sigma) - M\varepsilon \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \varepsilon\int_\Omega [u_0.u_1 + v_0.v_1]\, dx > 0.$$

Consequently, there exists $\Gamma > 0$ such that (3.29) becomes

$$L^{'}(t) \geq \varepsilon\Gamma(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}). \tag{3.30}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, we estimate

$$\Big|\int_\Omega uu_t(x,t)dx\Big| \leq ||u||_2||u_t||_2 \leq C||u||_{2(\rho+2)}||u_t||_2,$$

and

$$\Big|\int_\Omega vv_t(x,t)dx\Big| \leq ||v||_2||v_t||_2 \leq C||v||_{2(\rho+2)}||v_t||_2,$$

where here and in the sequel C denotes a positive constant which may change from line to line. Then, we have

$$\Big(\int_\Omega uu_t(x,t)dx + \int_\Omega vv_t(x,t)dx\Big)^{1/(1-\sigma)}$$

$$\leq C\left(||u||_{2(\rho+2)}^{1/(1-\sigma)}||u_t||_2^{1/(1-\sigma)}+||v||_{2(\rho+2)}^{1/(1-\sigma)}||v_t||_2^{1/(1-\sigma)}\right). \tag{3.31}$$

Again Young's inequality gives us

$$\begin{aligned}&\left(\int_\Omega uu_t(x,t)dx+\int_\Omega vv_t(x,t)dx\right)^{1/(1-\sigma)}\\&\leq C\left(||u||_{2(\rho+2)}^{\tau/(1-\sigma)}+||u_t||_2^{s/(1-\sigma)}+||v||_{2(\rho+2)}^{\tau/(1-\sigma)}+||v_t||_2^{s/(1-\sigma)}\right),\end{aligned} \tag{3.32}$$

for $1/\tau+1/s=1$. We take $s=2(1-\sigma)$, to get $\tau/(1-\sigma)=2/(1-2\sigma)$. By using (3.14) and (3.23) we get

$$\|u\|_{2(\rho+2)}^{2/(1-2\sigma)}\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H\left(t\right)\right),$$

and

$$\|v\|_{2(\rho+2)}^{2/(1-2\sigma)}\leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H\left(t\right)\right),\quad \forall t\geq 0.$$

Therefore, (3.32) becomes

$$\begin{aligned}&\left[\int_\Omega uu_t(x,t)dx+\int_\Omega vv_t(x,t)dx\right]^{1/(1-\sigma)}\\&\leq C\left[||u||_{2(\rho+2)}^{2(\rho+2)}+||v||_{2(\rho+2)}^{2(\rho+2)}+||u_t||_2^2+||v_t||_2^2+H(t)\right],\ \forall t\geq 0.\end{aligned} \tag{3.33}$$

Finally, by noting that

$$\begin{aligned}L^{1/(1-\sigma)}(t)&=\left(H^{1-\sigma}(t)+\varepsilon\int_\Omega uu_t+vv_t)(x,t)dx\right)^{1/(1-\sigma)}\\&\leq C\left(H(t)+|\int_\Omega uu_t(x,t)dx+vv_t(x,t)dx|^{1/(1-\sigma)}\right)\\&\leq C\left[H(t)+||u||_{2(\rho+2)}^{2(\rho+2)}+||v||_{2(\rho+2)}^{2(\rho+2)}+||u_t||_2^2+||v_t||_2^2\right],\quad \forall t\geq 0,\end{aligned} \tag{3.34}$$

a combination of (3.34) and (3.30) yields

$$L'(t)\geq\ a_0L^{1/(1-\sigma)}(t),\qquad \forall\ t\geq 0. \tag{3.35}$$

A simple application of Lemma 3.1 gives the desired result.

Remark 3.2. For $N=1,2,3$ and according to Theorem 2.1, if the solution (u,v) is smooth enough then it blows up in finite time. In fact, the norm of the solution in the energy space $H=H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$ defined by

$$\|(u,u_t,v,v_t)\|_H^2=\|\nabla u\|_2^2+\|u_t\|_2^2+\|\nabla v\|_2^2+\|v_t\|_2^2$$

blows up in finite time. Indeed, by using (3.34), (3.12), and the embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$, we obtain

$$L^{1/(1-\sigma)}(t) \le C(\|\nabla u\|_2^{2(\rho+2)} + \|\nabla v\|_2^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2). \tag{3.36}$$

Since our function $L(t)$ blows up in finite time, then the same result holds for the norm $\|(u, u_t, v, v_t)\|_H$.

Remark 3.3. It is clear that the more general degenerate system

$$\begin{cases} u_{tt} - \Delta u + |u|^k |u_t|^{m-1} u_t = f_1(u,v),\ x \in \Omega,\ t > 0 \\ v_{tt} - \Delta v + |u|^k |v_t|^{r-1} u_t = f_2(u,v),\ x \in \Omega,\ t > 0 \\ u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1),\ x \in \Omega \\ u(x) = v(x) = 0,\ x \in \partial\Omega \end{cases} \tag{3.37}$$

could be analyzed with the same method, for suitable conditions on the parameters k, m, p, r.

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