Differential and Integral Equations Volume 23, Numbers 1-2 (2010), 1–30

SEMILINEAR DIFFERENTIAL AND INTEGRODIFFERENTIAL EQUATIONS WITH HILLE-YOSIDA OPERATORS

Eugenio Sinestrari
Dipartimento di Matematica, Università di Roma "La Sapienza"
00185 Roma, Italy

(Submitted by: Giuseppe Da Prato)

Abstract. Existence and uniqueness of weak and strong solutions for semilinear differential and integrodifferential abstract equations with a Hille-Yosida operator are proved together with criteria for the global existence of the solutions. An application is given to a semilinear one-dimensional wave equation.

1. Introduction

Let X be a Banach space. A linear operator $A : D(A) \subset X \to X$ is called a Hille-Yosida operator if there exist ω, $M \in \mathbb{R}$ such that if $\lambda > \omega$, then there exists $(\lambda - A)^{-1} \in \mathcal{L}(X)$ and $\|(\lambda-\omega)^n(\lambda-A)^{-n}\| \leq M$ for each $n \in \mathbb{N}$. It is known that if in addition $\overline{D(A)} = X$, then A is the generator of a semigroup by virtue of the Hille-Yosida theorem.

Given $f : [0,T] \to X$ and $u_0 \in X$, the linear problem

$$u'(t) = Au(t) + f(t), \quad t \in [0,T]; \qquad u(0) = u_0, \tag{1.1}$$

has been studied in [2] where existence theorems for weak and strong solutions are proved. The aim of this paper is the study of the semilinear differential problem

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in [0,T]; \qquad u(0) = u_0, \tag{1.2}$$

where $F : [0,T] \times X \to X$, and of the semilinear integrodifferential problem

$$u'(t) = Au(t) + f(t) + \int_0^t K(t,s,u(s))ds, \quad t \in [0,T]; \qquad u(0) = u_0, \tag{1.3}$$

where $K : \Delta_T \times X \to X$ with $\Delta_T = \{(t,s) \in \mathbb{R}^2, 0 \leq s \leq t \leq T\}$.

Problem (1.2) was studied by H. Thieme in [14] under the assumption that $F(t,\cdot)$ is defined in a close convex set and has a linear growth. N. Tanaka

Accepted for publication: April 2009.
AMS Subject Classifications: 47D06, 35L05, 35L90.

in [13] assumes that A depends on $t \in [0,T]$ and studies the strict solutions of (1.1) as does Y. Lei in [7] with different methods. P. Magal in [8] finds weak solutions of (1.1) when $F(t,\cdot)$ is Lipschitz continuous on bounded sets which are strong when F is continuously differentiable.

The proofs of [14] are based on the Thieme's representation formula for the weak solution of (1.1); [13] uses the nonlinear semigroup theory and the other papers are based on the existence and estimates of the solutions of (1.1) proved in [2].

The integrodifferential problem was studied when $K(t,s,\cdot)$ is defined in $D(A)$ in [10] but our case seems to be new.

In the first part of this paper, we prove the existence and uniqueness of the weak solutions (defined in two equivalent ways) of (1.2) when $F(t,\cdot)$ is locally Lipschitz continuous and prove its regularity when $F(t,\cdot)$ is continuously differentiable. Moreover, a condition is given for global existence which is applied to the classical solutions of a semilinear wave equation.

In the second part, the study of the semilinear integrodifferential equation (1.3) concerns (for brevity) only the strict solutions but also in this case a condition for the existence in the large of a solution is given when the kernel is of convolution type. In addition to the usual notation, we use the following: if $(X, \|\cdot\|)$ is a Banach space, $u_0 \in X$ and $r > 0$ we set $B_X(u_0,r) := \{x \in X : \|x-u_0\| < r\}$ and $P_X(u_0,r) := \{x \in X : \|x-u_0\| \leq r\}$. We also define $W^{1,1}(0,T;X) := \{u : [0,T] \to X : \exists v \in L^1(0,T;X), u(t) = u(0) + \int_0^t v(s)ds, t \in [0,T]\}$.

2. The linear problem

Let us recall some definitions and results concerning the linear case which are the basic tool for the treatment of the semilinear ones. They have been established in [2] and we state them without proofs.

Definition 2.1. Let $A : D(A) \subset X \to X$ be a linear operator in a Banach space X. If there exist $\omega, M \in \mathbb{R}$ such that if $\lambda > \omega$, then $(\lambda - A)^{-1} \in \mathcal{L}(X)$ and $\|(\lambda-\omega)^n(\lambda-A)^{-n}\| \leq M$ for each $n \in \mathbb{N}$, then A is called a Hille-Yosida operator of type (M,ω).

If, in addition, $\overline{D(A)} = X$, then A is the generator of a semigroup in X by virtue of the Hille-Yosida theorem. More generally, we have the following result (see [5, Theorem 12.2.4]).

Theorem 2.2. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator. Setting $X_0 = \overline{D(A)}$, $A_0 : D(A_0) \subset X_0 \to X_0$, where $D(A_0) := \{x \in D(A) : Ax \in \overline{D(A)}\}$ we have that A_0 is the generator of a semigroup $e^{A_0 t}$ in X_0.*

In what follows, A, X_0 and A_0 will be defined as above.

For the linear problem

$$u'(t) = Au(t) + f(t), \quad t \in [0,T]; \qquad u(0) = u_0, \tag{2.1}$$

where $f : [0,T] \to X$ is continuous, several definitions of solution can be given (see [2]). A function $u \in C^1(0,T;X) \cap C(0,T;D(A))$ and satisfying (2.1) is called a strict solution. If $u \in C(0,T;X)$, $\int_0^t u(s)ds \in D(A)$ for $t \in [0,T]$ and

$$u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t f(s)ds, \qquad t \in [0,T], \tag{2.2}$$

then u is called an integral solution.

If there exist $u_n \in C^1(0,T;X) \cap C(0,T;D(A))$, $n \in \mathbb{N}$ such that

$$\begin{cases} \lim\limits_{n\to\infty} u_n(t) =: u(t) & \text{and } \lim_{n\to\infty} u_n'(t) - Au_n(t) = f(t) \\ & \text{uniformly for } t \in [0,T] \\ \lim\limits_{n\to\infty} u_n(0) = u_0, & \end{cases} \tag{2.3}$$

then u is called an F-solution (Friedrich's solution). In this case, $u \in C(0,T;X_0)$ and $u(0) = u_0$. We deduce that an integral and an F-solution have values in X_0, hence their existence implies that $u_0 \in X_0$. A strict solution is an integral and an F-solution: but these two coincide. In fact in [2] the following result is proved.

Theorem 2.3. *If $f \in C(0,T;X)$ and $u_0 \in X_0$, then there exists a unique integral solution u of* (2.1) *and*

$$\|u(t)\| \le Me^{\omega t}\|u_0\| + M\int_0^t e^{\omega(t-s)}\|f(s)\|ds, \quad t \in [0,T]. \tag{2.4}$$

An integral solution is an F-solution and conversely.

When $f \in L^1(0,T;E)$ we give a similar definition of F-solution and Theorem 2.3 holds also in this case.

In the same paper, a temporal regularity result is demonstrated.

Theorem 2.4. *If $f \in W^{1,1}(0,T;X), u_0 \in D(A)$ and $Au_0 + f(0) \in X_0$, then* (2.1) *has a unique strict solution and*

$$\|Au(t)\| \le \|f(t)\| + Me^{\omega t}\|Au_0 + f(0)\| + M\int_0^t e^{\omega(t-s)}\|f'(s)\|ds. \tag{2.5}$$

The conditions $u_0 \in D(A)$ and $Au_0 + f(0) \in X_0$ are necessary for the existence of such a solution.

3. The semilinear differential problem

Given a continuous nonlinear function $F : [0,T] \times X_0 \to X$ and $u_0 \in X_0$, consider the semilinear (differential) problem

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in [0,T]; \qquad u(0) = u_0. \tag{3.1}$$

Also, in this case we give three definitions of solution.

Definition 3.1. A strict solution of (3.1) is $u \in C^1(0,T;X) \cap C(0,T;D(A))$ satisfying (3.1).

Definition 3.2. *u is an integral solution of* (3.1) *if $u \in C(0,T;X)$, $\int_0^t u(s)ds \in D(A)$ for $t \in [0,T]$ and*

$$u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds, \qquad t \in [0,T]. \tag{3.2}$$

In this case, $u(t) \in X_0$ for each $t \in [0,T]$ and $u_0 \in X_0$.

Definition 3.3. *If there exist $u_n \in C^1(0,T;X) \cap C(0,T;D(A))$, $n \in N$, such that*

$$\begin{cases} \lim\limits_{n\to\infty} u_n(t) =: u(t) \quad & \text{and } \lim\limits_{n\to\infty} u_n'(t) - Au_n(t) - f(t,u_n(t)) = 0 \\ & \text{uniformly for } t \in [0,T] \\ \lim\limits_{n\to\infty} u_n(0) = u_0, \end{cases} \tag{3.3}$$

then u is called an F-solution of (3.1).

An F-solution is the uniform limit of strict solutions of a suitable approximating problems. In some cases the converse is true.

Theorem 3.4. *For each $n \in \mathbb{N}$ let $F_n : [0,T] \times X_0 \to X$ be a continuous function such that there exists $\lim_{n\to\infty} F_n(t,x) =: F(t,x)$ uniformly on bounded sets of $[0,T] \times X_0$ and let $u_{0n} \in X_0$, $n \in \mathbb{N}$, be such that $\lim_{n\to\infty} u_{0n} =: u_0$. If for $n \in \mathbb{N}$ the problem*

$$u_n'(t) = Au_n(t) + F_n(t, u_n(t)), \quad t \in [0,T]; \qquad u_n(0) = u_{0n} \tag{3.4}$$

has a strict solution u_n and there exists

$$\lim_{n\to\infty} u_n(t) =: u(t) \text{ uniformly for } t \in [0,T], \tag{3.5}$$

then u is an F-solution of (3.1).

Proof. We can suppose that $u_n(t)$ is in a bounded subset of X_0, for each $t \in [0,T]$ and $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} F_n(t,u_n(t) = F(t,u_n(t))$, uniformly for $t \in [0,T]$. From (3.4) we obtain

$$u_n'(t) - Au_n(t) - F(t,u_n(t)) = F_n(t,u_n(t)) - F(t,u_n(t)),$$

hence, (3.3) holds. □

Because of the next theorem, we can call weak solutions the F-solutions and the integral solutions.

Theorem 3.5. *An integral solution of* (3.1) *is an F-solution and conversely.*

Proof. Let u be an F-solution and (3.3) hold; for each $t \in [0,T]$ and $n \in \mathbb{N}$ we have

$$\int_0^t u_n(s)ds \in D(A) \quad \text{and} \quad \lim_{n\to\infty} \int_0^t u_n(s)ds = \int_0^t u(s)ds.$$

Setting $f_n(t) := u_n'(t) - Au_n(t) - F_n(t,u_n(t))$, $t \in [0,T]$, we get

$$u_n(t) = u_n(0) + A\int_0^t u_n(s)ds + \int_0^t F(s,u_n(s))ds + \int_0^t f_n(s)ds.$$

But $\lim_{n\to\infty} F(s,u_n(s)) = F(s,u(s))$ and $\lim_{n\to\infty} f_n(s) = 0$ uniformly for $s \in [0,t]$. Hence, there exists

$$\lim_{n\to\infty} A\int_0^t u_n(s)ds = u(t) - u_0 - \int_0^t F(s,u(s))ds;$$

as A is closed we deduce (3.2) and the fact that u is an integral solution of (3.1).

Let u be an integral solution of (3.1); then u is an integral solution of the linear problem (2.1) with $f(t) := F(t,u(t))$, $t \in [0,T]$. Hence, u is also an F-solution of the same problem (see Theorem 2.3). If $\{u_n\}$ and $\{u_{0n}\}$ satisfy (2.3) we deduce

$$\lim_{n\to\infty} u_n'(t) - Au_n(t) = F(t,u(t)) \quad \text{and} \quad \lim_{n\to\infty} F(t,u_n(t)) = F(t,u(t)),$$

uniformly for $t \in [0,T]$ and so

$$\lim_{n\to\infty} u_n'(t) - Au_n(t) - F(t,u_n(t) = 0,$$

uniformly for $t \in [0,T]$. In conclusion u is an F-solution of problem (3.1). □

4. Weak solutions of the semilinear problem

By virtue of Theorem 2.3 there is existence and uniqueness of a weak solution u of the linear problem (2.1) for each $f \in C(0,T;X)$ and $u_0 \in X_0$ and the pointwise estimate (2.4) holds for u. Hence, we can define an operator $S : C(0,T;X) \oplus X_0 \to C(0,T;X_0)$ such that, if $f \in C(0,T;X)$ and $u_0 \in X_0$, then

$$u := S(f, u_0) \tag{4.1}$$

is the weak solution of problem (2.1).

Let us first prove the existence of a weak solution of (3.1) under the assumption of global Lipschitz continuity of $F(t,\cdot)$; later we will relax this requirement.

Theorem 4.1. *If $F : [0,T] \times X_0 \to X$ is continuous and there exists $L > 0$ such that*

$$\|F(t,x') - F(t,x'')\| \le L\|x' - x''\|, \quad t \in [0,T]; \qquad x', x'' \in X_0, \tag{4.2}$$

then for each $u_0 \in X_0$ there exists a unique weak solution $u : [0,T] \to X_0$ of (3.1) *and*

$$\|u(t)\| \le Me^{(\omega+ML)t}\Big(\|u_0\| + \int_0^t e^{-\omega s}\|F(s,0)\|ds\Big). \tag{4.3}$$

If $u(t,u_0)$ denotes the weak solution of (3.1)*, then*

$$\|u(t,u_1) - u(t,u_2)\| \le Me^{(\omega+ML)t}\|u_1 - u_2\|, \; t \in [0,T], \; u_1, u_2 \in X_0, \tag{4.4}$$

and the mapping $(t,w) \to u(t,w)$ is continuous from $[0,T]\times X_0$ to $C(0,T;X_0)$. In particular,

$$\lim_{u_1 \to u_2} \|u(t,u_1) - u(t,u_2)\| = 0 \text{ uniformly for } t \in [0,T]. \tag{4.5}$$

Proof. Defining the operator $\Phi : C(0,T;X_0) \to C(0,T;X)$ for each $u \in C(0,T;X_0)$ as

$$\Phi(u)(t) := F(t,u(t)), \qquad t \in [0,T], \tag{4.6}$$

and $R : C(0,T;X_0) \to C(0,T;X_0)$ for each $u \in C(0,T;X_0)$ as $R(u) = S(\Phi(u), u_0)$ we see that the existence and uniqueness of a weak solution of (3.1) is equivalent to the existence of a unique fixed point of the operator R in $C(0,T;X_0)$.

Let us fix $\rho > \omega + ML$ and denote by Γ the space $C(0,T;X_0)$ with the norm $\|u\|_\Gamma = \sup_{s\le t\le T} e^{-\rho t}\|u(t)\|$. If u_1, $u_2 \in \Gamma$, we have (by using (4.1) and (2.4))

$$\|R(u_1) - R(u_2)\|_\Gamma = \|S\left(\Phi(u_1) - \Phi(u_2), 0\right)\|_\Gamma$$

$$
\begin{aligned}
&\leq M \sup_{0\leq t\leq T} e^{-\rho t}\int_0^t e^{\omega(t-s)}\|F(s,u_1(s))-F(s,u_2(s))\|ds\\
&\leq ML \sup_{0\leq t\leq T}\int_0^t e^{-(\rho-\omega)(t-s)}e^{-\rho s}\|u_1(s)-u_2(s)\|ds\\
&\leq ML \sup_{0\leq t\leq T}\int_0^t e^{-(\rho-\omega)(t-s)}ds\cdot\|u_1-u_2\|_\Gamma \leq \frac{ML}{\rho-\omega}\|u_1-u_2\|_\Gamma;
\end{aligned}
$$

hence, R is a contraction; this proves that there is a unique weak solution of (3.1) in $[0,T]$.

From (4.1) and (2.4) we deduce also

$$
\begin{aligned}
\|M^{-1}e^{-\omega t}u(t)\| &\leq \|u_0\| + \int_0^t e^{-\omega s}\|F(s,u(s))\|ds\\
&\leq \|u_0\| + \int_0^t e^{-\omega s}\|F(s,0)\|ds + L\int_0^t e^{-\omega s}\|u(s)\|ds,
\end{aligned}
$$

from Gronwall's lemma (see [3, page 76]) we deduce (4.3).

Given u_1, $u_2\in X_0$, set $v(t):=u(t,u_1)-u(t,u_2)$, $t\in[0,T]$. As v is a weak solution of the problem

$$
v'(t)=Av(t)+F(t,u_1(t))-F(t,u_2(t)),\quad t\in[0,T];\qquad v(0)=u_1-u_2,
$$

from (2.4) we get

$$
\begin{aligned}
\|v(t)\| &\leq Me^{\omega t}\|u_1-u_2\| + M\int_0^t e^{\omega(t-s)}\|F(s,u_1(s))-F(s,u_2(s))\|ds\\
&\leq Me^{\omega t}\|u_1-u_2\| + MLe^{\omega t}\int_0^t e^{-\omega s}\|v(s))\|ds.
\end{aligned}
$$

By using again Gronwall's lemma we have

$$
\|M^{-1}e^{-\omega t}v(t)\| \leq \|u_1-u_2\|e^{MLt},
$$

i.e., (4.4). If $t,s\in[0,T]$, from (4.4) we obtain

$$
\begin{aligned}
\|u(t,u_1)-u(s,u_2)\| &\leq \|u(t,u_1)-u(t,u_2)\| + \|u(t,u_2)-u(s,u_2)\|\\
&\leq Me^{|\omega+ML|T}\|u_1-u_2\| + \|u(t,u_2)-u(s,u_2)\|,
\end{aligned}
$$

and the conclusion follows. □

We will use later the following.

Theorem 4.2. *Let F be continuous and u_0, $u_1 \in X_0$. If $u(\cdot, u_1)$ is the weak solution of* (3.1) *and $e^{A_0 t}$ is the semigroup in X_0 associated to A (see Theorem* 2.2*), then, for $t \in [0,T]$,*

$$\|u(t,u_1) - u_0\| \leq \|e^{A_0 t}u_1 - u_0\| + M\int_0^t e^{-\omega(t-s)}\|F(s,u(s,u_1)\|ds. \quad (4.7)$$

Proof. As $u_1 \in X_0$ it is known that

$$\int_0^t e^{A_0 s}u_1 ds \in D(A_0) \quad \text{and} \quad A\int_0^t e^{A_0 s}u_1 ds = e^{A_0 t}u_1 - u_1.$$

Hence, $u(t) := e^{A_0 t}u_1$, $t \in [0,T]$ is the weak solution of the problem

$$u'(t) = Au(t), \quad t \in [0,T]; \qquad u(0) = u_1.$$

Setting $f(t) := F(t, u(t,u_1)$, $t \in [0,T]$, let v be the weak solution of the linear problem

$$v'(t) = Av(t) + f(t), \quad t \in [0,T]; \qquad v(s) = 0.$$

Hence, $u + v$ is the weak solution of the linear problem

$$w'(t) = Aw(t) + f(t), \quad t \in [0,T]; \qquad w(0) = u_1$$

and so $u(t,u_1) = u(t) + v(t)$. From (2.4) we get, for $t \in [0,T]$,

$$\begin{aligned}\|u(t,u_1) - u_0\| &\leq \|e^{A_0 t}u_1 - u_0\| + \|v(t)\| \\ &\leq \|e^{A_0 t}u_1 - u_0\| + M\int_0^t e^{-\omega(t-s)}\|F(s,u(s,u_1))\|ds,\end{aligned}$$

i.e., (4.7). □

Definition 4.3. *We say that $F : [0,T] \times X_0 \to X$ satisfies (LL) if F is continuous and given $u_0 \in X_0$ there exist r_0, $L_0 > 0$ such that*

$$\|F(t,x') - F(t,x'')\| \leq L_0\|x' - x''\|, \ t \in [0,T]; \ x', x'' \in P_{X_0}(u_0, r_0). \quad (4.8)$$

Under this condition we can prove a uniqueness result for the weak solutions of (3.1).

Theorem 4.4. *Let F satisfy (LL). If $u_1 : [0,T_1] \to X_0$, $u_2 : [0,T_2] \to X_0$ are solutions of problem* (3.1)*, then $u_1 \equiv u_2$ in $[0, T_1 \wedge T_2]$.*

Proof. Suppose that $T_1 \leq T_2$; as $u_1(0) = u_2(0)$ we can define $T_0 := \max\{T' \in [0,T_1] : u_1(t) = u_2(t), \ t \in [0,T']\}$ and so $u_1 \equiv u_2$ in $[0,T_0]$. If by contradiction $T_0 < T_1$ we have for each $\delta \in (0, T_1 - T_0)$

$$\sup\{\|u_1(T_0 + t) - u_2(T_0 + t)\| : t \in [0,\delta]\} > 0, \quad (4.9)$$

and there exist r, $L > 0$ such that

$$\|F(t,x) - F(t,y)\| \leq L\|x-y\|, \quad t \in [0,T]; \quad x, y \in P_{X_0}(u_1(T_0), r).$$

Let us choose $\delta > 0$ such that

$$\delta < T_1 - T_0,\ Me^{|\omega|T_2}L\delta < 1;\ u_1(T_0+t),\ u_2(T_0+t) \in P_{X_0}(u_1(T_0), r),\ t \in [0,\delta]. \tag{4.10}$$

As u_1 and u_2 are integral solutions which coincide on $[0, T_0]$ we have for $t \in [0, \delta]$

$$\begin{aligned} u_1(T_0+t) - u_2(T_0+t) &= A\int_0^t [u_1(T_0+s) - u_2(T_0+s)]ds \\ &+ \int_0^t [F(T_0+s, u_1(T_0+s)) - F(T_0+s, u_2(T_0+s))]ds. \end{aligned}$$

Hence, from (2.4)

$$\begin{aligned} &\|u_1(T_0+t) - u_2(T_0+t)\| \\ &\leq Me^{|\omega|T_2}\int_0^t \|F(T_0+s, u_1(T_0+s)) - F(T_0+s, u_2(T_0+s))\|ds \\ &\leq Me^{|\omega|T_2}L\delta \sup_{t\in[0,\delta]} \|u_1(T_0+t) - u_2(T_0+t)\|, \end{aligned}$$

and so

$$\sup_{t\in[0,\delta]} \|u_1(T_0+t) - u_2(T_0+t)\| \leq Me^{|\omega|T_2}L\delta \sup_{t\in[0,\delta]} \|u_1(T_0+t) - u_2(T_0+t)\|.$$

By using (4.9) we deduce $1 \leq Me^{|\omega|T_2}L\delta$ which contradicts (4.10). We conclude that $T_0 = T_1$. □

We can prove now the existence and uniqueness of the weak solution of the semilinear problem (3.1) when $F(t,\cdot)$ is locally Lipschitz continuous in X_0 (uniformly for $t \in [0,T]$). We examine also the dependence of the interval of existence of the solution upon the initial value.

Theorem 4.5. *Let $F : [0,T] \times X_0 \to X$ be continuous and suppose that, for each $u_0 \in X_0$, there exist r_0, $L_0 > 0$ such that*

$$\|F(t,x') - F(t,x'')\| \leq L_0\|x' - x''\|,\ t \in [0,T],\ x', x'' \in P_{X_0}(u_0, r_0); \tag{4.11}$$

then there exist $r_0' \in (0, r_0]$ and $T_0 \in (0,T]$ such that given $u_1 \in P_{X_0}(u_0, r_0')$ there exists a unique weak solution $u =: u(\cdot, u_1)$ of the problem

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in [0, T_0]; \qquad u(0) = u_1. \tag{4.12}$$

In addition we have the estimate

$$\|u(t)\| \le Me^{(\omega+ML_0)t}\Big(\|u_1\| + \int_0^t e^{-\omega s}\|F(s,0)\|ds\Big), \quad t \in [0,T_0], \tag{4.13}$$

and for $u_2 \in P_{X_0}(u_0, r_0')$

$$\|u(t,u_1) - u(t,u_2)\| \le Me^{(\omega+ML_0)t}\|u_1 - u_2\|, \quad t \in [0,T_0]. \tag{4.14}$$

Proof. Let us choose

$$\mu_0 \ge \sup\{\|F(t,x)\| : t \in [0,T], x \in P_{X_0}(u_0,r_0)\}, \tag{4.15}$$

and then $T_0 \in (0,T]$ such that

$$T_0 \le r_0/2Me^{|\omega|T}\mu_0,\ \sup\{\|e^{A_0 s}u_0 - u_0\| : 0 \le s \le T_0\} \le r_0/4, \tag{4.16}$$

and $r_0' \in (0,r_0]$ such that

$$4Me^{|\omega|T}r_0' \le r_0. \tag{4.17}$$

Taking $u_1 \in P_{X_0}(u_0,r_0')$ we have by virtue of (4.16) and (4.17) for $s \in [0,T_0]$

$$\begin{aligned}\|e^{A_0 s}u_1 - u_0\| &\le \|e^{A_0 s}u_1 - e^{A_0 s}u_0\| + \|e^{A_0 s}u_0 - u_0\| \\ &\le Me^{|\omega|T}\|u_1 - u_0\| + \|e^{A_0 s}u_0 - u_0\| \le \frac{r_0}{2}.\end{aligned} \tag{4.18}$$

Now, $F : [0,T] \times P_{X_0}(u_0,r_0) \to X$ can be extended to a function $\overline{F} : [0,T] \times X_0 \to X$ satisfying the assumption of Theorem 4.1 and so we deduce the existence of a weak solution $v : [0,T] \to X_0$ of the problem

$$v'(t) = Av(t) + \overline{F}(t,v(t)), \quad t \in [0,T]; \qquad v(0) = u_1. \tag{4.19}$$

If we prove that $v(t) \in P_{X_0}(u_0,r_0)$ for $t \in [0,T_0]$, then the restriction of v to $[0,T_0]$ is a weak solution of (4.12). Suppose by contradiction that there exists $t^* \in (0,T_0]$ such that $v(t^*) \notin P_{X_0}(u_0,r_0)$. As $\|v(0) - u_0\| = \|u_1 - u_0\| < r_0$ there exists $\hat{t} := \min\{t \in (0,\, T_0) :\ \|v(t) - u_0\| = r_0\}$. We have $0 < \hat{t} < T_0$ and $v(s) \in P_{X_0}(u_0,r_0)$ for $s \in [0,\hat{t}]$, hence

$$\|\overline{F}(s,v(s))\| = \|F(s,v(s))\| \le \mu_0, \qquad s \in [0,\hat{t}]. \tag{4.20}$$

From (4.18), (4.20) and Theorem 4.2 we deduce

$$\begin{aligned}r_0 = \|v(\hat{t}) - u_0\| &\le \|e^{A_0\hat{t}}u_1 - u_0\| + M\int_0^{\hat{t}} e^{-\omega(\hat{t}-s)}\|F(s,v(s))\|ds \\ &\le \frac{r_0}{2} + M\hat{t}e^{|\omega|T}\mu_0;\end{aligned}$$

hence, $\hat{t} \ge r_0/2Me^{|\omega|T}\mu_0$; from (4.16) we get $\hat{t} \ge T_0$ which is absurd and so (4.12) has a weak solution on $[0,T_0]$, which is unique by virtue of Theorem 4.4.

Estimates (4.13) and (4.14) are proved as were (4.3) and (4.4) because we have shown that $u(t,u_1)$, $u(t,u_2) \in P_{X_0}(u_0,r_0)$ for $t \in [0,T_0]$. □

We pass now to consider the maximal weak solutions of problem (3.1)

Definition 4.6. Given F satisfying (LL) and $u_0 \in \overline{D(A)}$ it is possible to define (by using Theorem 4.4 and 4.5)

$$\begin{cases} J(u_0) = \{\tau \in (0,T] : \text{ there exists a weak solution } u_\tau \text{of} (3.1) \text{ in } [0,\tau]\} \\ I(u_0) = \displaystyle\bigcup_{\tau \in J(u_0)} [0,\tau] \\ u(\cdot,u_0) : I(u_0) \to X_0 \text{ with } u(t,u_0) := u_\tau(t,u_0),\ t \in [0,\tau],\ \tau \in J(u_0). \end{cases} \tag{4.21}$$

If $I(u_0) \neq [0,T]$, then $I(u_0) = [0,T')$ for some $T' > 0$ by virtue of Theorem 4.5.

The function $u(\cdot,u_0)$ defined by (4.21) is a solution in $I(u_0)$ of (3.1) and is called the maximal weak solution.

To study the dependence of the maximal solution from the initial datum we need a lemma, whose proof can be obtained by the usual compactness arguments

Lemma 4.7. *Let F satisfy (LL). Given K, a compact subset of X_0, there exist $r, L > 0$ such that*

$$\|F(t,x') - F(t,x'')\| \leq L\|x' - x''\|, \quad t \in [0,T];$$
$$x', x'' \in K_r := \{x \in X_0 : \text{ dist}(x,K) \leq r\}.$$

Theorem 4.8. *Let F satisfy (LL) and $u(\cdot,u_0) : I(u_0) \to X_0$ be the maximal weak solution of (3.1). For each $[0,T_1] \subseteq I(u_0)$ there exists $r_1 > 0$ such that, choosing $u_1 \in P_{X_0}(u_0,r_1)$, there exists a weak solution $u := u(\cdot,u_1) : [0,T_1] \to X_0$ of*

$$u'(t) = Au(t) + F(t,u(t)), \quad t \in [0,T_1]; \qquad u(0) = u_1, \tag{4.22}$$

and

$$\lim_{u_1 \to u_0} u(t,u_1) = u(t,u_0) \text{ uniformly for } t \in [0,T_1]. \tag{4.23}$$

Proof. Given a compact subset K of X_0 we will prove the existence of $r_0', T_0 > 0$ such that given $u_0 \in K$ and $u_1 \in P_{X_0}(u_0,r_0')$ there exists a unique weak solution $u(\cdot,u_1)$ in $[0,T_0]$.

From Lemma 4.7 we deduce the existence of L, $r > 0$ such that

$$\|F(t,x') - F(t,x'')\| \leq L\|x' - x''\|,\ t \in [0,T];\ x', x'' \in P_{X_0}(u_0,r),\ u_0 \in K, \tag{4.24}$$

hence for $t \in [0,T]$, $u_0 \in K$ and $x \in P_{X_0}(u_0, r)$,

$$\begin{aligned}\|F(t,x)\| &\le \|F(t,x) - F(t,u_0)\| + \|F(t,u_0)\| \\ &\le Lr + \max\{\|F(t,u)\|,\ (t,u) \in [0,T] \times K\},\end{aligned}$$

and so

$$\mu := \sup\{\|F(t,x)\| : t \in [0,T],\ x \in P_{X_0}(u_0,r),\ u_0 \in K\} < \infty. \tag{4.25}$$

We deduce that if in the proof of Theorem 4.5 we suppose $u_0 \in K$, then we can set $r_0 = r$, $L_0 = L$ and $\mu_0 = \mu$ (independent of u_0): choosing $T_0 \in (0,T]$ such that

$$T_0 \le r/2Me^{|\omega|T}\mu, \quad \sup\{\|e^{A_0 s}u_0 - u_0\|, 0 \le s \le T_0, u_0 \in K\} \le \frac{r}{4},$$

and $r_0' \in (0,r)$ such that $4Me^{|\omega|T}r_0' \le r$ we deduce from the proof of Theorem 4.5 the existence of r_0' and T_0 with the required properties and so given $[0,T_1] \subseteq I(u_0)$ and set $K := \{u(t,u_0) : t \in [0,T_1]\}$ we can deduce that if $u_1 \in P_{X_0}(u_0, r_0')$ there exists a weak solution $u(\cdot, u_1)$ in $[0,T_0]$ and (from (4.14))

$$\lim_{u_1 \to u_0} u(t,u_1) = u(t,u_0) \text{ uniformly for } t \in [0,T_0].$$

If $T_0 \ge T_1$ the proof is finished; if $T_0 < T_1$, from (4.14) we get the existence of $r_0'' \in (0, r_0'[$ such that, if $u_1 \in P_{X_0}(u_0, r_0'')$, then $u(T_0,u_1) \in P_{X_0}(u(T_0,u_0), r_0')$. By application of Theorem 4.5 to the problem

$$u'(t) = Au(t) + F(T_0 + t, u(t)), t \in [0, T_1 - T_0], u(0) = u(T_0, u_1),$$

we find a weak solution in $[0, \min(T_0,T_1)]$, hence $u(\cdot,u_1)$ can be extended into a solution defined in $[0, \min(2T_0,T_1)]$ and converging uniformly to $u(\cdot,u_0)$. After a finite number of steps we get the conclusion. □

5. Strict solutions

In this section we want to extend the classical result by I. Segal (see [11, Chapter 6, Theorem 1.6]), which gives the regularity of the weak solution when the nonlinear term F is continuously differentiable. To this purpose we need three lemmas.

Lemma 5.1. *Let $B : [0,T] \to \mathcal{L}(X)$ be continuous, $f \in L^1(0,T;X)$ and $v_0 \in X_0$. There exists a unique $v \in C(0,T;X_0)$ such that $\int_0^t v(s)ds \in D(A)$, $t \in [0,T]$ and*

$$v(t) = v_0 + A\int_0^t v(s)ds + \int_0^t B(s)v(s)ds + \int_0^t f(s)ds, \quad t \in [0,T]. \tag{5.1}$$

Moreover, we have

$$\|v(t)\| \le M\Big(\|v_0\| + \int_0^t e^{-\omega s}\|f(s)\|ds\Big)e^{\omega t + M\int_0^t \|B(s)\|ds}, \quad t \in [0,T]. \quad (5.2)$$

Proof. Let S be the operator defined in (4.1). The solution of our problem is a fixed point in $C(0,T;X_0)$ of the operator $R(v) := S(f + B(\cdot)v(\cdot), v_0)$, $v \in C(0,T;X_0)$. Setting $\|B\| := \sup\{\|B(t)\|_{\mathcal{L}(X)} : t \in [0,T]\}$, choose $\rho > \omega + M\|B\|$ and denote by Γ the space $C(0,T;X_0)$ with the norm $\|v\|_\Gamma = \sup_{0\le t\le T}\|e^{-\rho t}v(t)\|$. By using (2.4) we get for $v_1, v_2 \in C(0,T;X_0)$

$$\begin{aligned}&\|R(v_1) - R(v_2)\|_\Gamma = \|S(B(\cdot)(v_1 - v_2), 0)\|_\Gamma \\ &\le \sup_{t\in[0,T]} Me^{-\rho t}\int_0^t e^{\omega(t-s)}\|B(s)(v_1(s) - v_2(s))\|ds \le \frac{M\|B\|}{\rho - \omega}\|v_1 - v_2\|_\Gamma.\end{aligned}$$

Hence, R is a contraction and the first part of the theorem is proved.

From (5.1) and (2.4) we obtain

$$\begin{aligned}\|M^{-1}e^{\omega t}v(t)\| &\le \|v_0\| + \int_0^t e^{-\omega s}\|f(s) + B(s)v(s)\|ds \\ &\le \|v_0\| + \int_0^t e^{-\omega s}\|f(s)\|ds + M\int_0^t \|B(s)\|M^{-1}e^{-\omega s}\|v(s)\|ds.\end{aligned}$$

From Gronwall's lemma (see [3, page 76]) we get

$$\|M^{-1}e^{-\omega t}v(t)\| \le (\|v_0\| + \int_0^t e^{-\omega s}\|f(s)\|ds)e^{M\int_0^t \|B(s)\|ds},$$

which is (5.2). □

Lemma 5.2. *Consider the Favard class of A_0, i.e.*

$$D_{A_0}(1,\infty) := \Big\{x \in X_0 : \sup_{t\in(0,1]}\frac{\|e^{A_0 t}x - x\|}{t} < \infty\Big\}$$

(see [1]). We have $D(A) \subset D_{A_0}(1,\infty)$ and if $f \in C(0,T;X)$, $u_0 \in D_{A_0}(1,\infty)$ and u is the weak solution of (2.1), then there exists $L_0 > 0$ such that

$$\|u(h) - u_0\| \le L_0 \cdot h, \qquad h \in [0,T]. \quad (5.3)$$

If in addition $u_0 \in D(A)$ and $Au_0 + f(0) \in X_0$, then there exists $u'(0) = Au_0 + f(0)$.

Proof. The inclusion $D(A) \subset D_{A_0}(1,\infty)$ is proved in [9], Proposition 3.2.

We can write $u = u_1 + u_2$ where u_1 and u_2 are weak solutions of

$$u_1'(t) = Au_1(t) + f(t), \quad t \in [0,T]; \qquad u_1(0) = 0,$$

and

$$u_2'(t) = Au_2(t), \quad t \in [0,T]; \qquad u_2(0) = u_0.$$

From (2.4) we deduce

$$\|u_1(t)\| \le Me^{|\omega|T} t \sup_{t\in[0,T]} \|f(t)\| =: C_1 \cdot t,$$

and in the proof of Corollary 4.2 we have seen that $u_2(t) = e^{A_0 t}u_0$; as $u_0 \in D_{A_0}(1,\infty)$ there exists $C_2 > 0$ such that $\|u_2(t) - u_0\| \le C_2 t$, $t \in [0,T]$. Hence, if $h \in [0,T]$,

$$\|u(h) - u_0\| \le \|u_1(h)\| + \|u_2(h) - u_0\| \le C_1 h + C_2 h,$$

i.e., (5.3) holds.

Writing $u = \hat{u}_1 + \hat{u}_2$ where $\hat{u}_1$ and $\hat{u}_2$ are weak solutions of

$$\hat{u}_1'(t) = A\hat{u}_1(t) + f(t) - f(0), \quad t \in [0,T]; \quad \hat{u}_1(0) = 0$$
$$\hat{u}_2'(t) = A\hat{u}_2(t) + f(0), \qquad t \in [0,T]; \quad \hat{u}_2(0) = u_0,$$

we deduce from (2.4) that

$$\|\hat{u}_1(t)\| \le Me^{|\omega|T} \int_0^t \|f(s) - f(0)\| ds, \qquad t \in [0,T],$$

hence, $\lim_{t\to 0^+} \frac{\|\hat{u}_1(t)\|}{t} = 0$. If $u_0 \in D(A)$ and $Au_0 + f(0) \in \overline{D(A)}$ from Theorem 2.4 we deduce the existence of $\hat{u}_2'(0) = Au_0 + f(0)$; in conclusion $\lim_{t\to 0^+} \|\frac{u(t)-u_0}{t} - Au_0 - f(0)\| = 0$. □

Lemma 5.3. *Let F be locally Lipschitz continuous on $[0,T]\times X_0$ and $u_0 \in D_{A_0}(1,\infty)$ (see Lemma 5.2). A weak solution $u : [0,T] \to X_0$ of problem (3.1) is Lipschitz continuous on $[0,T]$.*

Proof. As the graph of u is compact in $\mathbb{R}\times X_0$ there exists $L > 0$ such that

$$\|F(t+h, u(t+h)) - F(t, u(t))\| \le L(|h| + \|u(t+h) - u(t)\|), \quad 0 \le t, t+h \le T. \tag{5.4}$$

Fix $h \in (0,T)$ and set $I_h = [0, T-h]$, $u_h(t) := u(t+h)$, $t \in I_h$. Then $v := u_h - u$ is a weak solution of problem

$$v'(t) = Av(t) + F(t+h, u_h(t)) - F(t, u(t)), \quad t \in I_h; \quad v(0) = u(h) - u_0.$$

Hence, from (2.4) and (5.4), setting $C = Me^{|\omega|T}$, we get for $0 \le t, t+h \le T$,

$$\|u(t+h) - u(t)\|$$
$$\le C\|u(h) - u_0\| + C\int_0^t \|F(s+h, u(s+h)) - F(s, u(s))\| ds$$

$$\leq C\|u(h)-u_0\| + CLT|h| + CL\int_0^t \|u(s+h)-u(s)\|ds.$$

From Gronwall's lemma

$$\|u(t+h)-u(t)\| \leq C(\|u(h)-u_0\| + LT|h|)e^{cLt}, \quad 0 \leq t, t+h \leq T. \quad (5.5)$$

By virtue of Lemma 5.2 there exists $L_0 > 0$ such that $\|u(h)-u_0\| \leq L_0 \cdot h$ and so from (5.5) we get the conclusion. □

The next theorem extends a result by I. Segal to the case of Hille-Yosida operators, yielding the regularity of the weak solution of (3.1) when F is continuously differentiable. (It is known (see [15]) that Lipschitz continuity is not sufficient for this purpose.)

Theorem 5.4. *Let $F \in C^1([0,T]\times X_0; X)$, $u_0 \in D(A)$ and $u_1 := Au_0 + F(0,u_0) \in X_0$. If $u : [0,T_0] \to X_0$ is the weak solution of problem* (3.1)*, then u is a strict solution.*

Proof. As F satisfies (LL), we deduce the existence of a weak solution $u : [0,T_0] \to X_0$ from Theorem 4.5. To prove the theorem it is sufficient to show that $u \in C^1(0,T_0;X)$ because in this case u is a weak solution of the linear problem (2.1) with $f(t) := F(t,u(t))$, $t \in [0,T_0]$; as $f \in C^1(0,T_0;X)$ we deduce from Theorem 2.4 that the solution u is strict.

Setting $B(t) := F_u(t,u(t))$ and $\varphi(t) := F_t(t,u(t))$, $t \in [0,T_0]$, by virtue of Lemma 5.1 there exists a unique $v \in C(0,T_0;X_0)$ such that $\int_0^t v(s)ds \in D(A)$, $t \in [0,T_0]$ and

$$v(t) = u_1 + A\int_0^t v(s)ds + \int_0^t B(s)v(s)ds + \int_0^t \varphi(s)ds, \qquad t \in [0,T_0],$$

or (see (4.1))

$$v = S(B(\cdot)v(\cdot)+\varphi, u_1).$$

Fix $h \in (0,T_0[$ and set $I_h := [0,T_0-h]$ and for $t \in I_h$

$$u_h(t) := u(t+h), \quad f_h(t) := F(t+h, u(t+h)), \quad \tilde{v}(t) := hv(t),$$

we have (as $f = F(\cdot, u(\cdot))$)

$$u_h = S(f_h, u(h)), \quad u = S(f,u_0), \quad \tilde{v} = S(B(\cdot)\tilde{v}(\cdot)+h\varphi, hu_1),$$

hence for $t \in I_h$

$$\frac{u(t+h)-u(t)}{h} - v(t) = \frac{u_h(t)-u(t)-\tilde{v}(t)}{h}$$
$$= \frac{1}{h}S(f_h - f - B(\cdot)\tilde{v}(\cdot) - h\varphi, u(h)-u_0-hu_1).$$

From (2.4) and setting $C = 1/Me^{|\omega|T}$ we have

$$C\Big\|\frac{u(t+h)-u(t)}{h} - v(t)\Big\| \leq \Big\|\frac{u(h)-u(0)}{h} - u_1\Big\|$$
$$+\int_0^t \Big\|\frac{F(s+h,u(s+h))}{h} - \frac{F(s,u(s))}{h} - B(s)v(s) - \varphi(s)\Big\|ds \leq \sum_{i=1}^5 I_i,$$

where

$$I_1(h) := \Big\|\frac{u(h)-u(0)}{h} - u_1\Big\|$$
$$I_2(h) := \int_0^{T-h} \Big\|\frac{F(s+h,u(s+h)) - F(s,u(s))}{h} - F_s(s,u(s+h))\Big\|ds$$
$$I_3(h) := \int_0^{T-h} \Big\|\frac{F(s,u(s+h)) - F(s,u(s))}{h} - F_u(s,u(s))\cdot\frac{u(s+h)-u(s)}{h}\Big\|ds$$
$$I_4(h,t) := \int_0^t \Big\|F_u(s,u(s))\cdot\frac{u(s+h)-u(s)}{h} - F_u(s,u(s))\cdot v(s)\Big\|ds$$
$$I_5(h) := \int_0^{T-h} \|F_s(s,u(s+h)) - F_s(s,u(s))\|ds.$$

From Lemma 5.2 we obtain $\lim_{h\to 0^+} I_1(h) = 0$. As F_t and F_u are uniformly continuous on the graph of u and $\frac{\|u(s+h)-u(s)\|}{h}$ is bounded (see Lemma 5.3) we have $\lim_{h\to 0} I_2(h) = \lim_{h\to 0} I_3(h) = \lim_{h\to 0} I_5(h) = 0$.

Setting $M' = \sup\{\|F_u(s,u(s))\| : s \in [0,T_0]\}$ we have

$$I_4(h,t) \leq M' \int_0^t \Big\|\frac{u(s+h)-u(s)}{h} - v(s)\Big\|ds.$$

In conclusion, setting $\psi = I_1 + I_2 + I_3 + I_5$ we have

$$C\Big\|\frac{u(t+h)-u(t)}{h} - v(t)\Big\| \leq \psi(h) + M' \int_0^t \Big\|\frac{u(s+h)-u(s)}{h} - v(s)\Big\|ds, \ \ t \in I_h,$$

and so from Gronwall's lemma

$$\Big\|\frac{u(t+h)-u(t)}{h} - v(t)\Big\| \leq \frac{\psi(h)}{C} e^{\frac{M't}{c}}, \qquad 0 \leq t \leq T_0 - h < T_0.$$

As $\lim_{h\to 0^+} \psi(h) = 0$ we deduce the existence of $u'_+(t) = v(t)$ in $[0,T_0)$; as u, $v \in C(0,T_0;X_0)$ we conclude that u' exists continuous on $[0,T_0]$ (see [6], page 492). ☐

6. Global existence

In this section, we give a condition for the existence of a global solution of (3.1). This will be used in the application to partial differential equations that we will see in Section 7.

Let us start by describing the behavior of a maximal solution.

Theorem 6.1. *If F satisfies (LL) and in addition maps bounded subsets of $[0,T] \times X_0$ into bounded subsets of X, then a bounded maximal solution of* (3.1) *is global* (*i.e.,* $I(u_0) = [0,T]$).

Proof. Set $T' = \sup I(u_0)$ and suppose that the maximal solution $u(\cdot, u_0)$ of (3.1) is bounded in $[0,T')$; from our assumption we deduce that setting $f(t) \coloneqq F(t, u(t,u_0))$, $t \in [0,T')$, we have $f \in L^\infty(0,T';X)$; from Corollary 7.3 of [2] we obtain a unique weak solution $\hat{u} \in C(0,T';X_0)$ of the linear problem $\hat{u}'(t) = A\hat{u}(t) + f(t)$, $t \in [0,T']$; $\hat{u}(0) = u_0$. Therefore, $u(\cdot,u_0)$ can be extended into a weak solution of problem (3.1) in $[0,T']$; hence, $I(u_0)$ is closed and so coincides with $[0,T]$ (see Definition 4.6). □

Let us recall the definition of subdifferential of the norm and a theorem on the left derivative of the norm of a function (for a proof see *e.g.* [12, Lemma 3.1]).

Theorem 6.2. *Let X be a Banach space and X' its dual space. For each $x \in X$ set*

$$\partial\|x\| := \{\varphi \in X' : \|\varphi\| \le 1,\ \langle \varphi, x\rangle = \|x\|\}. \tag{6.1}$$

If $u \in C^1(0,T;X)$, setting $\gamma(t) := \|u(t)\|$, $t \in [0,T]$, there exists for $t \in (0,T]$

$$D_-\gamma(t) = \inf\{\langle \varphi, u'(t)\rangle :\ \varphi \in \partial\|u(t)\|\}, \tag{6.2}$$

where D_- denotes the left-derivative.

We give now a criterion to obtain an *a priori* estimate (hence the boundedness) of a weak solution of (3.1).

Theorem 6.3. *Let $A \in HY(M,\omega)$ and $F \in C([0,T] \times X_0; X)$ satisfy the following property*:

$$\begin{cases} \text{there exist } \omega', \mu \in \mathbb{R} \text{ such that given } x \in D(A) \\ \text{there exists } \varphi \in \partial\|x\| \quad \text{satisfying} \\ \langle \varphi, Ax\rangle \le \omega'\|x\|, \quad \langle \varphi, F(t,x) - F(t,0)\rangle \le \mu\|x\|, \quad t \in [0,T]. \end{cases} \tag{6.3}$$

If $u : J \to X$ is a weak solution of

$$u'(t) = Au(t) + F(t,u(t)), \quad t \in J, \quad u(0) = u_0, \tag{6.4}$$

where J is an interval in $[0,T]$ with $\min J = 0$ *then we have the estimate*

$$\|u(t)\| \le e^{t(\omega'+\mu)}\|u_0\| + \int_0^t e^{(t-s)(\omega'+\mu)}\|F(s,0)\|ds, \qquad t \in J. \tag{6.5}$$

Proof. Let $[0,\tau] \subseteq J$. As u is an F-solution of (6.4) in $[0,\tau]$ there exist $u_n \in C^1(0,\tau;X)\cap C(0,\tau;D(A))$ such that $\lim_{n\to\infty} u_n(t) = u(t)$ uniformly for $t \in [0,\tau]$, $\lim_{n\to\infty} u_n(0) = u_0$ and $f_n \in C(0,\tau;X)$ such that $\lim_{n\to\infty} f_n(t) = 0$ uniformly for $t \in [0,\tau]$ and

$$\begin{aligned} u_n'(t) &= Au_n(t) + F(t,u_n(t)) + f_n(t) \\ &= Au_n(t) + F(t,u_n(t)) - F(t,0) + F(t,0) + f_n(t), \quad t \in [0,\tau]. \end{aligned} \tag{6.6}$$

Given $n \in \mathbb{N}$ and $t \in [0,\tau]$, from (6.3) we deduce the existence of $\varphi \in \partial\|u_n(t)\|$ such that

$$\langle \varphi, Au_n(t)\rangle \le \omega'\|u_n(t)\|, \quad \langle \varphi, F(t,u_n(t)) - F(t,0)\rangle \le \mu\|u_n(t)\|.$$

Hence, from (6.6) we obtain

$$\langle \varphi, u_n'(t)\rangle \le \omega'\|u_n(t)\| + \mu\|u_n(t)\| + \|F(t,0) + f_n(t)\|.$$

Setting $\gamma_n(t) := \|u_n(t)\|$, $t \in [0,\tau]$, and using (6.2),

$$D'_-\gamma_n(t) \le (\omega'+\mu)\gamma_n(t) + \|F(t,0) + f_n(t)\|,$$

so that the function

$$v_n(t) := e^{-(\omega'+\mu)t}\gamma_n(t) - \int_0^t e^{-(\omega'+\mu)s}\|F(s,0) + f_n(s)\|ds$$

is continuous in $[0,\tau]$ and $D_-v_n(t) \le 0$, $t \in (0,\tau]$; hence $v_n(t) \le v_n(0)$, $t \in [0,\tau]$, *i.e.*,

$$\|u_n(t)\| \le e^{(\omega'+\mu)t}\|u_n(0)\| + \int_0^t e^{(\omega'+\mu)(t-s)}\|F(s,0) + f_n(s)\|ds, \quad t \in [0,\tau].$$

For $n \to \infty$ we obtain (6.5) when $t \in [0,\tau]$ and so when $t \in J$. □

From the preceding theorems a global existence criterion for problem (3.1) follows.

Theorem 6.4. *Let F satisfy* (*LL*), *map bounded subsets of* $[0,T] \times X_0$ *into bounded subsets of X and* (*together with A*) *satisfy property* (6.3)*. Then for each $u_0 \in X_0$ there exists a unique weak global solution of problem* (3.1)*.*

Proof. From Theorem 4.5 (and Definition 4.6) we deduce the existence of a maximal solution, which is bounded by virtue of Theorem 6.3; by application of Theorem 6.1 we obtain the conclusion. □

7. Semilinear wave equation

In this section, we want to apply the preceding abstract results to the study of a semilinear one-dimensional wave equation; for brevity we will limit ourselves to the classical (*i.e.,* nondistributional) solution. An application of the criterion given by Theorem 6.4 yields a curious example of global existence of the classical solution without restrictions on the norm of the initial datum or on the growth of the nonlinear term.

Given $T, \ell > 0$ set $I := [0, T]$, $J := [0, \ell]$, $Q : I \times J$, $Q' := I \times \{0, \ell\}$. We will consider the problem

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) + \varphi(t,x,w(t,x),w_t(t,x),w_x(t,x)), & (t,x) \in Q \\ w(t,x') = 0 & (t,x') \in Q' \\ w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x) & x \in J. \end{cases} \tag{7.1}$$

This can be reduced to a first-order problem according to the following result.

Theorem 7.1. *Let $w_0 \in C^2(J)$ and $w_1 \in C^1(J)$. If $w \in C^2(Q)$ is a solution of* (7.1)*, setting*

$$u = \tfrac{1}{2}(w_t + w_x), \quad v = \tfrac{1}{2}(w_t - w_x), \tag{7.2}$$

we have that u, $v \in C^1(Q)$ and $\binom{u}{v}$ is a solution of the system

$$\begin{cases} u_t(t,x) = u_x(t,x) + \dfrac{1}{2}\varphi\Big(t, x, \displaystyle\int_0^x [u(t,y) - v(t,y)]dy, \\ \qquad\qquad u(t,x) + v(t,x), u(t,x) - v(t,x)\Big), & (t,x) \in Q \\ v_t(t,x) = -v_x(t,x) + \dfrac{1}{2}\varphi\Big(t, x, \displaystyle\int_0^x [u(t,y) - v(t,y)]dy, \\ \qquad\qquad u(t,x) + v(t,x), u(t,x) - v(t,x)\Big), & (t,x) \in Q \\ u(0,x) = \tfrac{1}{2}(w_1(x) + w_0'(x)), & x \in J \\ v(0,x) = \tfrac{1}{2}(w_1(x) - w_0'(x)), & x \in J \\ u(t,x') + v(t,x') = 0, & (t,x') \in Q'. \end{cases} \tag{7.3}$$

Conversely, if

$$w_0(x') = 0, \qquad x' = 0, \ell, \tag{7.4}$$

and if problem (7.3) *has a solution $\binom{u}{v}$ with $u, v \in C^1(Q)$, then setting*

$$w(t,x) = \int_0^x [u(t,y) - v(t,y)]dy, \qquad (t,x) \in Q, \tag{7.5}$$

we have that $w \in C^2(Q)$ and is a solution of problem (7.3).

Proof. The proof can be obtained as in the case in which φ is independent of w (see [4, Theorem 2.3]). □

To write (7.3) in abstract form we will use the following results.

Theorem 7.2. *Let us define in $C(J)$ the sup-norm and in $X = C(J) \oplus C(J)$ the norm $\|\binom{u}{v}\| = \|u\| \vee \|v\|$. Define $A : D(A) \subset X \to X$ as*

$$\begin{cases} D(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X : u, v \in C^1(J), u(x') + v(x') = 0, x' = 0, \ell \right\} \\ A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u' \\ -v' \end{pmatrix}. \end{cases} \tag{7.6}$$

Then if $\lambda > 0$ there exists $(\lambda - A)^{-1} \in \mathcal{L}(X)$ and we have $\|\lambda(\lambda - A)^{-1}\| \le 1$. Hence, A is a Hille-Yosida operator and

$$X_0 = \overline{D(A)} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X, \quad u(x') + v(x') = 0, \quad x' = 0, \ell \right\}. \tag{7.7}$$

Proof. See [4, Theorem 2.1]. □

Definition 7.3. We will use the following bounded operators from X to $C(J)$: for $U = \binom{u}{v} \in X$ we set for $x \in J$

$$\begin{cases} P_1 U(x) = \displaystyle\int_0^x [u(y) - v(y)] dy \\ P_2 U(x) = u(x) + v(x), \quad P_3 U(x) = u(x) - v(x). \end{cases} \tag{7.8}$$

Given $\varphi : Q \times \mathbb{R}^3 \to \mathbb{R}$ continuous, define $F : I \times X \to X$ for $(t, U) \in I \times X$ and $x \in J$ as

$$F(t, u)(x) = \frac{1}{2} \begin{pmatrix} \varphi(t, x, P_1 U(x), P_2 U(x), P_3 U(x)) \\ \varphi(t, x, P_1 U(x), P_2 U(x), P_3 U(x)) \end{pmatrix}. \tag{7.9}$$

We consider now the classical solutions of the semilinear wave equation (7.1).

Theorem 7.4. *If $\varphi \in C^1(Q \times \mathbb{R}^3)$ and $w_0 \in C^2(J)$, $w_1 \in C^1(J)$ are such that*

$$w_0(x') = w_1(x') = w_0''(x') + \varphi(0, x', w_0(x'), w_1(x'), w_0'(x')) = 0; \;\; x' = 0, \ell, \tag{7.10}$$

then there exists a unique solution $w \in C^2\left([0, T_0] \times [0, \ell]\right)$ of problem (7.1). Note that (7.10) contains necessary conditions for the existence of such a solution.

Proof. By virtue of Theorem 7.1 problem (7.1) can be written as the following abstract semilinear problem in X:

$$U'(t) = AU(t) + F(t, U(t)), \quad t \in [0,T], \quad U(0) = U_0, \tag{7.11}$$

where

$$U_0 = \frac{1}{2}\begin{pmatrix} w_1 + w_0' \\ w_1 - w_0' \end{pmatrix}. \tag{7.12}$$

It can be checked that (7.10) implies $U_0 \in D(A)$ and $AU_0 + F(0,U_0) \in \overline{D(A)}$. In addition F is continuously differentiable and so we can use Theorem 5.4 to obtain a strict solution in an interval $[0,T_0]$ which yields a classical solution of problem (7.1). □

To use the criterion for the global existence of the solution given by Theorem 6.4 we need the following.

Lemma 7.5. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be continuous such that*

$$\begin{cases} \varphi(x,y) \le 0 & \text{for } x+y > 0 \text{ and } x-y \ge 0 \\ \varphi(x,y) \ge 0 & \text{for } x+y < 0 \text{ and } x-y \le 0 \\ \varphi(x,y) = 0 & \text{elsewhere,} \end{cases} \tag{7.13}$$

and define $F : X \to X$ for $U = \binom{u}{v} \in X, x \in J$ as

$$F(U)(x) = \begin{pmatrix} \varphi(u(x)+v(x), u(x)-v(x)) \\ \varphi(u(x)+v(x), u(x)-v(x)) \end{pmatrix}. \tag{7.14}$$

Then for each $U \in D(A)\backslash\{0\}$ and $\varphi \in \partial\|U\|$ we have $\langle \varphi, F(U)\rangle \le 0$.

Proof. Set $\psi(x,y) := \varphi(x+y, x-y)$, x, $y \in \mathbb{R}$ and choose $U = \binom{u}{v} \in D(A)\backslash\{0\}$ and $\varphi \in \partial\|U\|$. By using Theorem 10.1 of the appendix and [12, Theorem 1.1 and 2.1] we deduce that

(i) if $\|u\| > \|v\|$, then $\varphi = (\varphi_1, 0)$ with $\varphi_1 \in \partial\|u\|$ and we have

$$\psi(u(\overline{x}), v(\overline{x}) \operatorname{sgn} u(\overline{x})) \le 0,$$

if $\|u(\cdot)\|$ reaches its maximum in $\overline{x}$, hence

$$\langle \varphi, F(U)\rangle = \langle \varphi_1, \psi(u(\cdot), v(\cdot))\rangle \le 0;$$

(ii) if $\|u\| < \|v\|$ we get the same conclusion.

As we have shown that

$$\langle \varphi_1, \psi(u(\cdot), v(\cdot))\rangle \le 0, \quad \varphi_1 \in \partial\|u\|;$$
$$\langle \varphi_2, \psi(u(\cdot), v(\cdot)) >\le 0, \quad \varphi_2 \in \partial\|v\|$$

(iii) if $\|u\| = \|v\|$, then $\varphi = t\varphi_1 + (1-t)\varphi_2$ with $t \in [0,1]$, $\varphi_1 \in \partial\|u\|$, $\varphi_2 \in \partial\|v\|$ and we conclude again that

$$\langle \varphi, F(U)\rangle = t < \varphi_1, \psi(u(\cdot), v(\cdot))\rangle + (1-t)\langle \varphi_2, \psi(u(\cdot), v(\cdot))\rangle \leq 0. \qquad \square$$

We obtain now an example of existence in the large of a strict solution of a semilinear wave equation without conditions on the norm of the initial data or in the growth of the nonlinear term.

Theorem 7.6. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and satisfying (7.13). Let $w_0 \in C^2(J)$ and $w_1 \in C^1(J)$ be such that*

$$w_0(x') = w_1(x') = w_0''(x') + \varphi(w_1(x'), w_0'(x')) = 0 \qquad x' = 0, \ell. \tag{7.15}$$

Then for each $T > 0$ there exists a unique solution $w \in C^2([0,T] \times [0,\ell])$ of

$$\begin{cases} w_{tt}(t,x) = w_{xx}(t,x) + \varphi(w_t(t,x), w_x(t,x), & (t,x) \in [0,T] \times [0,\ell] \\ w(t,x') = 0 & (t,x') \in [0,T] \times \{0,\ell\} \\ w(0,x) = w_0(x) & x \in [0,\ell] \\ w_t(0,x) = w_1(x) & x \in [0,\ell]. \end{cases} \tag{7.16}$$

Proof. The existence of a strict solution in a set $[0,T_0]\times[0,\ell]$ with $T_0 \in (0,T]$ can be derived as in Theorem 7.4. If we define F as in Lemma 7.5 we see that property (6.3) is satisfied in our situation because given $U \in D(A)\backslash\{0\}$ we deduce from [12, Proposition 3.1] that there exists $\varphi \in \partial\|U\|$ such that $\langle \varphi, AU\rangle \leq 0$; from Lemma 7.5 we have $\langle \varphi, F(U)\rangle \leq 0$; as $F(0) = 0$, (6.3) holds with $\omega' = \mu = 0$. Hence an application of Theorem 6.4 yields a global solution. $\square$

8. Semilinear integrodifferential equations

We will study now the semilinear integrodifferential equation

$$u'(t) = Au(t) + f(t) + \int_0^t K(t,s,u(s))ds, \quad t \in [0,T]; \quad u(0) = u_0, \tag{8.1}$$

where, setting $\Delta_T := \{(t,s,) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$, $K : \Delta_T \times X_0 \to X$, $f : [0,T] \to X$ and $u_0 \in X_0$. For the sake of brevity we will consider only a strict solution of (8.1), *i.e.*, $u \in C^1(0,T;X) \cap C(0,T;D(A))$.

To prove its existence we will use the following result.

Lemma 8.1. *Suppose that*

$$K \text{ and } K_t \text{ are continuous in } \Delta_T \times X_0. \tag{8.2}$$

Then setting for each $u \in C(0,T;X_0)$

$$Ju(t) = \int_0^t K(t,s,u(s))ds, \qquad t \in [0,T], \tag{8.3}$$

there exists a unique strict solution $v := Pu$ *of*

$$v'(t) = Av(t) + Ju(t), \quad t \in [0,T]; \qquad v(0) = 0. \tag{8.4}$$

If in addition the following property holds: given $u_0 \in X_0$ *there exist* $L_0, r_0 > 0$ *such that*

$$\begin{cases} \|K(t,s,x') - K(t,s,x'')\| \le L_0\|x' - x''\|, \quad (t,s) \in \Delta_T; \\ x', x'' \in P_{X_0}(x_0, r_0), \\ \text{then there exists } \varphi_0 \in C(0,T) \text{ such that } \varphi_0(0) = 0 \\ \text{and if } u_1, u_2 \in C(0,T; P_{X_0}(x_0,r_0)) \text{ then} \\ \|Pu_1(t) - Pu_2(t)\| \le \varphi_0(t)\|u_1 - u_2\|_{C(0,T;Y)}, \quad t \in [0,T]. \end{cases} \tag{8.5}$$

Proof. Given $u \in C(0,T;X_0)$ we have $Ju \in C^1(0,T;X)$, hence, Theorem 2.4 gives a unique strict solution $v =: Pu$ of (8.4). By using (2.4) we get

$$\|Pu_1(t) - Pu_2(t)\| \le M \int_0^t e^{\omega(t-s)}\|Ju_1(s) - Ju_2(s)\|ds, \quad t \in [0,T],$$

and from (8.5) we deduce for $s \in [0,T]$

$$\begin{aligned} \|Ju_1(s) - Ju_2(s)\| &\le \int_0^s \|K(s,r,u_1(r)) - K(s,r,u_2(r))\|dr \\ &\le L_0 s\|u_1 - u_2\|_{C(0,T;Y)}, \end{aligned}$$

from which (8.5) follows by setting

$$\varphi_0(t) = ML_0 \int_0^t e^{\omega(t-s)} s ds, \qquad t \in [0,T]. \qquad \square$$

Theorem 8.2. *Let properties* (8.2) *and* (8.5) *hold. Given* $f \in W^{1,1}(0,T;X)$ *and* $u_0 \in D(A)$ *such that*

$$Au_0 + f(0) \in \overline{D(A)}, \tag{8.6}$$

there exists $T_0 \in (0,T]$ *such that problem* (8.1) *has a strict solution on* $[0,T_0]$. *In addition if* $u : [0,T_1] \to X_0$ *and* $u : [0,T_2] \to X_0$ *are solutions of* (8.1), *then* $u_1 \equiv u_2$ *on* $[0, T_1 \wedge T_2]$. *There exists a maximal solution of problem* (8.1).

Proof. For $\tau \in (0,T]$ let us define the following closed subset of $C(0,\tau;X_0)$: $\Gamma := \{u \in C(0,\tau;Y) : \|u(t)-u_0\| \le r_0, t \in [0,\tau]\}$, where r_0 is given by (8.5). Define (see Lemma 8.1), for $u \in \Gamma$, $Su := Pu + v_0$, where v_0 is the strict solution of the problem

$$v_0'(t) = Av_0(t) + f(t), \quad t \in [0,\tau]; \qquad v_0(0) = u_0.$$

A fixed point of S is a strict solution in $[0,\tau]$ of problem (8.1); to prove its existence it will be sufficient to show that

$$\|Su_1 - Su_2\|_\Gamma \le \tfrac{1}{2}; \qquad u_1, u_2 \in \Gamma \tag{8.7}$$

$$\|Su_0 - u_0\|_\Gamma \le \tfrac{r_0}{2}, \tag{8.8}$$

(when u_0 denotes a constant function with value u_0 in $[0,\tau]$).

To prove (8.7) it is sufficient to observe that, if u_1, $u_2 \in \Gamma$, then $Su_1 - Su_2 = Pu_1 - Pu_2$ and so (8.5) yields $\tau_1 > 0$ such that (8.7) holds if $\tau < \tau_1$. To prove (8.8) we see that $Su_0 - u_0 = Pu_0 + v_0 - u_0$ is a function in $C(0,\tau;X_0)$ which vanishes in 0, hence there exists $\tau_2 > 0$ such that (8.8) is true if $\tau < \tau_2$. In conclusion, we have found a strict solution of (8.1) in $[0,T_0]$ where $T_0 = \min(\tau_1,\tau_2,T)$.

To prove the uniqueness, suppose that u and v are two solutions of (8.1) in $[0,\hat{t}]$ with $\hat{t} \in (0,T]$. Set $t^* = \max\{t \in [0,\hat{t}] : u = v \text{ on } [0,t]\}$. Suppose by contradiction that $t^* < \hat{t}$. Set

$$y_0 := u(t^*) = v(t^*)$$

$$h(t) = \int_0^{t^*} K(t,s,u(s))ds + f(t) = \int_0^{t^*} K(t,s,v(s))ds + f(t), \quad t \in [t^*,\hat{t}].$$

We have $h \in W^{1,1}(t^*,\hat{t};X)$, $y_0 \in D(A)$ and $Ay_0 + h(t^*) \in X_0$, hence, the first part of the proof implies the existence of $\varepsilon \in (0,\hat{t}-t^*]$ such that the problem

$$u'(t) = Au(t) + h(t) + \int_{t^*}^{t} K(t,s,u(s))ds, \quad t \in [t^*,t^*+\varepsilon]; \quad u(T^*) = y_0,$$

has a unique strict solution; this is absurd by virtue of the definition of t^*. In conclusion, two solutions of (8.1) coincide in the common domain. The existence of a maximal solution now follows with the usual argument. □

We will examine in the next theorem the behavior of the maximal solution.

Theorem 8.3. *Suppose that K, f and u_0 satisfy the hypotheses of Theorem* 8.2 *and that K and K_t map bounded subsets of $\Delta_T \times X_0$ into bounded subsets of X. Then a bounded maximal solution of* (8.1) *is global* (*i.e., defined in* $[0,T]$).

Proof. Suppose that the maximal solution u of problem (8.1) is defined in $[0,\tau[$ and that $\sup\{\|u(t)\| : 0 \le t < \tau\} < \infty$. From our assumptions we deduce that

$$\begin{cases} C_1 := \sup\{\|K(t,s,u(s))\| : 0 \le s \le t < \tau\} < \infty \\ C_2 := \sup\{\|K_t(t,s,u(s))\| : 0 \le s \le t < \tau\} < \infty. \end{cases}$$

Set for $0 \le t < t+h < \tau$

$$v(t) := u(t+h) - u(t)$$

$$g(t) := f(t+h) - f(t) + \int_0^{t+h} K(t+h,s,u(s))ds - \int_0^t K(t,s,u(s))ds.$$

As v is a strict solution of the problem

$$v'(t) = Av(t) + g(t), \quad t \in [0,\tau - h], \quad v(0) = u(h) - u(0), \tag{8.9}$$

we deduce from (2.4)

$$\|u(t+h) - u(t)\| \le Me^{|\omega|T}\Big(\|u(h) - u(0)\| + \int_0^t \|g(s)\|ds\Big), \; 0 \le t < t+h < \tau. \tag{8.10}$$

As

$$\begin{aligned} \|g(t)\| \le \|f(t+h) - f(t)\| + \int_0^t \|K(t+h,s,u(s)) - K(t,s,u(s))\|ds \\ + \int_t^{t+h} \|K(t+h,s,u(s))\|ds \\ \le \int_t^{t+h} \|f'(s)\|ds + C_2\tau h + C_1 h, \end{aligned} \tag{8.11}$$

inserting in (8.10) we deduce the existence of $\lim_{t\to\tau} u(t)$ in X_0.

For $s \in [0,\tau - h)$ almost everywhere there exists

$$\begin{aligned} g'(s) = f'(s+h) - f'(s) + K(s+h,s+h,u(s+h)) - K(s,s,u(s)) \\ + \int_o^{s+h} K_t(s+h,r,u(r))dr - \int_0^s K_t(s,r,u(r))dr, \end{aligned} \tag{8.12}$$

and so

$$\begin{aligned} \int_0^t \|g'(s)\|ds \le \int_0^t \|f'(s+h) - f'(s)\|ds + \int_0^t \|K(s+h,s+h,u(s+h)) \\ - K(s,s,u(s))\|ds + \int_0^t ds \int_{-h}^0 \|K_t(s+h,r+h,u(r+h)\|dr \end{aligned} \tag{8.13}$$

$$+\int_0^t ds \int_0^s \|K_t(s+h, r+h, u(r+h)) - K_t(s, r, u(r))\| dr.$$

From (8.9) we deduce by virtue of (2.5)

$$\|Au(t+h) - Au(t)\| \leq \|g(t)\| + Me^{\omega t}\|u'(h) - u(0)\| + M\int_0^t e^{\omega(t-s)}\|g'(s)\|ds,$$

hence, (8.11) and (8.13) imply the existence of $\lim_{t\to\tau} Au(t)$. From equation (8.1) which is satisfied for $t \in [0, \tau)$ we see that u can be extended to a solution in $[0, \tau]$ hence $\tau = T$, *i.e.*, the solution is global. □

9. Existence in the large

We will examine now a class of semilinear integrodifferential equations of convolution type for which it is possible to give criteria for the existence in the large and stability of the solutions. More precisely, we will study the problem

$$u'(t) = Au(t) + \int_0^t g(t-s, u(s))ds, \quad t \geq 0, \quad u(0) = u_0, \tag{9.1}$$

under the assumption that there exists a null solution.

Theorem 9.1. *Let A be a Hille-Yosida operator with $\omega < 0$ and let $g : \mathbb{R}_+ \times B \to X$ with $B := B_{X_0}(0, \rho)$ satisfy*

$$\begin{cases} \text{(i)} & g \text{ is continuously differentiable} \\ \text{(ii)} & g(t,0) = 0, \quad t \geq 0 \\ \text{(iii)} & \text{there exists } h \in L^1(\mathbb{R}_+) \\ & \text{such that } \|g_u(t,x)\|_{\mathcal{L}(X_0,X)} \leq h(t)\|x\|, t \geq 0, x \in B. \end{cases} \tag{9.2}$$

Then given $\delta \in (0, \rho)$ there exists $r \in (0, \delta)$ such that if

$$u_0 \in D(A), \quad Au_0 \in X_0 \text{ and } \|u_0\| + \|Au_0\| \leq r, \tag{9.3}$$

then problem (9.1) has a unique solution $u \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+, D(A))$ and

$$\|u(t)\| + \|Au(t)\| \leq \delta, \qquad t \geq 0. \tag{9.4}$$

Proof. The uniqueness of the solution will be a consequence of a more general result proved in the next theorem.

To prove the existence, set $\|h\| = \int_0^{+\infty} h(s)ds$, fix $\delta \in (0, \rho)$ and define

$$\overline{\delta} := \min\Big(\frac{\delta}{2}, \frac{|\omega|}{4M\|h\|}\Big). \tag{9.5}$$

Choose $r \in (0, \rho)$ such that, if u_0 satisfies (9.3), then

$$\|u_0\| \leq \overline{\delta}, \quad \frac{M}{|\omega|}(\|u_0\|^2 + \|Au_0\|) \leq \frac{\overline{\delta}}{2}. \tag{9.6}$$

Fix an arbitrary $T > 0$ and consider the metric space (with the sup-distance) $\Gamma := \{u \in C(0, T; X_0) : \|u(t) - u_0\| \leq \overline{\delta},\ t \in [0, T]\}$. If $u \in \Gamma$, from (9.5) and (9.6) we deduce that

$$\|u(t)\| \leq \|u(t) - u_0\| + \|u_0\| \leq 2\overline{\delta} \leq \delta, \qquad t \in [0, T], \tag{9.7}$$

and so we can define for each $u \in \Gamma$

$$Ju(t) = \int_0^t g(t - s, u(s))ds = \int_0^t g(s, u(t - s))ds, \quad t \in [0, T].$$

By virtue of (9.7), the conclusion will be reached if we prove the existence of a strict solution $u \in \Gamma$ of

$$u'(t) = Au(t) + Ju(t), \quad t \in [0, T], \quad u(0) = u_0. \tag{9.8}$$

Given $u \in \Gamma$, we have $Ju \in C^1(0, T; X)$ and so there exists a unique strict solution $v =: Qu$ of the linear problem

$$v'(t) = Av(t) + Ju(t), \quad t \in [0, T], \quad v(0) = u_0.$$

Hence, (9.8) has a strict solution if Q has a fixed point in Γ.

If $0 \leq s \leq T$ and $u_1, u_2 \in \Gamma$ by virtue of (9.2) and (9.7) we have

$$\begin{aligned} \|Ju_1(s) - Ju_2(s)\| &= \left\| \int_0^s [g(r, u_1(s - r)) - g(r, u_2(s - r))]dr \right\| \\ &\leq 2\overline{\delta}\|h\|\ \|u_1 - u_2\|_\Gamma. \end{aligned} \tag{9.9}$$

As $w := Qu_1 - Qu_2$ is a strict solution of

$$w'(t) = Aw(t) + Ju_1(t) - Ju_2(t), \quad t \in [0, T]; \qquad w(0) = 0,$$

we deduce from (2.4) by using (9.9), (9.5) and $\omega < 0$, that for $t \in [0, T]$

$$\begin{aligned} \|Qu_1(t) - Qu_2(t)\| &\leq M \int_0^t e^{\omega(t-s)}\|Ju_1(s) - Ju_2(s)\|ds \\ &\leq \frac{M}{|\omega|}2\overline{\delta}\|h\|\ \|u_1 - u_2\|_\Gamma \leq \frac{1}{2}\|u_1 - u_2\|_\Gamma, \end{aligned} \tag{9.10}$$

hence, Q is a contraction.

To show that Q maps Γ into itself, observe that $Z := Qu_0 - u_0$ is a strict solution of the linear problem

$$Z'(t) = AZ(t) + Au_0 + \int_0^t g(s, u_0)ds, \quad t \in [0, T]; \qquad Z(0) = 0,$$

and so (2.4) implies

$$\|(Qu_0 - u_0)(t)\| \leq M \int_0^t e^{\omega(t-s)} \|Au_0\| ds + M \int_0^t e^{\omega(t-s)} ds \int_0^s \|g(r, u_0)\| dr,$$

$t \in [0, T]$. From (9.2) we deduce

$$\|g(r, u_0)\| = \|g(r, u_0) - g(r, 0)\| \leq h(r)\|u_0\|^2, \qquad 0 \leq r \leq T,$$

and so (by using (9.6) and $\omega < 0$)

$$\|(Qu_0 - u_0)(t)\| \leq \frac{M}{|\omega|}\|Au_0\| + \frac{M}{|\omega|}\|h\| \, \|u_0\|^2 \leq \frac{\overline{\delta}}{2}, \quad t \in [0, T],$$

which, together with (9.10), proves that $Q : \Gamma \to \Gamma$ and the conclusion follows. □

Theorem 9.2. *Under the assumptions of Theorem* 9.1 *if* u_i $(i = 1, 2)$ *are strict solutions of equation of* (9.1) *such that* $u_i(0) = u_{0i}$*, then we have*

$$\begin{aligned} &\|u_1(t) - u_2(t)\| \leq M\|u_{01} - u_{02}\| \exp(\omega t + t\gamma(t)), \\ &\textit{where } \gamma(t) = M\rho \int_0^t h(r)e^{-\omega r} dr, \quad t \in [0, T]. \end{aligned} \tag{9.11}$$

Proof. Setting $u(t) := u_1(t) - u_2(t)$, $t \in [0, T]$ and $u_0 := u_{01} - u_{02}$, we have

$$u'(t) = Au(t) + \int_0^t [g(t-s, u_1(s)) - g(t-s, u_2(s))] ds, \quad t \in [0, T]; \quad u(0) = u_0,$$

hence, from (2.4) and by virtue of (9.2)

$$\begin{aligned} \|u(t)\| &\leq Me^{\omega t}\Big(\|u_0\| + \int_0^t e^{-\omega r} dr \Big\| \int_0^r [g(r-s, u_1(s)) - g(r-s, u_2(s))] ds\Big) \\ &\leq Me^{\omega t}\Big(\|u_0\| + \rho \int_0^t e^{-\omega r} dr \int_0^r h(r-s)\|u(s)\| ds\Big) \\ &= Me^{\omega t}\Big(\|u_0\| + \rho \int_0^t \|e^{-\omega s}u(s)\| ds \int_s^t h(r-s)e^{-\omega(r-s)} dr\Big) \\ &\leq Me^{\omega t}\Big(\|u_0\| + \rho \int_0^t \|e^{-\omega s}u(s)\| ds \cdot \int_0^t h(r)e^{-\omega r} dr\Big). \end{aligned}$$

By application of Gronwall's lemma to the function

$$t \to \|e^{-\omega t}u(t)\| M^{-1}\gamma^{-1}(t)$$

we obtain (9.11). □

10. Appendix

We prove here a result used in Lemma 7.5 about the subdifferential of a product space.

Theorem 10.1. *Let X_1 and X_2 be Banach spaces and $X = X_1 \times X_2$ be endowed with the norm $\|(x_1, x_2\| = \|x_1\| \vee \|x_2\|$, $x_1 \in X_1, x_2 \in X_2$. We have $X' = X_1' \times X_2'$ and for $\varphi_1 \in X_1', \varphi_2 \in X_2'$, $\|(\varphi_1, \varphi_2)\|_{X'} = \|\varphi_1\|_{X_1'} + \|\varphi_1\|_{X_2'}$, $\langle(\varphi_1, \varphi_2), (x_1, x_2)\rangle = \langle\varphi_1, x_1\rangle + \langle\varphi_2, x_2\rangle$. Given $(x_1, x_2) \in X$,*

- (i) *if $\|x_1\| > \|x_2\|$ we have $\partial\|(x_1, x_2)\| = \partial\|x_1\| \times \{0\}$;*
- (ii) *if $\|x_1\| < \|x_2\|$ we have $\partial\|(x_1, x_2)\| = \{0\} \times \partial\|x_2\|$;*
- (iii) *if $\|x_1\| = \|x_2\| \neq 0$ we have $\partial\|(x_1, x_2\| = \{(t\psi_1, (1-t)\psi_2),$ $\psi_1 \in \partial\|x_1\|, \psi_2 \in \partial\|x_2\|, t \in [0, 1]\}$;*
- (iv) *if $x_1 = x_2 = 0$, we have* $\partial\|(x_1, x_2\| = \{(\varphi_1, \varphi_2) \in X', \|\varphi_1\| + \|\varphi_2\| \leq 1\}$.

Proof. (i) If $x = (x_1, x_2) \in X$ with $\|x_1\| > \|x_2\|$, then for $\varphi \in \partial\|x\|$ we have $\langle\varphi, y - x\rangle \leq \|y\| - \|x\| = \|y\| - \|x_1\|$ for each $y \in X$. Set $\varphi = (\varphi_1, \varphi_2)$ and $y = (y_1, y_2)$; if $y_2 = x_2$ we deduce

$$\langle\varphi_1, y_1 - x_1\rangle \leq \|y_1\| \vee \|x_2\| - \|x_1\|, \qquad y_1 \in X_1. \tag{10.1}$$

As $\delta := \|x_1\| - \|x_2\| > 0$, if $y_1 \in X_1$ is such that $\|x_1 - y_1\| \leq \delta$, then $\|x_2\| \leq \|x_1\| - \|x_1 - y_1\| \leq \|y_1\|$ hence (10.1) implies $\langle\varphi_1, y_1 - x_1\rangle \leq \|y_1\| - \|x_1\|$, $y_1 \in X_1$, $\|y_1 - x_1\| \leq \delta$ and so $\varphi_1 \in \partial\|x_1\|$. If we choose $y_1 = x_1$ we deduce

$$\langle\varphi_2, y_2 - x_2\rangle \leq \|x_1\| \vee \|y_2\| - \|x_1\|, \qquad y_2 \in X_2. \tag{10.2}$$

Choose $z_2 \in X_2$ and then $\lambda > 0$ such that $\lambda\|z_2\| \leq \delta = \|x_1\| - \|x_2\|$. Setting $y_2 := x_2 + \lambda z_2$ we have $\|y_2\| \leq \|x_2\| + \lambda\|z_2\| \leq \|x_1\|$ and so (10.2) implies $\lambda\langle\varphi_2, z_2\rangle \leq 0$ for each $z_2 \in X_2$ and so $\varphi_2 = 0$. In conclusion $\partial\|(x_1, x_2)\| \subseteq \partial\|x_1\| \times \{0\}$. If $\varphi = (\varphi_1, 0)$ with $\varphi_1 \in \partial\|x_1\|$ we have $\|\varphi\| = \|\varphi_1\| \leq 1$ and $\langle\varphi, x\rangle = \langle\varphi_1, x_1\rangle = \|x_1\| = \|x\|$ hence $\varphi \in \partial\|x\|$ and the case (i) is proved.

(ii) This can be dealt with analogously.

(iii) Let $x = (x_1, x_2)$ and $\|x_1\| = \|x_2\| = \|x\| \neq 0$. Given $\psi_1 \in \partial\|x_1\|$, $\psi_2 \in \partial\|x_2\|$ and $t \in [0, 1]$, set $\varphi_t = (t\psi_1, (1-t)\psi_2)$. We have $\langle\varphi_t, x\rangle = \|x\|$ and $\|\varphi_t\| \leq 1$ hence $\varphi_t \in \partial\|x\|$. Conversely, if $\varphi = (\varphi_1, \varphi_2) \in \partial\|(x_1, x_2)\|$ we must have

$$\begin{aligned} &\langle\varphi, x\rangle = \langle\varphi_1, x_1\rangle + \langle\varphi_2, x_2\rangle = \|x\| = \|x_1\| = \|x_2\| \\ &\|\varphi\| = \|\varphi_1\| + \|\varphi_2\| = 1. \end{aligned} \tag{10.3}$$

Hence,

$$\langle\varphi_1, x_1\rangle = \langle\varphi, x\rangle - \langle\varphi_2, x_2\rangle \geq \|x\| - \|\varphi_2\| \, \|x_2\|$$

$$= \|x_1\|(1-\|\varphi_2\|) = \|x_1\|\,\|\varphi_1\|,$$

i.e., $\langle \varphi_1, x_1\rangle = \|x_1\|\,\|\phi_1\|$. Analogously, one proves that $\langle \varphi_2, x_2\rangle = \|x_2\|\,\|\phi_2\|$. We have three possibilities for φ:

(α) $\varphi_1 \neq 0$, $\varphi_2 \neq 0$. Then setting $\psi_1 = \frac{\varphi_1}{\|\varphi_2\|}$ and $\psi_2 = \frac{\varphi_2}{\|\varphi_2\|}$ we have $\langle \psi_1, x_1\rangle = \|x_1\|$ and $\|\psi_1\| = 1$ hence $\psi_1 \in \partial\|x_1\|$; analogously, we get $\psi_2 \in \partial\|x_2\|$. From (10.3) setting $t = \|\varphi_1\|$ we have $t \in (0,1)$, $1-t = \|\varphi_2\|$ and so $\varphi_1 = t\psi_1$, $\varphi_2 = (1-t)\psi_2$.

(β) $\varphi_1 = 0$, $\varphi_2 \neq 0$. As $\varphi = (0, \varphi_2) \in \partial\|(x_1, x_2)\|$ we have $\|\varphi\| = \|\varphi_2\| \leq 1$ and $\langle \varphi_2, x_2\rangle = \langle \varphi, x\rangle = \|x\| = \|x_2\|$ hence $\varphi_2 \in \partial\|x_2\|$ and so $(\varphi_1, \varphi_2) = (t\psi_1, (1-t)\psi_2)$ with $t = 0$, arbitrary $\psi_1 \in \partial\|x_1\|$ and $\psi_2 = \varphi_2$.

(γ) $\varphi_1 \neq 0$, $\varphi_2 = 0$. Proceeding as in (β) we find $(\varphi_1, \varphi_2) = (t\psi_1, (1-t)\psi_2)$ with $t = 1$, $\psi_1 = \varphi_1$ and arbitrary $\psi_2 \in \partial\|x_2\|$.

(iv) This is a consequence of the definition of $\partial\|0\| = \{\varphi \in X' : \|\varphi\| \leq 1\}$.

References

[1] P.L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation", Springer-Verlag, Berlin, 1967.

[2] G. Da Prato and E. Sinestrari, *Differential operators with non dense domain*, Annali Sc. Norm. Sup. Pisa, **14** (1987), 285–344.

[3] R. Driver, "Ordinary and Delay Differential Equations", Springer-Verlag, New York, 1977.

[4] G. Grimmer and E. Sinestrari, *Maximum norm in one-dimensional hyperbolic problems*, Diff. Integral Eq., **5** (1992), 421-432.

[5] E. Hille and R. Phillips, "Functional Analysis and Semi-Groups", American Math. Soc., 1957.

[6] T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin, 1966.

[7] Y. Lei, *Semilinear functional differential equations*, in *Banach spaces with Hille-Yosida operators*, Nonlinear Analysis T.M.A., **41** (2000), 989–1004.

[8] P. Magal, *Compact attractors for time-periodic age-structured population model*, Electr. Jour. Diff. Eq., **65**, (2001), 1–35.

[9] R. Nagel and E. Sinestrari, *Inhomogeneous Volterra integrodifferential equations for Hille-Yosida operators*, Dekker Lect. Notes Pure Appl. Math, **150** (1994), 51–70.

[10] R. Nagel and E. Sinestrari, *Nonlinear hyperbolic Volterra integrodifferential equations*, Nonlinear Analysis T.M.A., **27** (1996), 167–186.

[11] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York, 1983.

[12] E. Sinestrari, *Accretive differential operators*, Boll. Un. Mat. It., **5** (1976), 19–31.

[13] N. Tanaka, *Semilinear equations in the "hyperbolic" case*, Nonlinear Analysis T.M.A., **24** (1995), 773–788.

[14] H. Thieme, *Semiflows generated by lipschitz perburbations of nondensely defined operators*, Diff. and Int. Eq., **3** (1990), 1035–1066.

[15] G. Webb, *Continuous nonlinear perturbations of linear accretive operators in Banach spaces*, J. Func. Anal., **10** (1972), 191–203.