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GLOBAL UNIQUENESS RESULTS FOR PARTIAL FUNCTIONAL AND NEUTRAL FUNCTIONAL EVOLUTION EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper, we provide sufficient conditions for the existence of a unique mild solution on a semi-infinite interval for two classes of first-order partial functional and neutral functional differential evolution equations with infinite delay using a recent nonlinear alternative of Leray Schauder type due to Frigon and Granas for contractions maps in Fréchet spaces, combined with semigroup theory.

1. Introduction

In this paper, we consider the existence of a unique mild solution, defined on a semi-infinite positive real interval $J := [0, +\infty)$, for two classes of first-order functional and neutral functional partial differential evolution equations with infinite delay in a real Banach space $(E, |\cdot|)$.

Firstly, in Section 3, we study the following partial functional differential evolution equation with infinite delay

$$y'(t) = A(t)y(t) + f(t, y_t), \quad \text{a.e.} \quad t \in J \tag{1.1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.2}$$

where $\mathcal{B}$ is an abstract phase space, to be specified later, $f : J \times \mathcal{B} \to E$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t, s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t < +\infty$.

For any continuous function y and any $t \le 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. Here, $y_t(\cdot)$ represents the history of the state from time $t \le 0$ up to the present time t. We assume that the histories y_t belongs to $\mathcal{B}$.

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In Section 4, we consider the following neutral functional differential evolution equation with infinite delay

$$\frac{d}{dt}[y(t) - g(t, y_t)] = A(t)y(t) + f(t, y_t), \quad \text{a.e.} \quad t \in J \tag{1.3}$$

$$y_0 = \phi \in \mathcal{B}, \tag{1.4}$$

where $A(\cdot)$, f and ϕ are as in problem $(1.1) - (1.2)$ and $g : J \times \mathcal{B} \to E$ is a given function. Finally in Section 5, examples are given to demonstrate the results.

It is a well-established idea to model the evolution of some physical, biological and economic systems using functional and partial functional differential equations, in which the response of the system depends not only on the current state of the system, but also on the past history of the system. For more details on this topic, see for example, the books of Kolmanovskii and Myshkis [32], Hale and Verduyn Lunel [22] and Wu [36], and the references therein.

Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations is the books by Hale [20] and Hale and Verduyn Lunel [22] and Kolmanovskii and Myshkis [32] and the references therein. During the last few decades, existence and uniqueness of mild, strong and classical solutions of semilinear functional differential equations and inclusions have been studied extensively by many authors using semigroup theory, fixed-point arguments, degree theory and measures of noncompactness. We mention, for instance, the books by Ahmed [4], Barbu [9], Engel and Nagel [15], Kamenski *et al* [30], Pazy [34] and Wu [36]. The pioneering Hernandez paper in [27] proved the existence of mild, strong and periodic solutions for neutral equations. Fu and Ezzinbi [19] considered the existence of mild and classical solutions for a class of neutral partial functional differential equations with nonlocal conditions. Recent results on neutral functional differential equations and inclusions were given by Arara *et al* [6], Benchohra *et al* in [11, 12, 13].

In the literature devoted to equations with finite delay, the state space is the space of all continuous functions on the finite interval $[-r, 0]$ for $r > 0$, endowed with the uniform norm topology. The past history plays an important role in the study of a system represented as functional and partial functional differential equations when the response of the system depends not only on the current state of the system, but also on its history. When the delay is infinite, we introduce the notion of the phase space $\mathcal{B}$ which plays an important role in the study of both qualitative and quantitative theory. A standard choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato in [21], see also Corduneanu and Lakshmikantham [14], Kappel and Schappacher [31] and Schumacher [35]. For detailed discussion and applications on this topic, we refer the reader to the book by Hale and Verduyn Lunel [22], Hino *et al.* [29] and Wu [36].

An extensive theory is developed to equation (1.1) with $A(t) = A$. We refer the reader to the books by [23] and the Hino and Murakami paper [28] and the papers by Adimy *et al* [1, 2, 3], Benchohra *et al* [10, 11, 12], Ezzinbi [16], Henriquez [24, 25] and Hernandez [26, 27], where existence and uniqueness, among other things, are derived.

When A is depending on the time t, Arara *et al* [6] considered control multivalued problem on a bounded interval $[0, b]$ and very recently Baghli and Benchohra [7, 8] provided uniqueness results for some classes of functional and neutral functional evolution equations with local and nonlocal conditions when the delay is finite.

Our main purpose in this paper is to extend some results from the above literature devoted to finite delay and those considered on a bounded interval. We provide sufficient conditions for the existence of a unique mild solution on a semi-infinite interval $J = [0, +\infty)$ for the two classes of first-order partial functional and neutral functional differential evolution equations $(1.1)-(1.2)$ and $(1.3)-(1.4)$ *when the delay is infinite* using the recent nonlinear alternative of Leray Schauder type due to Frigon and Granas for contraction maps in Fréchet spaces [17], combined with semigroup theory [5, 9, 34].

2. Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C([0, +\infty); E)$ be the space of continuous functions from $[0, +\infty)$ into E and $B(E)$ be the space of all bounded linear operators from E into E, with the norm $\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}$. A measurable function $y : [0, +\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see Yosida [37] for instance.)

Let $L^1([0, +\infty), E)$ denote the Banach space of measurable functions $y : [0, +\infty) \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| \, dt.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [21] and follow the terminology used in [29]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms:

- (A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold:
  (i) $y_t \in \mathcal{B}$;
  (ii) there exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
  (iii) there exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of $y(t)$ with K continuous and M locally bounded such that
  $$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{ \, |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$
- (A_2) For the function $y(.)$ in (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.
- (A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

Remark 2.1. (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

Remark 2.2. Since $\| \cdot \|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can satisfy $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.

Remark 2.3. From the equivalence in Remark 2.1, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$ we necessarily have that $\phi(0) = \psi(0)$.

Here are some examples of phase spaces. For other details we refer, for instance, to the book by Hino *et al* [29].

Example 2.4. The spaces BC, BUC, C^∞ and C^0. Let:

- BC be the space of bounded continuous functions defined from $(-\infty, 0]$ to E;
- BUC be the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E;
- $C^\infty := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\right\}$;
- $C^0 := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\right\}$, endowed with the uniform norm

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \leq 0\}.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(A_1) - (A_3)$. BC satisfies $(A_1), (A_3)$ but (A_2) is not satisfied.

Example 2.5. The spaces $C_g,\ UC_g,\ C_g^{\infty}$ and C_g^0.

Let g be a positive continuous function on $(-\infty, 0]$. We define:

$$C_g := \Big\{\phi \in C((-\infty,0],E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty,0]\Big\};$$

$$C_g^0 := \Big\{\phi \in C_g : \lim_{\theta\to-\infty} \frac{\phi(\theta)}{g(\theta)} = 0\Big\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\Big\{\frac{|\phi(\theta)|}{g(\theta)} :\ \theta \le 0\Big\}.$$

We consider the following condition on the function g.

(g_1) For all $a > 0$, $\displaystyle\sup_{0\le t\le a}\sup\Big\{\frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \le -t\Big\} < \infty.$

Then we have that the spaces C_g and C_g^0 satisfy conditions (A_3). They satisfy conditions (A_1) and (A_2) if g_1 holds.

Example 2.6. The space C_γ. For any real constant γ, we define the functional space C_γ by

$$C_\gamma := \Big\{\phi \in C((-\infty,0],E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } E\Big\},$$

endowed with the following norm:

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| :\ \theta \le 0\}.$$

Then in the space C_γ the axioms $(A_1) - (A_3)$ are satisfied.

Definition 2.7. *A function $f : J\times\mathcal{B} \to E$ is said to be an L^1 -Carathéodory function if it satisfies*

(i) *for each $t \in J$ the function $f(t,.) : \mathcal{B} \to E$ is continuous;*
(ii) *for each $y \in \mathcal{B}$ the function $f(.,y) : J \to E$ is measurable;*
(iii) *for every positive integer k there exists $h_k \in L^1(J;\mathbb{R}^+)$ such that*

$$|f(t,y)| \le h_k(t) \quad \textit{for all } \|y\|_{\mathcal{B}} \le k \quad \textit{and almost each } t \in J.$$

In what follows, we assume that $\{A(t) : t \ge 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E with domain $D(A(t))$ independent of t.

Definition 2.8. *A family of bounded linear operators $\{U(t,s)\}_{(t,s)\in\Delta}$, $U(t,s) : E \to E(t,s) \in \Delta := \{(t,s) \in J\times J : 0 \le s \le t < +\infty\}$, is called an evolution system if the following properties are satisfied:*

(1) *$U(t,t) = I$, where I is the identity operator in E,*
(2) *$U(t,s)\ U(s,\tau) = U(t,\tau)$ for $0 \le \tau \le s \le t < +\infty$,*

(3) $U(t,s) \in B(E)$, *the space of bounded linear operators on* E, *where for every* $(t,s) \in \Delta$ *and for each* $y \in E$, *the mapping* $(t,s) \to U(t,s)\, y$ *is continuous.*

More details on evolution systems and their properties can be found in the books of Ahmed [4], Engel and Nagel [15] and Pazy [34].

Let X be a Fréchet space with a family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$; we say that Y is bounded if, for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad \textit{for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by $x \sim_n y$ if and only if $\|x-y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote by $[x]_n$ the equivalence class of x of subset X^n and we define $Y^n = \{[x]_n : x \in Y\}$. We denote by $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. We assume that the family of seminorms $\{\|\cdot\|_n\}$ satisfies

$$\|x\|_1 \leq \|x\|_2 \leq \ \|x\|_3 \leq ... \quad \textit{for every } x \in X.$$

Now, we introduce most of the definitions and theorems used in this paper.

Definition 2.9. [17] *A function* $f : X \to X$ *is said to be a contraction if for each* $n \in \mathbb{N}$ *there exists* $k_n \in (0,1)$ *such that*

$$\|f(x) - f(y)\|_n \leq k_n\ \|x-y\|_n \quad \textit{for all } x, y \in X.$$

Theorem 2.10. *(Nonlinear Alternative of Granas-Frigon,* [17]*). Let* X *be a Fréchet space and* $Y \subset X$ *a closed subset and let* $N : Y \to X$ *be a contraction such that* $N(Y)$ *is bounded. Then one of the following statements holds:*

$(C1)$ *N has a unique fixed point;*

$(C2)$ *there exists* $\lambda \in [0,1)$, $n \in \mathbb{N}$ *and* $x \in \partial_n Y^n$ *such that* $\|x - \lambda\, N(x)\|_n = 0$.

3. SEMILINEAR EVOLUTION EQUATIONS

Before stating and proving the main result, we give first the definition of mild solution of the semilinear evolution problem $(1.1) - (1.2)$.

Definition 3.1. *We say that the function $y(\cdot):\mathbb{R}\to E$ is a mild solution of* $(1.1)-(1.2)$ *if $y(t)=\phi(t)$ for all $t\le 0$ and y satisfies the following integral equation:*

$$y(t)=U(t,0)\ \phi(0)+\int_0^t U(t,s)\ f(s,y_s)\ ds \qquad for\ each\ t\geq 0. \tag{3.1}$$

We will need to introduce the following hypotheses which are assumed here and thereafter:

$(H1)$ There exists a constant $M\geq 1$ such that

$$\|U(t,s)\|_{B(E)}\leq M \quad \text{for every } (s,t)\in\Delta.$$

$(H2)$ There exists a function $p\in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+\to(0,\infty)$ such that

$$|f(t,u)|\leq p(t)\ \psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B}.$$

$(H3)$ For all $R>0$, there exists $l_R\in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)-f(t,v)|\leq l_R(t)\ \|u-v\|_{\mathcal{B}},$$

for all $u,v\in\mathcal{B}$ with $\|u\|_{\mathcal{B}}\leq R$ and $\|v\|_{\mathcal{B}}\leq R$.

Consider the following space:

$$B_{+\infty}=\left\{y:\mathbb{R}\to E : y|_{[0,T]} \text{ continuous for } T>0 \text{ and } y_0\in\mathcal{B}\right\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau>1$. For every $n\in\mathbb{N}$, we define in $B_{+\infty}$ seminorms by

$$\|y\|_n:=\sup\{e^{-\tau\ L^*_n(t)}\ |y(t)| : t\in[0,n]\},$$

where $L^*_n(t)=\int_0^t \bar{l}_n(s)\ ds$, $\bar{l}_n(t)=K_nMl_n(t)$ and l_n is the function from $(H3)$.

Then $B_{+\infty}$ is a Fréchet space with the family of seminorms $\|\cdot\|_{n\in\mathbb{N}}$.

Theorem 3.2. *Suppose that hypotheses* $(H1)-(H3)$ *are satisfied and moreover*

$$\int_{c_n}^{+\infty}\frac{ds}{\psi(s)}>K_nM\int_0^n p(s)\ ds \quad for\ each\ n\in\mathbb{N}, \tag{3.2}$$

with $c_n=(K_nMH+M_n)\|\phi\|_{\mathcal{B}}$. Then the problem $(1.1)-(1.2)$ *has a unique mild solution.*

Proof. Transform the problem $(1.1)-(1.2)$ into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0) + \displaystyle\int_0^t U(t,s)\ f(s,y_s)\ ds, & \text{if } t \in J. \end{cases} \tag{3.3}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(1.1)-(1.2)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t). \tag{3.4}$$

It is obvious that y satisfies (3.1) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t U(t,s)\ f(s, z_s + x_s)\ ds \quad \text{for } t \in J.$$

Let $B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}$. Define the operator $F : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)\ f(s, z_s + x_s)\ ds \quad \text{for } t \in J. \tag{3.5}$$

Obviously the operator N having a fixed point is equivalent to F having one, so we turn to proving that F has a fixed point.

Let $z \in B^0_{+\infty}$ be a possible fixed point of the operator F. By the hypotheses $(H1)$ and $(H2)$, we have for each $t \in [0,n]$

$$|z(t)| \leq \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s)|\ ds \leq M \int_0^t p(s)\psi\left(\|z_s + x_s\|_{\mathcal{B}}\right) ds.$$

Assumption $(A1)$ gives

$$\begin{aligned} \|z_s + x_s\|_{\mathcal{B}} &\leq \|z_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} \\ &\leq K(s)|z(s)| + M(s)\|z_0\|_{\mathcal{B}} + K(s)|x(s)| + M(s)\|x_0\|_{\mathcal{B}} \\ &\leq K_n|z(s)| + K_n\|U(s,0)\|_{B(E)}|\phi(0)| + M_n\|\phi\|_{\mathcal{B}} \\ &\leq K_n|z(s)| + K_n M|\phi(0)| + M_n\|\phi\|_{\mathcal{B}}. \end{aligned}$$

Using (ii), we obtain

$$\begin{aligned} \|z_s + x_s\|_{\mathcal{B}} &\leq K_n|z(s)| + K_n M H\|\phi\|_{\mathcal{B}} + M_n\|\phi\|_{\mathcal{B}} \\ &\leq K_n|z(s)| + (K_n M H + M_n)\|\phi\|_{\mathcal{B}}. \end{aligned}$$

Using the nondecreasing character of ψ, we get

$$|z(t)| \leq M \int_0^t p(s)\psi\left(K_n|z(s)| + (K_nMH + M_n)\|\phi\|_{\mathcal{B}}\right) ds.$$

Thus

$$K_n|z(t)| + (K_nMH + M_n)\|\phi\|_{\mathcal{B}}$$
$$\leq K_nM \int_0^t p(s)\psi(K_n|z(s)| + (K_nMH + M_n)\|\phi\|_{\mathcal{B}})ds + (K_nMH + M_n)\|\phi\|_{\mathcal{B}}.$$

We consider the function μ defined by

$$\mu(t) := \sup\{K_n|z(s)| + (K_nMH + M_n)\|\phi\|_{\mathcal{B}} : 0 \leq s \leq t\}, \quad 0 \leq t < +\infty.$$

Let $t^\star \in [0, t]$ be such that $\mu(t) = K_n|z(t^\star)| + (K_nMH + M_n)\|\phi\|_{\mathcal{B}}$. By the previous inequality, we have

$$\mu(t) \leq K_nM \int_0^t p(s)\ \psi(\mu(s))\ ds + (K_nMH + M_n)\|\phi\|_{\mathcal{B}} \quad \text{for} \quad t \in [0, n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have $\mu(t) \leq v(t)$ for all $t \in [0, n]$. From the definition of v, we have

$$c_n := v(0) = (K_nMH + M_n)\|\phi\|_{\mathcal{B}},$$

and

$$v'(t) = K_nMp(t)\ \psi(\mu(t)) \quad \text{a.e.} \;\; t \in [0, n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq K_nM\ p(t)\ \psi(v(t)) \;\; \text{a.e. } t \in [0, n].$$

Using the condition (3.2) this implies that for each $t \in [0, n]$ we have

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \leq K_nM \int_0^t p(s)\ ds \leq K_nM \int_0^n p(s)\ ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

Thus, for every $t \in [0, n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$. Set

$$Z = \{\ z \in B^0_{+\infty}\ :\ \sup\{\ |z(t)|\ :\ 0 \leq t \leq n\} \leq \Lambda_n + 1 \quad \text{for all} \;\; n \in \mathbb{N}\}.$$

Clearly, Z is a closed subset of $B^0_{+\infty}$.

We shall show that $F : Z \to B^0_{+\infty}$ is a contraction operator. Indeed, consider $z, \overline{z} \in B^0_{+\infty}$; using $(H1)$ and $(H3)$ for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$|F(z)(t) - F(\overline{z})(t)| = \left| \int_0^t U(t, s)\ [f(s, z_s + x_s) - f(s, \overline{z}_s + x_s)]\ ds \right|$$

$$\leq \int_0^t \|U(t,s)\|_{B(E)} \ |f(s, z_s + x_s) - f(s, \overline{z}_s + x_s)| \ ds$$
$$\leq \int_0^t M \ |f(s, z_s + x_s) - f(s, \overline{z}_s + x_s)| \ ds$$
$$\leq \int_0^t M l_n(s) \|z_s + x_s - \overline{z}_s - x_s\|_{\mathcal{B}} ds \leq \int_0^t M l_n(s) \|z_s - \overline{z}_s\|_{\mathcal{B}} ds.$$

Using $(A1)$, we obtain

$$|F(z)(t) - F(\overline{z})(t)| \leq \int_0^t M l_n(s)(K(s)|z(s) - \overline{z}(s)| + M(s)\|z_0 - \overline{z}_0\|_{\mathcal{B}}) ds$$
$$\leq \int_0^t M \ K_n \ l_n(s) \ |z(s) - \overline{z}(s)| \ ds$$
$$\leq \int_0^t [\bar{l}_n(s) \ e^{\tau \ L_n^*(s)} \ ] \ [e^{-\tau \ L_n^*(s)} \ |z(s) - \overline{z}(s)|] \ ds$$
$$\leq \int_0^t [\bar{l}_n(s) \ e^{\tau \ L_n^*(s)}] \ ds \ \|z - \overline{z}\|_n \leq \int_0^t \ \frac{1}{\tau} [e^{\tau \ L_n^*(s)}]' \ ds \ \|z - \overline{z}\|_n$$
$$\leq \frac{1}{\tau} \ e^{\tau \ L_n^*(t)} \ \|z - \overline{z}\|_n.$$

Therefore,

$$\|F(z) - F(\overline{z})\|_n \leq \frac{1}{\tau} \ \|z - \overline{z}\|_n.$$

So, the operator F is a contraction for all $n \in \mathbb{N}$. From the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda \ F(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 2.10 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. Thus, we deduce that the operator F has a unique fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator N, which is the unique mild solution of the problem (1.1) – (1.2).

4. Semilinear Neutral Evolution Equations

In this section, we give an existence and uniqueness result for the neutral functional differential evolution problem with infinite delay (1.3) – (1.4). Firstly we define its mild solution.

Definition 4.1. *We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of* (1.3) – (1.4) *if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral* *equation:*

$$
\begin{aligned}
y(t) &= U(t,0)[\phi(0) - g(0,\phi)] + g(t,y_t) + \int_0^t U(t,s)A(s)g(s,y_s)ds \\
&+ \int_0^t U(t,s)f(s,y_s)\, ds \qquad \textit{for each } t \geq 0.
\end{aligned}
\tag{4.1}
$$

We consider the hypotheses $(H1)-(H3)$ and we will need the following assumptions:

$(H4)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad \textit{for all } t \in J.$$

$(H5)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_n}$, such that

$$|A(t)\ g(t,\phi)| \leq L\ (\|\phi\|_{\mathcal{B}} + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

$(H6)$ There exists a constant $L_* > 0$ such that

$$|A(s)\ g(s,\phi) - A(\overline{s})\ g(\overline{s},\overline{\phi})| \leq L_*\ (|s - \overline{s}| + \|\phi - \overline{\phi}\|_{\mathcal{B}})$$

for all $s, \overline{s} \in J$ and $\phi, \overline{\phi} \in \mathcal{B}$.

Theorem 4.2. *Suppose that hypotheses $(H1)-(H6)$ are satisfied and moreover*

$$\int_{\delta_n}^{+\infty} \frac{ds}{s + \psi(s)} > \frac{MK_n}{1 - \overline{M}_0 L K_n} \int_0^n \max(L, p(s))ds \quad \textit{for each } n \in \mathbb{N}, \tag{4.2}$$

with

$$\delta_n = c_n + K_n L \frac{\overline{M}_0(1+M) + Mn + \overline{M}_0[c_n + M\|\phi\|_{\mathcal{B}}]}{1 - \overline{M}_0 L K_n};$$

then the problem $(1.3)-(1.4)$ *has a unique mild solution.*

Proof. Consider the operator $\widetilde{N} : B_{+\infty} \to B_{+\infty}$ defined by

$$
\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0; \\ U(t,0)\ [\phi(0) - g(0,\phi)] + g(t,y_t) & \\ + \displaystyle\int_0^t U(t,s)A(s)g(s,y_s)ds + \int_0^t U(t,s)f(s,y_s)ds, & \text{if } t \in J. \end{cases}
\tag{4.3}
$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(1.3)-(1.4)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \leq 0; \\ U(t,0)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t). \tag{4.4}$$

It is obvious that y satisfies (4.1) if and only if z satisfies $z_0 = 0$ and, for $t \in J$, we get

$$z(t) = g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds$$
$$+ \int_0^t U(t,s)f(s, z_s + x_s)ds.$$

Define the operator $\widetilde{F} : B^0_{+\infty} \to B^0_{+\infty}$ by

$$\widetilde{F}(z)(t) = g(t, z_t + x_t) - U(t,0)g(0,\phi) + \int_0^t U(t,s)A(s)g(s, z_s + x_s)ds$$
$$+ \int_0^t U(t,s)f(s, z_s + x_s)ds. \tag{4.5}$$

Obviously the operator $\widetilde{N}$ having a fixed point is equivalent to $\widetilde{F}$ having one, so we turn to proving that $\widetilde{F}$ has a fixed point.

Let $z \in B^0_{+\infty}$ be a possible $\widetilde{F}$ fixed point of the operator. Then, using $(H1) - (H5)$, we have for each $t \in [0, n]$

$$|z(t)| \leq |g(t, z_t + x_t)| + |U(t,0)g(0,\phi)| + |\int_0^t U(t,s)A(s)g(s, z_s + x_s)ds|$$
$$+ |\int_0^t U(t,s)f(s, z_s + x_s)ds|$$
$$\leq \|A^{-1}(t)\|_{B(E)}\|A(t)\ g(t, z_t + x_t)\| + \|U(t,0)\|_{B(E)}\|A^{-1}(0)\|\ \|A(0)\ g(0,\phi)\|$$
$$+ \int_0^t \|U(t,s)\|_{B(E)}\|A(s)\ g(s, z_s + x_s)\|ds + \int_0^t \|U(t,s)\|_{B(E)}|f(s, z_s + x_s)|ds$$
$$\leq \overline{M}_0 L(\|z_t + x_t\|_{\mathcal{B}} + 1) + M\overline{M}_0 L(\|\phi\|_{\mathcal{B}} + 1)$$
$$+ M\int_0^t L(\|z_s + x_s\|_{\mathcal{B}} + 1)ds + M\int_0^t p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})ds$$
$$\leq \overline{M}_0 L\|z_t + x_t\|_{\mathcal{B}} + \overline{M}_0 L(1 + M) + MLn + M\overline{M}_0 L\|\phi\|_{\mathcal{B}}$$

$$+ML\int_0^t \|z_s+x_s\|_{\mathcal{B}}ds + M\int_0^t p(s)\psi(\|z_s+x_s\|_{\mathcal{B}})ds.$$

Since

$$\|z_s+x_s\|_{\mathcal{B}} \le K_n|z(s)| + (K_nMH+M_n)\|\phi\|_{\mathcal{B}} = K_n|z(s)|+c_n,$$

we obtain

$$\begin{aligned}|z(t)| &\le \overline{M}_0L(K_n|z(t)|+c_n) + \overline{M}_0L(1+M) + MLn + M\overline{M}_0L\|\phi\|_{\mathcal{B}}\\ &\quad + ML\int_0^t (K_n|z(s)|+c_n)ds + M\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds\\ &\le \overline{M}_0LK_n|z(t)| + \overline{M}_0L(1+M) + MLn + \overline{M}_0Lc_n + M\overline{M}_0L\|\phi\|_{\mathcal{B}}\\ &\quad + ML\int_0^t (K_n|z(s)|+c_n)ds + M\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds.\end{aligned}$$

Then

$$\begin{aligned}(1-\overline{M}_0LK_n)|z(t)| &\le L\left(\overline{M}_0(1+M) + Mn + \overline{M}_0c_n + M\overline{M}_0\|\phi\|_{\mathcal{B}}\right)\\ &+ ML\int_0^t (K_n|z(s)|+c_n)ds + M\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds.\end{aligned}$$

Set

$$\delta_n := c_n + \frac{LK_n}{1-\overline{M}_0LK_n}\left[\overline{M}_0(1+M) + Mn + \overline{M}_0c_n + M\overline{M}_0\|\phi\|_{\mathcal{B}}\right].$$

Thus,

$$\begin{aligned}K_n|z(t)|+c_n &\le \delta_n + \frac{MLK_n}{1-\overline{M}_0LK_n}\int_0^t (K_n|z(s)|+c_n)ds\\ &\quad + \frac{MK_n}{1-\overline{M}_0LK_n}\int_0^t p(s)\psi(K_n|z(s)|+c_n)ds.\end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup\{\ K_n|z(s)|+c_n\ :\ 0\le s\le t\ \},\quad 0\le t<+\infty.$$

Let $t^\star\in[0,t]$ be such that $\mu(t)=K_n|z(t^\star)|+c_n$. By the previous inequality, we have

$$\mu(t) \le \delta_n + \frac{MK_n}{1-\overline{M}_0LK_n}\Big[\int_0^t L\ \mu(s)ds + \int_0^t p(s)\ \psi(\mu(s))ds\Big]\quad \text{for } t\in[0,n].$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have $\mu(t)\le v(t)$ for all $t\in[0,n]$. From the definition of v, we have $v(0)=\delta_n$

and

$$v'(t) = \frac{MK_n}{1 - \overline{M}_0 L K_n} \ [\ L\ \mu(t) + p(t)\ \psi(\mu(t))\ ] \quad \text{a.e.} \ \ t \in [0, n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq \frac{MLK_n}{1 - \overline{M}_0 L K_n} \ [\ L\ v(t) + p(t)\ \psi(v(t))\ ] \quad \text{a.e.} \ \ t \in [0, n].$$

Using the condition (4.2) this implies that for each $t \in [0, n]$ we have

$$\begin{aligned}\int_{\delta_n}^{v(t)} \frac{ds}{s + \psi(s)} &\leq \frac{MK_n}{1 - \overline{M}_0 L K_n} \int_0^t \max(L, p(s)) ds \\ &\leq \frac{MK_n}{1 - \overline{M}_0 L K_n} \int_0^n \max(L, p(s)) ds < \int_{\delta_n}^{+\infty} \frac{ds}{s + \psi(s)}.\end{aligned}$$

Thus, for every $t \in [0, n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$.

Now, we shall show that $\widetilde{F} : Z \to B^0_{+\infty}$ is a contraction operator. Indeed, consider $z, \overline{z} \in Z$; for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned}|\widetilde{F}(z)(t) - \widetilde{F}(\overline{z})(t)| &\leq |g(t, z_t + x_t) - g(t, \overline{z}_t + x_t)| \\ &\quad + \int_0^t \|U(t,s)\|_{B(E)} |A(s)[g(s, z_s + x_s) - g(s, \overline{z}_s + x_s)]| ds \\ &\quad + \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s) - f(s, \overline{z}_s + x_s)| ds \\ &\leq \|A^{-1}(t)\|_{B(E)}\ |A(t) g(t, z_t + x_t) - A(t) g(t, \overline{z}_t + x_t)| \\ &\quad + \int_0^t \|U(t,s)\|_{B(E)} |A(s) g(s, z_s + x_s) - A(s) g(s, \overline{z}_s + x_s)| ds \\ &\quad + \int_0^t \|U(t,s)\|_{B(E)} |f(s, z_s + x_s) - f(s, \overline{z}_s + x_s)| ds \\ &\leq \overline{M}_0 L_\star \|z_t + x_t - \overline{z}_t - x_t\|_{\mathcal{B}} + \int_0^t M L_\star \|z_s + x_s - \overline{z}_s - x_s\|_{\mathcal{B}} ds \\ &\quad + \int_0^t M l_n(s) \|z_s + x_s - \overline{z}_s - x_s\|_{\mathcal{B}} ds \\ &\leq \overline{M}_0 L_\star \|z_t - \overline{z}_t\|_{\mathcal{B}} + \int_0^t M[L_\star + l_n(s)] \|z_s - \overline{z}_s\|_{\mathcal{B}} ds.\end{aligned}$$

Using $(A1)$, we obtain

$$\begin{aligned}|\widetilde{F}(z)(t)-\widetilde{F}(\overline{z})(t)| &\leq \overline{M}_0 L_*(K(t)\ |z(t)-\overline{z}(t)| + M(t)\ \|z_0-\overline{z}_0\|_{\mathcal{B}}) \\ &\quad + \int_0^t M[L_* + l_n(s)](K(s)\ |z(s)-\overline{z}(s)| + M(s)\ \|z_0-\overline{z}_0\|_{\mathcal{B}})ds \\ &\leq \overline{M}_0 L_* K_n\ |z(t)-\overline{z}(t)| + \int_0^t MK_n[L_\star + l_n(s)]\ |z(s)-\overline{z}(s)|ds \\ &\leq \overline{M}_0 L_* K_n\ |z(t)-\overline{z}(t)| + \int_0^t \bar{l}_n(s)\ |z(s)-\overline{z}(s)|ds.\end{aligned}$$

Let us take here $\bar{l}_n(t) = MK_n[L_* + l_n(t)]$ for the family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$, then

$$\begin{aligned}|\widetilde{F}(z)(t)-\widetilde{F}(\overline{z})(t)| &\leq \left[\overline{M}_0 L_* K_n\ e^{\tau\ L_n^*(t)}\right]\ \left[e^{-\tau\ L_n^*(t)}\ |z(t)-\overline{z}(t)|\right] \\ &\quad + \int_0^t \left[\bar{l}_n(s)\ e^{\tau\ L_n^*(s)}\right]\left[e^{-\tau\ L_n^*(s)}\ |z(s)-\overline{z}(s)|\right]ds \\ &\leq \overline{M}_0 L_* K_n\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n + \int_0^t \left[e^{\tau\ L_n^*(s)}\right]'\ ds\ \|z-\overline{z}\|_n \\ &\leq \overline{M}_0 L_* K_n\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n + \frac{1}{\tau}\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n \\ &\leq \left[\overline{M}_0 L_* K_n + \frac{1}{\tau}\right]\ e^{\tau\ L_n^*(t)}\ \|z-\overline{z}\|_n.\end{aligned}$$

Therefore,

$$\|\widetilde{F}(z)-\widetilde{F}(\overline{z})\|_n \leq \left[\overline{M}_0 L_* K_n + \frac{1}{\tau}\right]\ \|z-\overline{z}\|_n.$$

So, for $[\overline{M}_0 L_* K_n + \frac{1}{\tau}] < 1$, the operator $\widetilde{F}$ is a contraction for all $n \in \mathbb{N}$. From the choice of Z there is no $z \in \partial Z^n$ such that $z = \lambda\ \widetilde{F}(z)$ for some $\lambda \in (0,1)$. Then the statement $(C2)$ in Theorem 2.10 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that $(C1)$ holds. We deduce that the operator $\widetilde{F}$ has a unique fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in (-\infty, +\infty)$ is a fixed point of the operator $\widetilde{N}$, which is the unique mild solution of the problem $(1.3) - (1.4)$.

5. Applications

To illustrate the previous results, we give in this section two applications.

Example 1. Consider the following partial functional differential equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = a(t,\xi)\dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) & \\ \quad + \displaystyle\int_{-\infty}^{0} P(\theta) r(t, v(t+\theta,\xi))d\theta, & t \geq 0, \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, & t \geq 0, \\ v(\theta,\xi) = v_0(\theta,\xi), & -\infty < \theta \leq 0, \xi \in [0,\pi], \end{cases} \tag{5.1}$$

where $a(t,\xi)$ is a continuous function which is uniformly Hölder continuous in t and $P : \mathbb{R}_- \to \mathbb{R}$, $r : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $v_0 : \mathbb{R}_- \times [0,\pi] \to \mathbb{R}$ are given functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,\xi)w''$ with domain $D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}$. Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumption $(H1)$ (see [18, 33]).

For the phase space $\mathcal{B}$, we choose the well-known space $BUC(\mathbb{R}_-;E)$, the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\varphi\| = \sup_{\theta\leq 0} |\varphi(\theta)| \quad \text{for} \quad \varphi \in \mathcal{B}.$$

If $\varphi \in BUC(\mathbb{R}_-;E)$ and $\xi \in [0,\pi]$ we put

$$y(t)(\xi) = v(t,\xi),\ t \geq 0,\ \xi \in [0,\pi],$$

$$\phi(\theta)(\xi) = v_0(\theta,\xi),\ -\infty < \theta \leq 0,\ \xi \in [0,\pi],$$

and

$$f(t,\varphi)(\xi) = \int_{-\infty}^{0} P(\theta) r(t, \varphi(\theta)(\xi))d\theta,\ -\infty < \theta \leq 0,\ \xi \in [0,\pi].$$

Thus, under the above definitions of f, ϕ and $A(\cdot)$ the system (5.1) takes the abstract partial functional evolution equation form $(1.1) - (1.2)$. In order to show the existence of the unique mild solution of system (5.1), we suppose the following assumptions:

- There exists a continuous function $p \in L^1(\mathbb{R}_+,\mathbb{R}_+)$ and a nondecreasing continuous function $\psi : \mathbb{R}_+ \to (0,+\infty)$ such that

$$|r(t,u)| \leq p(t)\psi(|u|),\ \text{for } t \in \mathbb{R}_+,\ \text{and } u \in \mathbb{R}.$$

- P is integrable on $\mathbb{R}_-$.

By the dominated convergence theorem, one can show that f is a continuous function from $\mathcal{B}$ to E. In fact, we have for $\varphi \in \mathcal{B}$ and $\xi \in [0,\pi]$

$$|f(t,\varphi)(\xi)| \leq \int_{-\infty}^{0} |p(t)P(\theta)|\,\psi(|(\varphi(\theta))(\xi)|)d\theta.$$

Since the function ψ is nondecreasing, it follows that

$$|f(t,\varphi)| \leq p(t)\int_{-\infty}^{0} |P(\theta)|\,d\theta\psi(|\varphi|), \text{ for } \varphi \in \mathcal{B}.$$

Thus, the condition (3.2) of Theorem 3.2 ensures the existence of a unique mild solution of the system (5.1) by Theorem 2.10.

Example 2. Consider the following semilinear neutral evolution equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[z(t,\xi) - \displaystyle\int_{-\infty}^{0} T(\theta)w(t,z(t+\theta,\xi))d\theta\Big] = a(t,\xi)\dfrac{\partial^2 z}{\partial \xi^2}(t,\xi) \\ \qquad + \displaystyle\int_{-\infty}^{0} P(\theta)r(t,z(t+\theta,\xi))d\theta, & t \geq 0, \xi \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, & t \geq 0, \\ z(\theta,\xi) = z_0(\theta,\xi), & -\infty < \theta \leq 0, \xi \in [0,\pi], \end{cases} \tag{5.2}$$

where $a(t,\xi)$ is a continuous function which is uniformly Hölder continuous in t, $T, P : \mathbb{R}_- \to \mathbb{R}$; $w, r : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $z_0 : \mathbb{R}_- \times [0,\pi] \to \mathbb{R}$.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)w = a(t,\xi)w''$ with domain $D(A) = \{w \in E : w, w'$ are absolutely continuous $, w'' \in E,\ w(0) = w(\pi) = 0\}$. Then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumptions $(H1)$ and $(H4)$ (see [18, 33]).

As before, for the phase space $\mathcal{B}$ we choose $BUC(\mathbb{R}_-;E)$.

If $\varphi \in BUC(\mathbb{R}_-;E)$ and $\xi \in [0,\pi]$ we put

$$y(t)(\xi) = z(t,\xi),\ t \geq 0,\ \xi \in [0,\pi],$$

$$\phi(\theta)(\xi) = z_0(\theta,\xi),\ -\infty < \theta \leq 0,\ \xi \in [0,\pi],$$

$$g(t,\varphi)(\xi) = \int_{-\infty}^{0} T(\theta)w(t,\varphi(\theta)(\xi))d\theta,\ -\infty < \theta \leq 0,\ \xi \in [0,\pi],$$

and

$$f(t,\varphi)(\xi) = \int_{-\infty}^{0} P(\theta)r(t,\varphi(\theta)(\xi))d\theta,\ -\infty < \theta \leq 0,\ \xi \in [0,\pi].$$

Thus, under the above definitions of f, g, ϕ and $A(\cdot)$, the system (5.2) takes the abstract neutral functional evolution equation form $(1.3) - (1.4)$. In order to show the existence of the unique mild solution of (5.2), we suppose the following assumptions:

- w is Lipschitz with respect to its second argument. Let $lip(w)$ denotes the Lipschitz constant of w.
- There exist a continuous function $p \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and a nondecreasing continuous function $\psi : \mathbb{R}_+ \to (0, +\infty)$ such that

$$|r(t,u)| \leq p(t)\psi(|u|), \text{ for } t \in \mathbb{R}_+, \text{ and } u \in \mathbb{R}.$$

- T and P are integrable on $\mathbb{R}_-$.

By the dominated convergence theorem, one can show that f is a continuous function from $\mathcal{B}$ to E. Moreover the mapping g is Lipschitz continuous in its second argument; in fact, we have

$$|g(t,\varphi_1) - g(t,\varphi_2)| \leq \overline{M}_0 L_\star lip(w) \int_{-\infty}^{0} |T(\theta)|\, d\theta\, |\varphi_1 - \varphi_2|\,, \text{ for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

On the other hand, we have for $\varphi \in \mathcal{B}$ and $\xi \in [0, \pi]$

$$|f(t,\varphi)(\xi)| \leq \int_{-\infty}^{0} |p(t)P(\theta)|\, \psi(|(\varphi(\theta))(\xi)|) d\theta.$$

Since the function ψ is nondecreasing, it follows that

$$|f(t,\varphi)| \leq p(t) \int_{-\infty}^{0} |P(\theta)|\, d\theta \psi(|\varphi|), \text{ for } \varphi \in \mathcal{B}.$$

Thus, by Theorem 2.10 the condition (4.2) of Theorem 4.2 ensures the existence of the unique mild solution of the neutral evolution system (5.2).

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