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FUNCTIONS WITH ORTHOGONAL HESSIAN

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Abstract. A Dirichlet problem for orthogonal Hessians in two dimensions is explicitly solved, by characterizing all piecewise C^2 functions $u:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ with orthogonal Hessian in terms of a property named "*second order angle condition*" as in (1.1).

1. Introduction

We consider a piecewise C^2 function u defined on an open set $\Omega\subset\mathbb{R}^2$, with almost everywhere orthogonal Hessian matrix D^2u. We denote by Σ_u the singular set where the Hessian is discontinuous. It turns out that Σ_u is a union of segments and at every vertex of Σ_u meet exactly four consecutive angles $\alpha^+,\alpha^-,\beta^+,\beta^-$ with

$$\alpha^+ + \beta^+ = \pi \quad\text{and}\quad \alpha^- = \beta^- = \frac{\pi}{2}\,. \tag{1.1}$$

We call this condition the "*second order angle condition.*"

This unexpected property allows us to characterize all piecewise C^2 functions $u:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ with orthogonal Hessian. It also allows us to exhibit solutions to some second-order Dirichlet problems related to differential systems (see (3.1)). These and some further properties are discussed below (cf. Theorems 1 and 4).

We would also like to contrast the second order angle condition (1.1) with the "*angle condition*" considered in a previous paper [3]. This last property states that if $w:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ is any piecewise C^1 vector field with orthogonal gradient Dw, then at any vertex of the singular set Σ_w it meets exactly $2m$ consecutive angles $\alpha_1,\cdots,\alpha_{2m}$ with the property that

$$\alpha_1+\alpha_3+\cdots+\alpha_{2m-1} = \alpha_2+\alpha_4+\cdots+\alpha_{2m} = \pi.$$

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The angle property was first discovered in the framework of *origami*, the ancient Japanese art of folding papers, by Kawasaki (see [5]).

2. NECESSARY AND SUFFICIENT CONDITIONS FOR SECOND DERIVATIVES

Let Ω be an open subset of $\mathbb{R}^2$. We say that a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is *piecewise* C^2 in Ω (and we write $u \in C^2_{\text{piec}}(\Omega)$) if $u \in W^{2,\infty}(\Omega)$ and there exists a locally finite partition of Ω into open sets Ω_k with piecewise C^1 boundary and a set of measure zero; i.e.,

$$\Omega_h \cap \Omega_k = \emptyset, \quad \text{for every } h \neq k,$$
$$\text{meas}\left(\Omega - \bigcup_k \Omega_k\right) = 0,$$

such that u is $C^2(\overline{\Omega_k})$ for every k.

Notation Let $u \in C^2_{\text{piec}}(\Omega)$ with almost everywhere orthogonal Hessian. The *regular set* is the set where the Hessian is continuous and we denote by Ω^+ (respectively Ω^-) the open subset of the regular set where $\det D^2u = 1$ (respectively $\det D^2u = -1$). The *singular set*, denoted Σ, is the complement of the regular set, so that $\Omega = \Omega^+ \cup \Omega^- \cup \Sigma$.

In the theorem below, we say that C is a *convex polygon* (respectively a *rectangle*) *with respect to* Ω, if there exists a polygon (respectively a rectangle) $K \subset \mathbb{R}^2$ so that $C = K \cap \Omega$.

Theorem 1 (Necessary condition)**.** *Let $\Omega \subset \mathbb{R}^2$ be open and convex and let $u \in C^2_{\text{piec}}(\Omega)$ with almost everywhere orthogonal Hessian. Then the Hessian D^2u is constant on every connected component of Ω^+ and Ω^-. Moreover, the following properties hold:*

(i) the connected components of Ω^+ are convex polygons with respect to Ω and their closures meet in at most one common vertex which belongs to Σ; the connected components of Ω^- are rectangles with respect to Ω and their closures meet in at most one common vertex which belongs to Σ;

(ii) at every interior vertex exactly two components of Ω^+ and two of Ω^- meet in an alternated way with angles $\alpha^+, \alpha^-, \beta^+, \beta^-$ with

$$\alpha^+ + \beta^+ = \pi \quad \textit{and} \quad \alpha^- = \beta^- = \tfrac{\pi}{2}\,.$$

Proof. Step 1. We first prove that the Hessian D^2u is constant on every connected component of Ω^+ and Ω^-. This can be deduced from the classical Liouville theorem, but, for the sake of completeness, we give here a proof that uses the more restrictive structure of Hessians.

1) We start with a component of Ω^+. First note that a symmetric orthogonal matrix with positive determinant can be only of the form

$$\pm I = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$D^2 u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}$$

is such that $u_{xy} = 0$ and $u_{xx} = u_{yy} = \pm 1$ and thus $D^2 u$ is indeed constant on every connected component of Ω^+.

2) We now discuss the case of a component of Ω^-. Note that since $\det D^2 u = -1$ and $D^2 u$ is orthogonal, we have, for some function $\theta = \theta(x, y)$,

$$D^2 u = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Thus, $\Delta u = 0$, which implies that $u \in C^\infty$ there. Therefore, $\theta \in C^\infty$ and we get from

$$\begin{cases} (\cos\theta)_y = (\sin\theta)_x \\ (\sin\theta)_y = (-\cos\theta)_x\,, \end{cases}$$

that $\theta_x = \theta_y = 0$, as desired.

Step 2. On either side of a smooth part of the boundary of a connected component $D^2 u$ assumes the values A and B which differ by a rank one matrix. More precisely, since A and B are constant, Du is affine on both sides of the boundary and the boundary itself is locally a segment (with normal ν). Thus, A and B differ by a rank one matrix and because of symmetry

$$A - B = \pm\, \nu \otimes \nu. \tag{2.1}$$

Thus, any connected component of Ω^+ and Ω^- is a polygon with respect to Ω. The properties of these polygons are established in Step 4.

Step 3. Let us prove that two connected components of Ω^- (and similarly for Ω^+) cannot touch along a segment. Let $D^2 u = A, B$ on each side of the discontinuity segment. We know that A and B are necessarily of the form

$$A = \begin{pmatrix} \cos a & \sin a \\ \sin a & -\cos a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \cos b & \sin b \\ \sin b & -\cos b \end{pmatrix}.$$

Since Du should be continuous on the segment of discontinuity of the Hessian, we must have (see (2.1))

$$0 = \det(A - B) = -(\cos a - \cos b)^2 - (\sin a - \sin b)^2\,,$$

which is possible only if $A = B$.

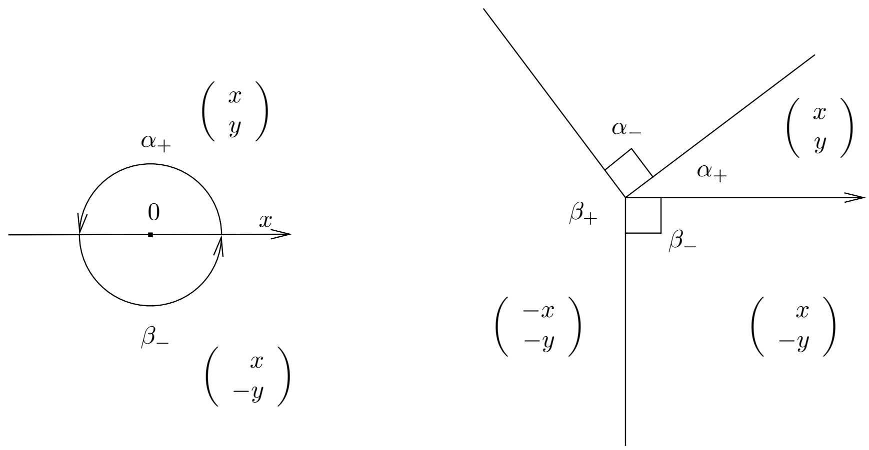

FIGURE 1. Second order angle condition

Step 4. We now prove (ii) of the theorem. Without loss of generality (we refer to Figure 1), we can assume that the interior vertex is at $(0,0)$, $Du\,(0,0) = (0,0)$, that a line of discontinuity of the Hessian is on the half $x-$axis, namely $L_1^+ = \{(x,0) : x > 0\}$ and that in the right upper half-plane near $(0,0)$

$$Du\,(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}.$$

By Step 3 and continuity of Du we should have below L_1^+ that

$$Du\,(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

Appealing once more to Step 3, we see that only two possibilities (since by Step 1, $D^2u = \pm I$) can happen about the next half-line of discontinuity (computed clockwise):

1) if $D^2u = I$ (see Figure 1, left), then the gradient on the other side of the discontinuity has to be

$$Du\,(x,y) = \begin{pmatrix} x \\ y \end{pmatrix};$$

thus, the half-line of discontinuity (where however Du must be continuous) is where $y=0$; i.e., $L_1^- = \{(x,0) : x<0\}$ and $\beta^- = \pi$; hence, this implies that $(0,0)$ is not a vertex, since in this case we would have $\alpha^+ = \pi$ by symmetry;

2) if $D^2u = -I$ (see Figure 1, right), the gradient on the other side of the discontinuity has to be

$$Du\,(x,y) = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

In this case $x=0$; i.e., the half-line of discontinuity is $L_2^- = \{(0,y) : y<0\}$; then $\beta^- = \pi/2$. By symmetry we deduce that $\alpha^- = \pi/2$ and thus $\alpha^+ + \beta^+ = \pi$.

Since Ω is convex we conclude that the connected components of Ω^- are rectangles. Since the angles in the connected components of Ω^+ are less than π, these components are convex polygons. $\square$

Remark 2. *We can see that, up to a rotation, if the vertex is at* $(0,0)$, $Du\,(0,0) = (0,0)$, *and the angle* α^+ *is measured with origin in the half* $x-$*axis* $L_1^+ = \{(x,0) : x>0\}$, *then the gradient* Du, *with respect to the angle* θ *in polar coordinates locally around the vertex has the analytic expression*

$$Du = \begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } 0 \le \theta \le \alpha^+ \\ \begin{pmatrix} x\cos 2\alpha^+ + y \sin 2\alpha^+ \\ x \sin 2\alpha^+ - y\cos 2\alpha^+ \end{pmatrix} & \text{if } \alpha^+ \le \theta \le \alpha^+ + \frac{\pi}{2} \\ \begin{pmatrix} -x \\ -y \end{pmatrix} & \text{if } \alpha^+ + \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \\ \begin{pmatrix} x \\ -y \end{pmatrix} & \text{if } \frac{3\pi}{2} \le \theta \le 2\pi. \end{cases} \tag{2.2}$$

In the proof of the theorem below, we will use the following result.

Lemma 3. *Let* Ω *be an open connected subset of* $\mathbb{R}^2$. *Let* $w : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *be a piecewise vector field of class* $C^1_{\text{piec}}\left(\Omega;\mathbb{R}^2\right)$ *with singular set* Σ *satisfying (i) and (ii) of Theorem 1. If its gradient* Dw *is a symmetric orthogonal matrix at one connected component of* $\Omega \setminus \Sigma$, *then it is a symmetric matrix almost everywhere in* Ω.

Proof. The argument is similar to the one of Theorem 1 and we do not enter into many details. Along a segment in Σ, the situation is even more direct than at a vertex and we therefore only discuss this last case. Up to a rotation and a translation, locally at any interior vertex of Σ we have the situation described in Remark 2, with the vector field w as in (2.2) and its gradient Dw being the symmetric matrix

$$Dw = \begin{cases} I & \text{if } 0 \leq \theta \leq \alpha^+ \\ \begin{pmatrix} \cos 2\alpha^+ & \sin 2\alpha^+ \\ \sin 2\alpha^+ & -\cos 2\alpha^+ \end{pmatrix} & \text{if } \alpha^+ \leq \theta \leq \alpha^+ + \frac{\pi}{2} \\ -I & \text{if } \alpha^+ + \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \frac{3\pi}{2} \leq \theta \leq 2\pi \end{cases} .$$

That is, the gradient Dw, being a symmetric orthogonal matrix at one connected component around the interior vertex $(0,0) \in \Omega$, either has positive determinant (and in this case it is forced to be either equal to the identity matrix I or to its opposite $-I$), or it has negative determinant (and in this case it has the form $\begin{pmatrix} \cos 2\alpha^+ & \sin 2\alpha^+ \\ \sin 2\alpha^+ & -\cos 2\alpha^+ \end{pmatrix}$ for some α^+. In any case, the matrix Dw is symmetric around the vertex $(0,0)$. □

Theorem 4 (Sufficient condition)**.** *Let $\Omega \subset \mathbb{R}^2$ be open and simply connected. Let $\Omega = \Omega^+ \cup \Omega^- \cup \Sigma$, where Ω^+, Ω^- are open and Σ is a locally finite union of segments, satisfying* (*i*) *and* (*ii*) *of Theorem* 1. *Then there exists $u \in C^2_{\text{piec}}(\Omega)$ with almost everywhere orthogonal Hessian, whose singular set is Σ and $\det D^2u = \pm 1$ in $\Omega^\pm$. Moreover, u is uniquely determined up to fixing the values $u(x_0)$, $Du(x_0)$ and, for example, $D^2u(x_0) = I$ at a point $x_0 \in \Omega^+$.*

Proof. Step 1. We first use Theorem 4.9 in [3] and find that there exists a vector field $w : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $w = \binom{w^1}{w^2} \in C^1_{\text{piec}}(\Omega; \mathbb{R}^2)$, whose singular set is $\Sigma = \Sigma_w$ where Dw is almost everywhere an orthogonal matrix. Moreover, we can fix Dw to be the identity at one given point in Ω^+.

Step 2. We apply Lemma 3 to the above w. We find that Dw is symmetric. We regularize w with the standard convolution $w^\epsilon = w * \rho^\epsilon$ and observe that Dw^ϵ is a symmetric matrix. Therefore, there exists u^ϵ so that $w^\epsilon = Du^\epsilon$. Passing to the limit we find that $u^\epsilon \to u$ uniformly for a certain $u \in W^{2,\infty}$ and $D^2u^\epsilon \to D^2u = Dw$ almost everywhere in Ω. Thus, $u \in C^2_{\text{piec}}(\Omega)$ and the result is obtained. □

3. SOLUTIONS TO DIRICHLET PROBLEMS

We consider the second-order Dirichlet problem for the unknown function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, where Ω is a rectangle,

$$\begin{cases} D^2 u \in O(2) & \text{a.e. in } \Omega \\ Du = \varphi & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

or the first-order Dirichlet problem for the unknown map $w : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $w = \binom{w^1}{w^2}$,

$$\begin{cases} Dw \in O(2) & \text{a.e. in } \Omega \\ w = \varphi & \text{on } \partial\Omega \end{cases} . \tag{3.2}$$

For general considerations on second-order Dirichlet problems as in (3.1), we refer to [2]. For explicit constructions of solutions to (3.2), we refer, among others, to [1], [3] and [4].

It turns out, by the characterization results of the previous section, that in fact the Dirichlet problem (3.2) contains, as a special case, some solutions to the Dirichlet problem (3.1).

That is, for suitable φ, there exist solutions w to (3.2) which have the gradient representation $w = Du$ for a certain function u. In fact, we proved in [3] that there exists a solution w to the Dirichlet problem (3.2) for linear boundary data and, without loss of generality, when

$$\varphi\,(x, y) = \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix},$$

for $\alpha, \beta \in (-1, 1)$ (note that the compatibility constraint on α, β is necessary). In this case we proved the existence of a map $w : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $w \in C^1_{\text{piec}}\left(\Omega; \mathbb{R}^2\right)$, whose singular set is the set $\Sigma = \Sigma_w$ represented in Figure 2. We can easily see that the singular set satisfies the necessary and sufficient conditions of Theorems 1 and 4; thus there exists $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ such that $w = Du$. This function u solves the Dirichlet problem (3.1).

Our aim here is to show how to read the boundary value from the singular set, namely how to relate the $\alpha, \beta \in (-1, 1)$ values with the singular set $\Sigma = \Sigma_w$ represented in Figure 2. The set Σ is contained in the rectangle Ω of sizes a and b, where

$$\frac{b^2}{a^2} = \frac{1 - \alpha^2}{1 - \beta^2}.$$

For the sake of clarity we start to describe the construction when $\alpha = \beta$. In this case the rectangle is a square and the *triangular set* in Figure 3 is a real triangle delimited by the diagonals of the square. In the general case, the triangular set is a perturbation of the above one. More precisely, the

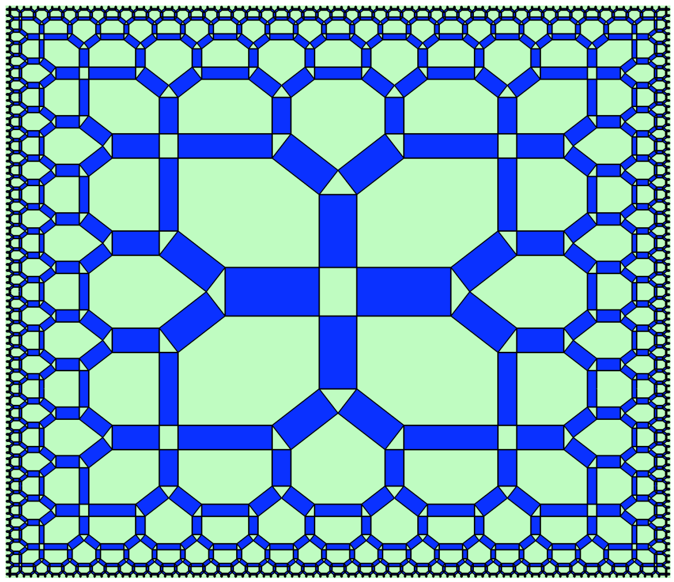

FIGURE 2. Singular set Σ

triangular set is delimited by two polygonal paths joining the center with two consecutive vertices of the rectangle, passing through vertices of Σ which are "close" to the diagonals of Ω (as in Figure 3).

We consider a generic vertical segment, parallel to the $y-$axis, in the triangular set. The segment intersects periodically the structure represented in Figure 3, in particular the singular set Σ and the connected components Ω^- (respectively Ω^+) of the regular set where the determinant of the matrix Dw in (3.2), or the matrix D^2u in (3.1), is equal to -1 (respectively equal to $+1$). All the connected components of Ω^- are rectangles (dark regions), while the connected components of Ω^+ are the other polygons (light regions), as in Figures 2 and 3.

We further limit ourselves to the segments parallel to the $y-$axis which meet orthogonally the boundary of the connected components of Ω^+ and Ω^-, see Figure 3. According to Remark 2, the Jacobian Dw in (3.2), or the Hessian D^2u in (3.1), in Ω^+ and Ω^- respectively, is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

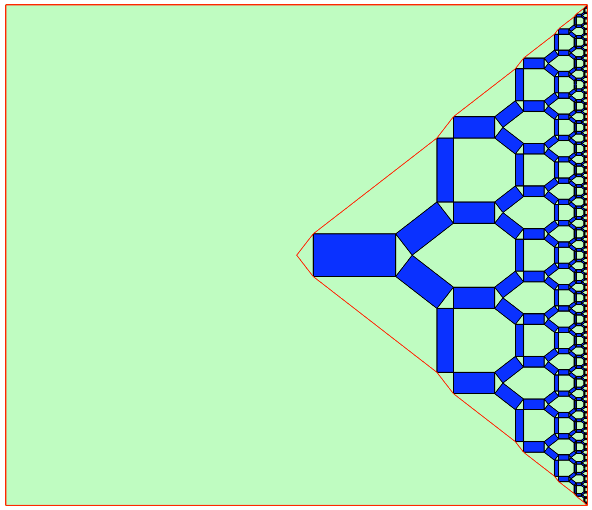

FIGURE 3. The triangular set

Then, under the notation $w = \binom{w^1}{w^2}$, we have that $\partial w^2/\partial y$ assumes the values $+1$ and -1 with proportions given by the vertical lengths of the connected components of Ω^+ and Ω^-. These proportions, limited to a period, up to a normalization, define the average of $\partial w^2/\partial y$, i.e., the slope of w^2 at the boundary parallel to the $y-$axis; therefore

$$\beta = \frac{\text{vertical length of } \Omega^+ - \text{vertical length of } \Omega^-}{\text{vertical length of } \Omega^+ + \text{vertical length of } \Omega^-}.$$

We proceed similarly for the $x-$axis and α and we get

$$\alpha = \frac{\text{horizontal length of } \Omega^+ - \text{horizontal length of } \Omega^-}{\text{horizontal length of } \Omega^+ + \text{horizontal length of } \Omega^-}.$$

We should note that the above construction for the solution of the Dirichlet problem is the same for the first and second order cases. This has been made possible from the special choice of the singular set Σ.

In Figure 4, we propose two examples (one of them being as in Figure 2) of singular sets which satisfy the second-order angle condition and therefore, lead to solutions of the first as well as the second-order case.

In contrast, in Figure 5, we propose two singular sets that satisfy the angle condition but not the second-order one. Therefore, they can be used to solve the first-order system but not the second-order one.

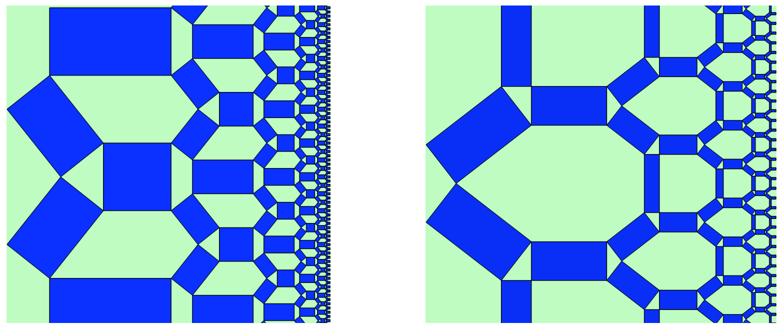

FIGURE 4. Two examples of second order angle condition

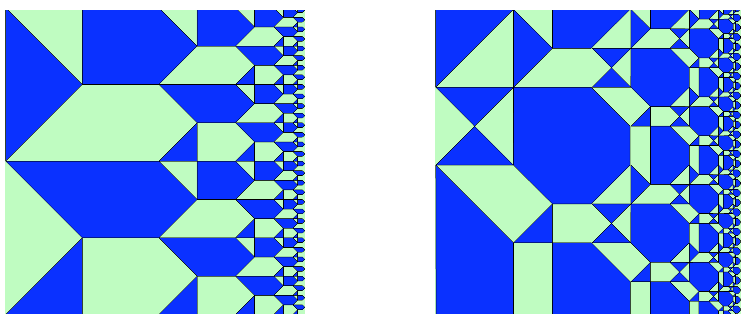

FIGURE 5. Two examples of first order angle condition

References

[1] A. Cellina and S. Perrotta, *On a problem of potential wells*, J. Convex Analysis, **2** (1995), 103–115.

[2] B. Dacorogna and P. Marcellini, "Implicit Partial Differential Equations," Progress in Nonlinear Differential Equations and Their Applications, vol. 37, Birkhäuser, 1999.

[3] B. Dacorogna, P. Marcellini, and E. Paolini, *Lipschitz-continuous local isometric immersions: rigid maps and origami*, J. Math. Pures Appl., **90** (2008), 66–81.

[4] T. Iwaniec, G. Verchota, and A. Vogel, *The failure of rank one connections*, Arch. Ration. Mech. Anal., **163** (2002), 125–169.

[5] T. Kawasaki, *On the relation between mountain-creases and valley creases of a flat origami*, Proceedings of the 1st International Meeting of Origami Science and Technology, Ferrara (H. Huzita, ed.), 1989, 229–237.