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ON THE PRIMITIVE EQUATIONS OF LARGE-SCALE OCEAN WITH RANDOM BOUNDARY

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Abstract. This paper is concerned with global well posedness and the existence of attractors for the three-dimensional viscous primitive equations, describing large-scale oceanic motion under random wind stress. We prove global well posedness of the primitive equations with white-noise boundary conditions. Moreover, by studying the asymptotic behavior of strong solutions, we obtain the existence of random attractors for the corresponding random dynamical system.

1. Introduction

Understanding the mechanism of climate change as well as long-term weather prediction has been of particular interest during the past several decades. The corresponding systems can be well modeled by 3D viscous primitive equations of the atmosphere and ocean. After a series of pioneering articles by J. L. Lions, R. Temam and S. Wang [12, 13], there have been many works about global well posedness and long-time dynamics of the deterministic primitive oceanic or atmospheric equations, see, e.g., [4, 11, 25] and references therein. In [4], Cao and Titi obtained global well posedness for 3D viscous primitive equations of the large-scale ocean.

Despite the great success of the above development, a comprehensive understanding of the long-time dynamics of atmospheric and oceanic motions

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seems to be still incomplete. For example, the non-trivial effects of randomness are not included among all the former literature for the 3D primitive equations. As usual, the wind stress on the surface of the ocean, which is caused by the atmospheric motions, must be regarded as random, see, e.g., [8, 16, 17, 19, 23]. Therefore, in studying the primitive equations of the large-scale ocean, taking the stochastic external factors into account is reasonable and necessary. There are many works about the mathematical study of some stochastic climate models, see, e.g., [5, 6, 10, 14, 15, 16]. [6] is one of the first works on a 3D stochastic quasi-geostrophic flow model. In [10], we consider global well posedness and long-time dynamics for the three-dimensional viscous primitive equations describing the large-scale oceanic motion under a random forcing. However, to our knowledge, no one has considered long-time dynamics for the 3D primitive equations of the large-scale ocean motion under a random wind stress on the surface of the ocean.

In the present paper, we are interested in considering the stochastic boundary value problem of 3D primitive equations (2.1)-(2.9), which, when $\tau(t, x, y)$ in (2.7) is equal to zero, is the boundary-value problem of 3D deterministic primitive equations considered in [4]. After introducing an auxiliary Ornstein-Uhlenbeck process Z, we study global well posedness and long-time behavior of global strong solutions to (3.4)-(3.11), which is a new formulation of the system (2.1)-(2.9). Our main results are as follows.

Theorem 1.1. (Global well posedness of (3.4)-(3.11)) *If $Q_1, Q_2 \in H^1(\Omega)$ and $\mathcal{U}_{t_0} = (w_{t_0}, T_{t_0}, S_{t_0})$ satisfies: $\mathcal{U}_{t_0} \in H$, $\tilde{w}_{t_0} \in (L^4(\Omega))^2$, $T_{t_0}, S_{t_0} \in L^4(\Omega)$, $\partial_z w_{t_0} \in (L^2(\Omega))^2$, $\partial_z T_{t_0}, \partial_z S_{t_0} \in L^2(\Omega)$, then, for any given $\mathcal{T} > t_0$, there exists a unique weakly strong solution U of the system* (3.4)-(3.11) *on the interval $[t_0, \mathcal{T}]$; moreover, the weakly strong solution $\mathcal{U}$ is dependent continuously on the initial data.*

The definitions of the space H, $\tilde{w}_{t_0}$, the weakly strong solution to (3.4)-(3.11) and some assumptions on random boundary are given in Section 3.

Theorem 1.2. (Existence of the random pull-back attractor for the dynamical system (3.4)-(3.10)) *If the auxiliary Ornstein-Uhlenbeck process Z satisfies condition (C) in Subsection* 3.2, *then the system* (3.4)-(3.10) *possesses a unique random pull-back attractor $\mathcal{A}(\omega)$ that captures all the trajectories started at time $-\infty$ and evolved, under the action of the shift ϑ_t, from $-\infty$ to the present time $t = 0$.*

The attractor $\mathcal{A}(\omega)$ has the following properties:

(i) (weak compact) $\mathcal{A}(\omega)$ is bounded and weakly closed in V;
(ii) (invariant) for any $t \geq 0$, $\psi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\vartheta_t\omega)$;

(iii) (attracting) for any deterministic bounded set B in V, the sets $\psi(t, \vartheta_{-t}\omega)B$ converge to $\mathcal{A}(\vartheta_t\omega)$ with respect to the V-weak topology as $t \to +\infty$; i.e.,

$$\lim_{t\to+\infty} d_V^w(\psi(t, \vartheta_{-t}\omega)B, \mathcal{A}(\omega)) = 0,\ P - a.s.,$$

where the definition of the space V is given in Section 3, the definitions of the random attractor, $\vartheta_t, t \in \mathbb{R}$ and $\psi(t, \omega), t \geq 0$ given in Subsection 5.1, and the distance d_V^w is induced by the V-weak topology.

One of the difficulties in considering global well posedness and long-time dynamics of the 3D primitive equations with random boundary is the irregularity of the wind stress. Since the wind stress $\tau(t, x, y)$ is so irregular in time, it is still unknown how to handle

$$\int_{t_0}^{\mathcal{T}} \Big[\int_{\Gamma_u} \tau(s, x, y)v(s, x, y, 0)\Big] ds;$$

it is impossible to obtain necessary energy inequalities with the usual energy methods. However, inspired by methods in [20] and [21, chapter 13], we introduce an auxiliary Ornstein-Uhlenbeck process Z, which is given in Section 3. Then we obtain a new formulation (3.4)-(3.11) of the system (2.1)-(2.9). (3.4)-(3.11) is the initial-value problem of stochastic partial differential equations with deterministic boundary conditions. Since the system (3.4)-(3.11) is similar to (4.18)-(4.22) in [10], we can use the methods in [10] to study global well posedness and large-time behavior of solutions to (3.4)-(3.11).

However, there are two main differences between the present paper and our former work [10]. The first one is that the conditions of the initial data for global well posedness, here, is weaker than those in [10]. Thus, the proof of global well posedness is different from that in [10].

The key step in the proof of Theorem 1.1 is to make uniform estimates of $\|\partial_z w(t)\|_{(L^2(\Omega))^2}$, $\|\partial_z T(t)\|_{L^2(\Omega)}$ and $\|\partial_z S(t)\|_{L^2(\Omega)}$. On the basis of uniform estimates of $\|\tilde{w}(t)\|_{(L^4(\Omega))^2}$, which are obtained by estimates similar to those in [10], we prove that $\|\partial_z w(t)\|_{(L^2(\Omega))^2}$, $\|\partial_z T(t)\|_{L^2(\Omega)}$ and $\|\partial_z S(t)\|_{L^2(\Omega)}$ are bounded uniformly in t. Here, we need no more regularity of $\bar{w}$ than $\bar{w} \in L^4(0, \mathcal{T}; (L^4(\Omega))^2)$ which is satisfied when $U = (w, T, S)$ is a global weak solution to (3.4)-(3.11) in the usual sense. Therefore, we do not make uniform estimates of $\|\bar{w}(t)\|_{(H^1(M))^2}$ as in [10]. The second difference is that the equation satisfied by Z here does not consist of a damping term. Therefore, in order to obtain the existence of a random pull-back attractor to the primitive equations with random boundary, we should add a condition (C)

on the random boundary. The condition means that the action of the wind stress can not be too strong.

The paper is organized as follows. In Section 2, the stochastic boundary-value problem of the 3D primitive equations is introduced. The auxiliary Ornstein-Uhlenbeck process Z and a new formulation of the stochastic boundary-value problem for the primitive equations are given in Section 3. We prove global well posedness for (3.4)-(3.11) in Section 4. The proof of the existence of a random attractor to the dynamical system (3.4)-(3.10) is given in Section 5.

2. The 3D primitive equations with random boundary

The non-dimensional form of 3D viscous primitive equations of large-scale oceans (cf. [13, 18]), in a Cartesian coordinate system, is written as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \theta \frac{\partial v}{\partial z} + f k \times v + \nabla p - \frac{1}{Re_1} \triangle v - \frac{1}{Re_2} \frac{\partial^2 v}{\partial z^2} = 0, \tag{2.1}$$

$$\frac{\partial p}{\partial z} + \beta_1 \rho = 0, \tag{2.2}$$

$$\nabla \cdot v + \frac{\partial \theta}{\partial z} = 0, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla) T + \theta \frac{\partial T}{\partial z} - \frac{1}{Rt_1} \triangle T - \frac{1}{Rt_2} \frac{\partial^2 T}{\partial z^2} = Q_1, \tag{2.4}$$

$$\frac{\partial S}{\partial t} + (v \cdot \nabla) S + \theta \frac{\partial S}{\partial z} - \frac{1}{Rs_1} \triangle S - \frac{1}{Rs_2} \frac{\partial^2 S}{\partial z^2} = Q_2, \tag{2.5}$$

$$\rho = 1 - \beta_2 (T - 1) + \beta_3 (S - 1), \tag{2.6}$$

where the unknown functions are $v,\ \theta,\ p,\ T,\ S,\ \rho,\ v = (v^{(1)}, v^{(2)})$ is horizontal velocity, θ vertical velocity, p pressure, T temperature, S salinity, ρ density, $f = \frac{f_0}{R_0}(\beta + y)$ Coriolis parameter, $\beta_1, \beta_2, \beta_3$ are positive constants, k vertical unit vector, Re_1, Re_2 Reynolds numbers, Rt_1, Rt_2 the horizontal and vertical heat diffusivity respectively, Rs_1, Rs_2 the horizontal and vertical salinity diffusivity respectively, $Q_1(x,y,z),\ Q_2(x,y,z)$ given functions, $\nabla = (\partial_x,\ \partial_y),\ \triangle = \partial_x^2 + \partial_y^2$.

The space domain for $(2.1) - (2.6)$ is

$$\Omega = \{(x,y,z) :\ (x,y) \in M \text{ and } z \in (-g(x,y), 0)\},$$

where M is a smooth bounded domain in $\mathbb{R}^2$ and g is a sufficiently smooth function. Here, we assume $g=1$; that is, $\Omega = M \times (-1,0)$. The boundary-value conditions for (2.1) – (2.6) are given by

$$\frac{\partial v}{\partial z} = \tau,\ \theta = 0,\ \frac{\partial T}{\partial z} = -\alpha_u T,\ \frac{\partial S}{\partial z} = 0 \qquad \text{on } M \times \{0\} = \Gamma_u, \tag{2.7}$$

$$\frac{\partial v}{\partial z} = 0,\ \theta = 0,\ \frac{\partial T}{\partial z} = 0,\ \frac{\partial S}{\partial z} = 0 \qquad \text{on } M \times \{-1\} = \Gamma_b, \tag{2.8}$$

$$v \cdot \vec{n} = 0,\ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0,\ \frac{\partial T}{\partial \vec{n}} = 0,\ \frac{\partial S}{\partial \vec{n}} = 0 \qquad \text{on } \partial M \times [-1,0] = \Gamma_l, \tag{2.9}$$

where α_u is a positive constant and $\vec{n}$ is the norm vector to Γ_l.

In this article, we assume that the wind stress τ is random. The form of τ is given in the next section.

Remark 2.1 Our results are valid when the boundary-value condition

$$\frac{\partial T}{\partial z}|_{\Gamma_u} = -\alpha_u T$$

is replaced by

$$\frac{\partial T}{\partial z}|_{\Gamma_u} = -\alpha_u (T - T^*),$$

for smooth enough T^* satisfying the compatible boundary condition:

$$\frac{\partial T^*}{\partial \vec{n}}|_{\partial M} = 0.$$

Integrating (2.3) from -1 to z and using (2.7), (2.8), we have

$$\theta(t,x,y,z) = \Phi(v)(t,x,y,z) = -\int_{-1}^{z} \nabla \cdot v(t,x,y,z')\ dz'; \tag{2.10}$$

moreover,

$$\int_{-1}^{0} \nabla \cdot v\ dz = 0.$$

Supposing that p_b is a certain unknown function at Γ_b, and integrating (2.2) from -1 to z, we rewrite (2.1) – (2.6) as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \Phi(v)\frac{\partial v}{\partial z} + fk \times v + \nabla p_b + \int_{-1}^{z} \nabla(\nu_1 T - \nu_2 S)dz' - \frac{1}{Re_1}\triangle v - \frac{1}{Re_2}\frac{\partial^2 v}{\partial z^2} = 0, \tag{2.11}$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla)T + \Phi(v)\frac{\partial T}{\partial z} - \frac{1}{Rt_1}\triangle T - \frac{1}{Rt_2}\frac{\partial^2 T}{\partial z^2} = Q_1, \tag{2.12}$$

$$\frac{\partial S}{\partial t}+(v\cdot\nabla)S+\Phi(v)\frac{\partial S}{\partial z}-\frac{1}{Rs_1}\triangle S-\frac{1}{Rs_2}\frac{\partial^2 S}{\partial z^2}=Q_2, \tag{2.13}$$

$$\int_{-1}^{0}\nabla\cdot v\,dz=0, \tag{2.14}$$

where $\nu_1=\beta_1\beta_2$ and $\nu_2=\beta_1\beta_3$. The boundary-value conditions for (2.11) – (2.14) are given by

$$\frac{\partial v}{\partial z}=\tau,\ \frac{\partial T}{\partial z}=-\alpha_u T,\ \frac{\partial S}{\partial z}=0 \qquad \text{on } \Gamma_u, \tag{2.15}$$

$$\frac{\partial v}{\partial z}=0,\ \ \frac{\partial T}{\partial z}=0,\ \frac{\partial S}{\partial z}=0 \qquad \text{on } \Gamma_b, \tag{2.16}$$

$$v\cdot\vec{n}=0,\ \frac{\partial v}{\partial\vec{n}}\times\vec{n}=0,\ \frac{\partial T}{\partial\vec{n}}=0,\ \frac{\partial S}{\partial\vec{n}}=0 \qquad \text{on } \Gamma_l, \tag{2.17}$$

and the initial-value condition is

$$(v|_{t=t_0},T|_{t=t_0},S|_{t=t_0})=(v_{t_0},T_{t_0},S_{t_0}). \tag{2.18}$$

3. New formulation of (2.11)-(2.18)

3.1. **Some function spaces.** $L^p(\Omega)$ is the usual Lebesgue space with the norm $|\cdot|_p$, $1\le p\le\infty$. $H^m(\Omega)$ is the usual Sobolev space (m is a positive integer) with the norm

$$\|h\|_m=\Big[\int_\Omega\Big(\sum_{1\le k\le m}\ \sum_{i_j=1,2,3;j=1,\cdots,k}|\nabla_{i_1}\cdots\nabla_{i_k}h|^2+|h|^2\Big)\Big]^{\frac{1}{2}},$$

where

$$\nabla_1=\frac{\partial}{\partial x},\quad \nabla_2=\frac{\partial}{\partial y},\quad \nabla_3=\frac{\partial}{\partial z}.$$

$\int_\Omega\cdot d\Omega$ and $\int_M\cdot dM$ are denoted by $\int_\Omega\cdot$ and $\int_M\cdot$ respectively.

We define our working spaces. Let

$$\mathcal{V}_1:=\Big\{w\in(C^\infty(\Omega))^2:\frac{\partial w}{\partial z}|_{\Gamma_u,\Gamma_b}=0,w\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial w}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0,\int_{-1}^{0}\nabla\cdot w dz=0\Big\},$$

$$\mathcal{V}_2:=\Big\{T\in C^\infty(\Omega):\frac{\partial T}{\partial z}|_{\Gamma_u}=-\alpha_u T,\ \frac{\partial T}{\partial z}|_{\Gamma_b}=0,\ \frac{\partial T}{\partial\vec{n}}|_{\Gamma_l}=0\Big\},$$

$$\mathcal{V}_3:=\Big\{S\in C^\infty(\Omega):\frac{\partial S}{\partial z}|_{\Gamma_u}=0,\ \frac{\partial S}{\partial z}|_{\Gamma_b}=0,\ \frac{\partial S}{\partial\vec{n}}|_{\Gamma_l}=0,\ \int_\Omega S=0\Big\},$$

$V_1=$ the closure of $\mathcal{V}_1$ with respect to the norm $\|\cdot\|_1$,

$V_2=$ the closure of $\mathcal{V}_2$ with respect to the norm $\|\cdot\|_1$,

$V_3=$ the closure of $\mathcal{V}_3$ with respect to the norm $\|\cdot\|_1$,

$H_1=$ the closure of $\mathcal{V}_1$ with respect to the norm $|\cdot|_2$,

$$V = V_1 \times V_2 \times V_3, \qquad H = H_1 \times L^2(\Omega) \times \dot{L}^2(\Omega),$$

where

$$\dot{L}^2(\Omega) = \Big\{ S : S \in L^2(\Omega), \int_\Omega S = 0 \Big\}.$$

The inner products and norms on $V,\ H$ are given by

$$(\mathcal{U},\mathcal{U}_1)_V = (w,w_1)_{V_1} + (T,T_1)_{V_2} + (S,S_1)_{V_3},$$

$$(\mathcal{U},\mathcal{U}_1) = (w^{(1)},(w_1)^{(1)}) + (w^{(2)},(w_1)^{(2)}) + (T,T_1) + (S,S_1),$$

$$\|\mathcal{U}\| = (w,w)_{V_1}^{\frac{1}{2}} + (T,T)_{V_2}^{\frac{1}{2}} + (S,S)_{V_3}^{\frac{1}{2}} = \|w\| + \|T\| + \|S\|,\ |\mathcal{U}|_2 = (\mathcal{U},\mathcal{U})^{\frac{1}{2}},$$

where $\mathcal{U} = (w,T,S),\ \mathcal{U}_1 = (w_1,T_1,S_1) \in V$, and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

3.2. **An auxiliary Ornstein-Uhlenbeck process.** At first, we define the functionals $a : V \times V \to \mathbb{R},\ a_1 : V_1 \times V_1 \to \mathbb{R},\ a_2 : V_2 \times V_2 \to \mathbb{R},\ a_3 : V_3 \times V_3 \to \mathbb{R}$, and their corresponding linear operators $A : V \to V',\ A_1 : V_1 \to V_1',\ A_2 : V_2 \to V_2',\ \ A_3 : V_3 \to V_3'$ by

$$a(\mathcal{U},\mathcal{U}_1) = (A\mathcal{U},\mathcal{U}_1) = a_1(w,w_1) + a_2(T,T_1) + a_3(S,S_1),$$

where

$$a_1(w,w_1) = (A_1 w,w_1) = \int_\Omega (\frac{1}{Re_1}\nabla w \cdot \nabla w_1 + \frac{1}{Re_2}\frac{\partial w}{\partial z}\cdot\frac{\partial w_1}{\partial z}),$$

$$a_2(T,T_1) = (A_2 T,T_1) = \int_\Omega (\frac{1}{Rt_1}\nabla T \cdot \nabla T_1 + \frac{1}{Rt_2}\frac{\partial T}{\partial z}\frac{\partial T_1}{\partial z}) + \frac{\alpha_u}{Rt_2}\int_{\Gamma_u} TT_1.$$

$$a_3(S,S_1) = (A_3 S,S_1) = \int_\Omega (\frac{1}{Rs_1}\nabla S \cdot \nabla S_1 + \frac{1}{Rs_2}\frac{\partial S}{\partial z}\frac{\partial S_1}{\partial z}).$$

Lemma 3.1. (1) *a is coercive and continuous, and $A : V \to V'$ is isomorphism. Moreover,*

$$a(\mathcal{U},\mathcal{U}_1) \le c\|w\|\|w_1\| + c\|T\|\|T_1\| + c\|S\|\|S_1\| \le c\|U\|\|U_1\|,$$
$$a(\mathcal{U},\mathcal{U}) \ge c\|w\|^2 + c\|T\|^2 + c\|S\|^2 \ge c\|U\|^2. \tag{3.1}$$

Here, c will denote positive constants and can be determined in concrete conditions.

(2) *The isomorphism $A : V \to V'$ can be extended to a self-adjoint unbounded linear operator on H with a compact inverse $A^{-1} : H \to H$ and with $D(A) = V \cap [(H^2(\Omega))^2 \times H^2(\Omega) \times H^2(\Omega)]$.*

Proof. By

$$\|w\|^2_{L^2(M)} \le C_M \|\nabla w\|^2_{L^2(M)},$$

(cf. [9, page 55]) and the Poincaré inequality, we prove (3.1). Since the operator A is similar to the usual positive symmetric Laplacian operator on H_0^1, the other part of Lemma 3.1 can be proven with the usual argument. Therefore, we omit the details of the proof. For more details, the reader can refer to [13, Lemma 2.4]. □

We denote by $0 < \lambda_1 < \lambda_2 \leq \cdots$ the eigenvalues of A_1 and by $e_1, e_2, \cdots$ a corresponding complete orthonormal system of eigenvectors. We remark that $\|w\|^2 \geq \lambda_1 |w|_2^2$ for any $w \in V_1$. We denote by $e^{t(-A_1)}$, $t \geq 0$, the semigroup on H_1 generated by $-A_1$.

In the present paper, we assume that the random wind stress $\tau(t,x,y)$ is an additive white in time noise with the form

$$\tau(t,x,y) = G^{\frac{1}{2}} \frac{\partial W}{\partial t}, \tag{3.2}$$

where the derivative is in the Itô integral sense, the random process W a two-sided in time cylindrical Wiener process in

$$\Big\{ u : u \in L^2(M), u \cdot \vec{n}|_{\partial M} = 0,\ \frac{\partial u}{\partial \vec{n}} \times \vec{n}|_{\partial M} = 0 \Big\} = \mathrm{Span}\{f_i\},$$

with the form

$$W(t) = \sum_{i=1}^{+\infty} \omega_i(t,\omega) f_i,$$

and G is a nonnegative bounded operator on $L^2(M)$, with

$$\sum_{i=1}^{+\infty} \lambda_i^{2+2\gamma_0+1} |G^{\frac{1}{2}} \mathcal{D}^* e_i|_2^2 < +\infty;$$

for the definition of Wiener process, see, e.g., [22]. Here, $\omega_1, \omega_2, \cdots$ is a sequence of independent standard one-dimensional Brownian motions on a complete probability space $(\mathbf{\Omega}, \mathcal{F}, P)$ with expectation E and $\mathcal{D}^*$ is the dual operator of $\mathcal{D}$ given later.

Let μ be a real number such that the elliptic boundary-value problem

$$\begin{aligned}
&-\frac{1}{Re_1} \triangle Y_1 - \frac{1}{Re_2} \frac{\partial^2 Y_1}{\partial z^2} + \nabla p_b = \mu Y_1, \\
&\frac{\partial Y_1}{\partial z}|_{\Gamma_u} = u,\ \frac{\partial Y_1}{\partial z}|_{\Gamma_b} = 0,\ Y_1 \cdot \vec{n}|_{\Gamma_l} = 0,\ \frac{\partial Y_1}{\partial \vec{n}} \times \vec{n}|_{\Gamma_l} = 0, \\
&\int_{-1}^{0} \nabla \cdot Y_1 dz = 0
\end{aligned}$$

has a unique solution $Y_1 = \mathcal{D}u$ for any $u \in L^2(M)$ and $u \cdot \vec{n}|_{\partial M} = 0$, $\frac{\partial u}{\partial \vec{n}} \times \vec{n}|_{\partial M} = 0$. Now, we define an auxiliary Ornstein-Uhlenbeck process. Using similar arguments as in [20] or [21, Chapter 13], we know that the process

$$Z(t) = (-\mu + A_1) \int_{-\infty}^{t} e^{(t-s)(-A_1)} \mathcal{D} G^{\frac{1}{2}} dW(s) \tag{3.3}$$

is the unique Markovian generalized solution of the problem

$$\frac{\partial Z}{\partial t} = \frac{1}{Re_1} \triangle Z + \frac{1}{Re_2} \frac{\partial^2 Z}{\partial z^2} - \nabla p_{b_1},$$
$$\frac{\partial Z}{\partial z}|_{\Gamma_u} = G^{\frac{1}{2}} \frac{\partial W}{\partial t}, \ \frac{\partial Z}{\partial z}|_{\Gamma_b} = 0, \ Z \cdot \vec{n}|_{\Gamma_l} = 0, \ \frac{\partial Z}{\partial \vec{n}} \times \vec{n}|_{\Gamma_l} = 0,$$
$$\int_{-1}^{0} \nabla \cdot Z dz = 0, \quad Z(0) = (-\mu + A_1) \int_{-\infty}^{0} e^{-s(-A_1)} \mathcal{D} G^{\frac{1}{2}} dW(s).$$

According to the definition, we have a lemma about the properties of $Z(t)$.

Lemma 3.2. [21, Proposition 13.2.4] *If $Z(t)$ is the solution for the above problem, then the process $Z(t)$ is a stationary ergodic solution with continuous trajectories, taking values in $H^{2+2\gamma}(\Omega) \times H^{2+2\gamma}(\Omega)$, for any $\gamma < \gamma_0$.*

In order to obtain the existence of random attractors for (3.4)-(3.10), we should assume that the process Z satisfies the following condition:

$$-\lambda_1 + cE(|Z(0)|_4^8 + \|Z(0)\|^4 + \|Z(0)\|^2 \|Z(0)\|_2^2) < 0. \tag{C}$$

In the following two examples, if the coefficients δ_i and μ_i are small enough, then the condition (C) is satisfied.

Now, we give two examples of τ.

Example 3.3. An example of τ is $\tau = \frac{\partial W}{\partial t}$, where W is a two-sided in time finite-dimensional Brownian motion with the form

$$W = \sum_{i=1}^{m} \delta_i \omega_i(t, \omega) f_i.$$

In the above formula, $\omega_1, \cdots, \omega_m$ are independent standard one-dimensional Brownian motions on a complete probability space $(\mathbf{\Omega}, \mathcal{F}, P)$ (with expectation denoted by E), and δ_i are real coefficients. In this case, Z is a stationary ergodic solution with continuous trajectories which takes values in $H^k(\Omega)$ for any $k \in \mathbb{N}$.

Example 3.4. Another example of τ is $\tau = \frac{\partial W}{\partial t}$, where W is a two-sided in time infinite-dimensional Brownian motion with the form

$$W(t) = \sum_{i=1}^{+\infty} \mu_i \omega_i(t, \omega) f_i.$$

Here, $\omega_1, \omega_2, \cdots$ is a sequence of independent standard one-dimensional Brownian motions on a complete probability space $(\mathbf{\Omega}, \mathcal{F}, P)$ (with expectation denoted by E) and the coefficients μ_i satisfy

$$\sum_{i=1}^{+\infty} \lambda_i^{3+2\gamma_0} \mu_i^2 |\mathcal{D}^* e_i|_2^2 < +\infty,$$

for some $\gamma_0 > 0$. In this case, for $C > 0$,

$$\begin{aligned} E\|Z(t)\|_{2+2\gamma}^2 &\leq CE \sum_{i=1}^{+\infty} \left| \int_{-\infty}^{t} \lambda_i^{2+\gamma} e^{(t-s)(-\lambda_i)} \mu_i d\omega_i \right|^2 |\mathcal{D}^* e_i|_2^2 \\ &= \sum_{i=1}^{+\infty} \int_{-\infty}^{t} \lambda_i^{4+2\gamma} e^{2(t-s)(-\lambda_i)} \mu_i^2 ds |\mathcal{D}^* e_i|_2^2 \\ &= \sum_{i=1}^{+\infty} \frac{\lambda_i^{3+2\gamma} \mu_i^2}{2} |\mathcal{D}^* e_i|_2^2 < +\infty, \text{ for any } \gamma < \gamma_0. \end{aligned}$$

3.3. **New formulation of the system (2.11)-(2.18).** Let $w(t) = v(t) - Z(t)$. According to the definition of Z, we derive from (2.11)-(2.18) that

$$\begin{aligned} &\frac{\partial w}{\partial t} + [(w+Z)\cdot\nabla](w+Z) + \Phi(w+Z)\frac{\partial (w+Z)}{\partial z} + fk\times(w+Z) \\ &\qquad + \nabla p_{b_2} + \int_{-1}^{z} \nabla(\nu_1 T - \nu_2 S) dz' - \frac{1}{Re_1}\triangle w - \frac{1}{Re_2}\frac{\partial^2 w}{\partial z^2} = 0, \end{aligned} \tag{3.4}$$

$$\frac{\partial T}{\partial t} + [(w+Z)\cdot\nabla]T + \Phi(w+Z)\frac{\partial T}{\partial z} - \frac{1}{Rt_1}\triangle T - \frac{1}{Rt_2}\frac{\partial^2 T}{\partial z^2} = Q_1, \tag{3.5}$$

$$\frac{\partial S}{\partial t} + [(w+Z)\cdot\nabla]S + \Phi(w+Z)\frac{\partial S}{\partial z} - \frac{1}{Rs_1}\triangle S - \frac{1}{Rs_2}\frac{\partial^2 S}{\partial z^2} = Q_2, \tag{3.6}$$

$$\int_{-1}^{0} \nabla\cdot w \, dz = 0, \tag{3.7}$$

$$\frac{\partial w}{\partial z} = 0, \ \frac{\partial T}{\partial z} = -\alpha_u T, \ \frac{\partial S}{\partial z} = 0 \qquad \text{on } \Gamma_u, \tag{3.8}$$

$$\frac{\partial w}{\partial z} = 0, \ \frac{\partial T}{\partial z} = 0, \ \frac{\partial S}{\partial z} = 0 \qquad \text{on } \Gamma_b, \tag{3.9}$$

$$w\cdot\vec{n} = 0, \ \frac{\partial w}{\partial \vec{n}} \times \vec{n} = 0, \ \frac{\partial T}{\partial \vec{n}} = 0, \ \frac{\partial S}{\partial \vec{n}} = 0 \qquad \text{on } \Gamma_l, \tag{3.10}$$

$$(w|_{t=t_0}, T|_{t=t_0}, S|_{t=t_0}) = (w_{t_0}, T_{t_0}, S_{t_0}) = (v_{t_0} - Z_{t_0}, T_{t_0}, S_{t_0}). \tag{3.11}$$

Before the definition of weakly strong solutions, we define the baroclinic flow $\tilde{w}$ and find the equation satisfied by $\tilde{w}$ as in [4]. For any $w \in V_1$, denote the fluctuation of the horizontal velocity by $\tilde{w} = w - \bar{w}$, where

$$\bar{w} = \int_{-1}^{0} w dz.$$

As is shown similarly in [10], $\tilde{w}$ satisfies the equation

$$\begin{aligned}
&\frac{\partial \tilde{w}}{\partial t} - \frac{1}{Re_1}\triangle \tilde{w} - \frac{1}{Re_2}\frac{\partial^2 \tilde{w}}{\partial z^2} + [(\tilde{w} + \tilde{Z}) \cdot \nabla](\tilde{w} + \tilde{Z}) + \Phi(\tilde{w} + \tilde{Z})\frac{\partial(\tilde{w} + \tilde{Z})}{\partial z} \\
&\quad + [(\tilde{w} + \tilde{Z}) \cdot \nabla](\bar{w} + \bar{Z}) + [(\bar{w} + \bar{Z}) \cdot \nabla](\tilde{w} + \tilde{Z}) + fk \times (\tilde{w} + \tilde{Z}) \\
&\quad - \overline{(\tilde{w} + \tilde{Z})\nabla \cdot (\tilde{w} + \tilde{Z}) + [(\tilde{w} + \tilde{Z}) \cdot \nabla](\tilde{w} + \tilde{Z})} + \int_{-1}^{z} \nabla(\nu_1 T - \nu_2 S) dz' \\
&\quad - \int_{-1}^{0}\int_{-1}^{z} \nabla(\nu_1 T - \nu_2 S) dz' dz = 0 \quad \text{in } \Omega, \qquad (3.12)
\end{aligned}$$

$$\frac{\partial \tilde{w}}{\partial z} = 0 \quad \text{on} \quad \Gamma_u, \ \frac{\partial \tilde{w}}{\partial z} = 0 \quad \text{on} \quad \Gamma_b, \ \tilde{w} \cdot \vec{n} = 0, \ \frac{\partial \tilde{w}}{\partial \vec{n}} \times \vec{n} = 0 \quad \text{on} \quad \Gamma_l. \qquad (3.13)$$

Definition 3.5. *If Z is defined above, $\mathcal{U}_{t_0} = (w_{t_0}, T_{t_0}, S_{t_0})$ satisfies $\mathcal{U}_{t_0} \in H$, $\tilde{w}_{t_0} \in (L^4(\Omega))^2$, $T_{t_0}, S_{t_0} \in L^4(\Omega)$, $\partial_z w_{t_0} \in (L^2(\Omega))^2$, $\partial_z T_{t_0}, \partial_z S_{t_0} \in L^2(\Omega)$. Let $\mathcal{T} > t_0$ be a fixed time. For P-a.e. $\omega \in \mathbf{\Omega}$, (w, T, S) is called a weakly strong solution of the system* (3.4)-(3.11) *on the time interval* $[t_0, \mathcal{T}]$ *if it satisfies* (3.4)-(3.7) *in a weak sense such that*

$$w \in L^2(t_0, \mathcal{T}; V_1) \cap L^\infty(t_0, \mathcal{T}; H_1), \quad \tilde{w} \in L^\infty(t_0, \mathcal{T}; (L^4(\Omega))^2),$$

$$\partial_z w \in L^\infty(t_0, \mathcal{T}; (L^2(\Omega))^2) \cap L^2(t_0, \mathcal{T}; (H^1(\Omega))^2),$$

$$T \in L^\infty(t_0, \mathcal{T}; L^4(\Omega)) \cap L^2(0, \mathcal{T}; V_2), S \in L^\infty(t_0, \mathcal{T}; L^4(\Omega)) \cap L^2(t_0, \mathcal{T}; V_3),$$

$$\partial_z T \in L^\infty(t_0, \mathcal{T}; L^2(\Omega)) \cap L^2(t_0, \mathcal{T}; H^1(\Omega)),$$

$$\partial_z S \in L^\infty(t_0, \mathcal{T}; L^2(\Omega)) \cap L^2(t_0, \mathcal{T}; H^1(\Omega)),$$

$$\frac{\partial v}{\partial t} \in L^2(t_0, V_1'), \ \frac{\partial T}{\partial t} \in L^2(t_0, \mathcal{T}; V_2'), \frac{\partial S}{\partial t} \in L^2(t_0, \mathcal{T}; V_3'),$$

where V_i' is the dual space of V_i for $i = 1, 2, 3$.

Remark 3.6 For almost all given paths of the process $Z(t)$, we can study the equations (3.4)-(3.7) as deterministic evolution equations.

4. Global well-posedness (3.4)-(3.11)

4.1. **Global existence of weakly strong solutions.** We prove the global existence of weakly strong solutions by the well-known Faedo-Galerkin method. Since the procedure is now standard, we only give *a priori* estimates. Before making *a priori* estimates about the weakly strong solutions to (3.4)-(3.11), we recall some interpolation inequalities (see, e.g., [9]) and give a lemma used frequently later.

i) For $h_1 \in H^1(M)$,

$$\|h_1\|_{L^4} \le c\|h_1\|_{L^2}^{\frac{1}{2}}\|h_1\|_{H^1}^{\frac{1}{2}}, \tag{4.1}$$

$$\|h_1\|_{L^5} \le c\|h_1\|_{L^3}^{\frac{3}{5}}\|h_1\|_{H^1}^{\frac{2}{5}}, \tag{4.2}$$

$$\|h_1\|_{L^6} \le c\|h_1\|_{L^4}^{\frac{2}{3}}\|h_1\|_{H^1}^{\frac{1}{3}}. \tag{4.3}$$

ii)
$$|h_2|_4 \le c|h_2|_2^{\frac{1}{4}}\|h_2\|_1^{\frac{3}{4}}, \quad \text{for } h_2 \in H^1(\Omega). \tag{4.4}$$

Lemma 4.1. *If $v_1 \in H^2(\Omega)\times H^2(\Omega)$, $v_2 \in H^2(\Omega)\times H^2(\Omega)$ or $H^2(\Omega)$, $v_3 \in L^2(\Omega)\times L^2(\Omega)$ or $L^2(\Omega)$, then*

i) $\displaystyle\Big|\int_\Omega v_3\cdot(v_1\cdot\nabla)v_2\Big| \le c(|v_1|_4^2+|v_1|_4^8)|\nabla v_2|_2^2+\varepsilon\Big[|v_3|_2^2+\int_\Omega(|\nabla v_{2z}|^2+|\triangle v_2|^2)\Big]$,

ii) $\displaystyle\Big|\int_\Omega \Phi(v_1)v_{2z}\cdot v_3\Big| \le c|\nabla v_1|_2^{\frac{1}{2}}(|\nabla v_1|_2^2+|\triangle v_1|_2^2)^{\frac{1}{4}}|v_{2z}|_2^{\frac{1}{2}}|\nabla v_{2z}|_2^{\frac{1}{2}}|v_3|_2$.

In this assertion, ε is a small enough positive constant.

Proof. With the Hölder inequality (4.4) and Young inequality,

$$\begin{aligned}\int_\Omega |v_1||\nabla v_2||v_3| &\le c|v_1|_4^2\Big(\int_\Omega|\nabla v_2|^4\Big)^{\frac{1}{2}}+\varepsilon|v_3|_2^2\\ &\le c(|v_1|_4^2+|v_1|_4^8)|\nabla v_2|_2^2+\varepsilon\Big[|v_3|_2^2+\int_\Omega(|\nabla v_{2z}|^2+|\triangle v_2|^2)\Big].\end{aligned}$$

Using the Hölder inequality, the Minkowski inequality and (4.1), we obtain

$$\begin{aligned}&\int_M\Big(\int_{-1}^0|\nabla v_1|dz\int_{-1}^0|v_{2z}||v_3|dz\Big)\\ &\le \int_M\Big[\Big(\int_{-1}^0|\nabla v_1|^2dz)^{\frac{1}{2}}\Big(\int_{-1}^0|v_{2z}|^2dz\Big)^{\frac{1}{2}}\Big(\int_{-1}^0|v_3|^2dz\Big)^{\frac{1}{2}}\Big]\\ &\le c\Big[\int_{-1}^0\Big(|\nabla v_1|_{L^2(M)}|\nabla v_1|_{H^1(M)}\Big)\int_{-1}^0\Big(|v_{2z}|_{L^2(M)}|v_{2z}|_{H^1(M)}\Big)\Big]^{\frac{1}{2}}|v_3|_2. \qquad\square\end{aligned}$$

In this subsection, $\omega \in \mathbf{\Omega}$ is fixed. Now, we give *a priori* estimates of weakly strong solutions (w, T, S) to (3.4)-(3.11).

L^2 **estimates about** T, S**,** w**.** Taking the inner product of equation (3.5) with T in $L^2(\Omega)$, by integration by parts,

$$T(t,x,y,z) = -\int_z^0 \frac{\partial T}{\partial z'} dz' + T|_{z=0};$$

using the Hölder inequality and Cauchy-Schwarz inequality, we obtain

$$\frac{d|T|_2^2}{dt} + \frac{1}{Rt_1}\int_\Omega |\nabla T|^2 + \frac{1}{Rt_2}\int_\Omega |\frac{\partial T}{\partial z}|^2 + \frac{\alpha_u}{Rt_2}|T|_{z=0}|_2^2 \le c|Q_1|_2^2. \tag{4.5}$$

With (4.5) and the Gronwall inequality,

$$|T(t)|_2^2 \le e^{-ct}|T_{t_0}|_2^2 + c|Q_1|_2^2, \tag{4.6}$$

where $t \ge t_0$. From (4.5) and (4.6), for $\mathcal{T} > t_0$ given, there exists a positive constant $C_1(\mathcal{T}, |T_{t_0}|_2, |Q_1|_2)$ such that

$$\int_{t_0}^{\mathcal{T}} [\int_\Omega (|\nabla T|^2 + |\frac{\partial T}{\partial z}|^2 + |T|^2) + |T|_{z=0}|_2^2] + |T(t)|_2^2 \le C_1, \tag{4.7}$$

where $t \in [t_0, \mathcal{T})$ and $\int_{t_0}^{\mathcal{T}} \cdot ds$ is denoted by $\int_{t_0}^{\mathcal{T}} \cdot$. Throughout this paper, $C_i(\cdot,\cdot)$ denotes the positive constants dependent only on the quantities appearing in parentheses, $i \in \mathbb{N}$.

Taking the inner product of equation (3.6) with S in $L^2(\Omega)$, using the Poincaré inequality and Gronwall inequality, we obtain

$$|S(t)|_2^2 \le e^{-ct}|S_{t_0}|_2^2 + c|Q_2|_2^2, \tag{4.8}$$

where $t \ge t_0$. Moreover, for $\mathcal{T} > t_0$ given, there exists a positive constant $C_2(\mathcal{T}, |T_{t_0}|_2, |Q_2|_2)$ such that

$$\int_{t_0}^{\mathcal{T}} \int_\Omega (|\nabla S|^2 + |\frac{\partial S}{\partial z}|^2 + |S|^2) + |S(t)|_2^2 \le C_2, \tag{4.9}$$

where $t \in [t_0, \mathcal{T})$.

Choosing w as a test function in (3.4), using Lemma 4.1 and the Hölder and Young inequalities, we obtain

$$\begin{aligned} &\frac{d|w|_2^2}{dt} + \frac{1}{Re_1}\int_\Omega |\nabla w|^2 + \frac{1}{Re_2}\int_\Omega |\frac{\partial w}{\partial z}|^2 \\ &\le c(|Z|_4^8 + \|Z\|^4 + \|Z\|^2\|Z\|_2^2)|w|_2^2 + c(\|Z\|^2 + |T|_2^2 + |S|_2^2). \end{aligned} \tag{4.10}$$

Using the fact that $\lambda_1|w|_2^2 \le \|w\|^2$ and the Gronwall inequality, for $t \ge t_0$, we derive from (4.10) that

$$|w(t)|_2^2 \le e^{\int_{t_0}^t[-\lambda_1+c(|Z|_4^8+\|Z\|^4+\|Z\|^2\|Z\|_2^2)]d\tau}|w(t_0)|_2^2 + c\int_{t_0}^t e^{\int_\sigma^t[-\lambda_1+c(|Z|_4^8+\|Z\|^4+\|Z\|^2\|Z\|_2^2)]d\tau}(\|Z\|^2+|T|_2^2+|S|_2^2)d\sigma. \quad (4.11)$$

By Lemma 3.2 and (4.11), for $\mathcal{T} > t_0$ given, there exists a positive constant $C_3(\mathcal{T},\mathcal{U}_{t_0},Q_1,Q_2,Z_{t_0})$ such that

$$\int_{t_0}^{\mathcal{T}}\int_\Omega(|\nabla w|^2+|\frac{\partial w}{\partial z}|^2)+|w(t)|_2^2 \le C_3, \text{ for any } \ t\in[t_0,\mathcal{T}). \quad (4.12)$$

By the Minkowski and Hölder inequalities, we derive from (4.12) that

$$\int_{t_0}^{\mathcal{T}}\int_M(|\nabla\bar{w}|^2+|\bar{w}|^2)+\|\bar{w}(t)\|_{L^2}^2 \le C_3, \forall t\in[t_0,\mathcal{T}), \quad (4.13)$$

which implies

$$\int_{t_0}^{\mathcal{T}}|\bar{v}|_4^4 = \int_{t_0}^{\mathcal{T}}\|\bar{v}\|_{L^4(M)}^4 \le \int_{t_0}^{\mathcal{T}}\|\bar{v}\|_{L^2(M)}^2\|\bar{v}\|_{H^1(M)}^2 \le C_3^2. \quad (4.14)$$

Remark 4.2. In the following, the result of L^3 estimates about $\tilde{w}$ will be used in studying the long-time behavior of strong solutions to the stochastic primitive equations. Since L^3 estimates about $\tilde{w}$ are similar to L^4 estimates about $\tilde{w}$, we give L^3 estimates about $\tilde{w}$ here.

L^3, L^4 **estimates about** $\tilde{w}$**.** As is shown similarly in [10],

$$\frac{d|\tilde{w}|_3^3}{dt}+\frac{1}{Re_1}\int_\Omega(|\nabla\tilde{w}|^2|\tilde{w}|+\frac{4}{9}|\nabla|\tilde{w}|^{\frac{3}{2}}|^2)+\frac{1}{Re_2}\int_\Omega(|\partial_z\tilde{w}|^2|\tilde{w}|+\frac{4}{9}|\partial_z|\tilde{w}|^{\frac{3}{2}}|^2)$$
$$\le c(1+\|\bar{w}\|_{L^2}^2\|\bar{w}\|_{H^1}^2+\|\tilde{w}\|^2+|w|_2^{\frac{3}{2}}\|w\|^{\frac{3}{2}}\|Z\|^{\frac{3}{2}}\|Z\|_2^{\frac{3}{2}}+|Z|_2^{\frac{3}{2}}\|Z\|^3\|Z\|_2^{\frac{3}{2}}$$
$$+\|Z\|^2+|Z|_4^8+\|Z\|^4+\|Z\|\|Z\|_2^3+\|Z\|^2\|Z\|_2^2+|Z|_6^6)|\tilde{w}|_3^3+c|w|_2^2\|w\|^2$$
$$+c\|w\|^{\frac{3}{2}}+c\|w\|^2+c\|Z\|^3+c\|Z\|^8+c|T|_2^2\|T\|^2+c|S|_2^2\|S\|^2+c\|Z\|^2\|Z\|_2^2,$$

which implies that there exists $C_4(\mathcal{T},\mathcal{U}_{t_0},Q_1,Q_2,Z_{t_0})$ such that

$$|\tilde{w}(t)|_3^3 \le C_4, \quad \text{for any } t\in[t_0,\mathcal{T}). \quad (4.15)$$

Similarly, we obtain

$$\frac{d|\tilde{w}|_4^4}{dt}+\frac{1}{Re_1}\int_\Omega(|\nabla\tilde{w}|^2|\tilde{w}|^2+\frac{1}{2}|\nabla|\tilde{w}|^2|^2)+\frac{1}{Re_2}\int_\Omega(|\partial_z\tilde{w}|^2|\tilde{w}|^2+\frac{1}{2}|\partial_z|\tilde{w}|^2|^2)$$

$$\le c(1 + |w|_2^2\|w\|^2 + |Z|_2^2\|Z\|^2 + |Z|_8^4 + \|w\|^2 + \|Z\|^4 + \|Z\|^{\frac{4}{3}}\|Z\|_2^{\frac{4}{3}}$$
$$+ \|Z\|^{\frac{2}{3}}\|Z\|_2^2 + \|Z\|^{\frac{8}{5}}\|Z\|_2^{\frac{8}{5}} + \|Z\|\|Z\|_2^3 + \|Z\|^2\|Z\|_2^2)|\tilde{w}|_4^4 + c|w|_2^2\|w\|^2$$
$$+ c|Z|_2^2\|Z\|^2 + c|Z|_4^4 + c|T|_2^2\|T\|^2 + c|S|_2^2\|S\|^2 + \|Z\|^{\frac{8}{5}}\|Z\|_2^{\frac{8}{5}},$$

which implies that there exists $C_5(\mathcal{T}, \mathcal{U}_{t_0}, Q_1, Q_2, Z_{t_0})$ such that

$$|\tilde{w}(t)|_4^4 \le C_5, \tag{4.16}$$

where $t_0 \le t < \mathcal{T}$.

L^2 **estimates about** $\partial_z w$**.** Taking the derivative, with respect to z, of equation (3.4), we get

$$\begin{aligned}
&\frac{\partial w_z}{\partial t} - \frac{1}{Re_1}\triangle w_z - \frac{1}{Re_2}\frac{\partial^2 w_z}{\partial z^2} + [(w+Z)\cdot\nabla](w_z + Z_z) \\
&+ \Phi(w+Z)\frac{\partial(w_z+Z_z)}{\partial z} + [(w_z+Z_z)\cdot\nabla](w+Z) \\
&- (w_z+Z_z)\nabla\cdot(w+Z) + fk\times(w_z+Z_z) + \nabla(\nu_1 T - \nu_2 S) = 0.
\end{aligned} \tag{4.17}$$

From integrations by parts, the Hölder, Minkowski, and Sobolev inequalities and (4.4),

$$\begin{aligned}
&-\int_\Omega \{[(w+Z)\cdot\nabla]Z_z + \Phi(w+Z)\frac{\partial Z_z}{\partial z}\}\cdot w_z \\
&\le \varepsilon\|w_z\|^2 + c\|Z\|_2^2 + c(|\tilde{w}+\tilde{Z}|_4^8 + |\bar{w}+\bar{Z}|_{L^4}^4)|w_z|_2^2 + c\|w+Z\|^2\|Z\|_{2+2\gamma}^2,
\end{aligned} \tag{4.18}$$

where $0 < \gamma < \gamma_0$, γ_0 is given in the definition of τ. By integration by parts, the Hölder inequality, (4.4) and the Young inequality,

$$\begin{aligned}
&-\int_\Omega \{[(w_z+Z_z)\cdot\nabla](w+Z) - (w_z+Z_z)\nabla\cdot(w+Z)\}\cdot w_z \\
&\le \varepsilon\|w_z\|^2 + c\|Z\|_2^2 + c(|\tilde{w}+\tilde{Z}|_4^8 + |\bar{w}+\bar{Z}|_{L^4}^4)|Z_z|_2^2|w_z|_2^2.
\end{aligned} \tag{4.19}$$

Taking the inner product of equation (4.17) with w_z in $L^2(\Omega)\times L^2(\Omega)$, by integration by parts, the Hölder and Poincaré inequalities, (4.18), (4.19), and choosing ε small enough, we obtain

$$\begin{aligned}
&\frac{d|w_z|_2^2}{dt} + \frac{1}{Re_1}\int_\Omega|\nabla w_z|^2 + \frac{1}{Re_2}\int_\Omega|\frac{\partial w_z}{\partial z}|^2 \le c(|\bar{w}|_2^2\|\bar{w}\|_2^2 + |Z|_4^4 + |\tilde{w}|_4^8 + |Z|_4^8) \\
&\times(|w_z|_2^2 + |Z_z|_2^2) + c\|Z\|_2^2 + c\|w+Z\|^2\|Z\|_{2+2\gamma}^2 + c|T|_2^2 + c|S|_2^2.
\end{aligned} \tag{4.20}$$

By the Gronwall inequality, Lemma 3.2, (4.7), (4.9) and (4.16), for $\mathcal{T} > t_0$ given, there exists $C_6(\mathcal{T}, \mathcal{U}_{t_0}, Q_1, Q_2, Z_{t_0})$ such that

$$\int_{t_0}^{\mathcal{T}} \|w_z\|^2 + |w_z(t)|_2^2 \leq C_6, \text{ for any } \ t \in [t_0, \mathcal{T}). \tag{4.21}$$

L^2 **estimates about** $\partial_z T, \partial_z S$**.** Taking the derivative, with respect to z, of equation (3.5), we get

$$\begin{aligned} \frac{\partial T_z}{\partial t} - \frac{1}{Rt_1} \triangle T_z - \frac{1}{Rt_2} \frac{\partial^2 T_z}{\partial z^2} + [(w+Z)\cdot\nabla] T_z + \Phi(w+Z) \frac{\partial T_z}{\partial z} \\ + [(w_z + Z_z)\cdot\nabla]T - T_z \nabla\cdot(w+Z) = Q_{1z}. \end{aligned} \tag{4.22}$$

By integration by parts and the Hölder, Poincaré, and Young inequalities, we obtain

$$\begin{aligned} &\Big| \int_\Omega [((w_z + Z_z)\cdot\nabla)T - T_z\nabla\cdot(w+Z)]T_z \Big| \qquad (4.23) \\ &\leq c\int_\Omega \Big[(|\nabla w_z| + |\nabla Z_z|)|T||T_z| + |w_z + Z_z||T||\nabla T_z| \\ &+ (|\tilde{w}| + |\bar{w}| + |Z|)|\nabla T_z||T_z| \Big] \\ &\leq c\int_\Omega (|\nabla w_z|^2 + |\nabla Z_z|^2) + \frac{\varepsilon}{2}|\nabla T_z|_2^2 + c(|T|_4^2 + |\tilde{w}|_4^2)|T_z|_4^2 \\ &+ c(|w_z|_4^2 + |Z_z|_4^2)|T|_4^2 + c\|\bar{w}\|_{L^4(M)}^2 \int_{-1}^0 \Big(\int_M |T_z|^4 \Big)^{\frac{1}{2}} + |Z|_4^2 |T_z|_4^2 \\ &\leq \varepsilon(|T_{zz}|_2^2 + |\nabla T_z|_2^2) + c\Big(|w_{zz}|_2^2 + \int_\Omega |\nabla w_z|^2 \Big) + c\|Z\|_2^2 + c\|Z\|_2^2 |T|_4^2 \\ &+ c|T|_4^8 |w_z|_2^2 + c(|T|_4^8 + |Z|_4^8 + |\tilde{w}|_4^8 + \|\bar{w}\|_{L^4(M)}^4)|T_z|_2^2. \end{aligned}$$

Taking the trace on $z = 0$ of equation (3.5), we have

$$\frac{1}{Rt_2} T_{zz}|_{z=0} = \frac{\partial T|_{z=0}}{\partial t} + [(w+Z)\cdot\nabla]T|_{z=0} - \frac{1}{Rt_1}\triangle T|_{z=0} - Q_1|_{z=0}. \tag{4.24}$$

From (3.8) and (4.24), we get

$$\begin{aligned} &-\frac{1}{Rt_2}\int_M (T_z|_{z=0} T_{zz}|_{z=0}) \qquad (4.25) \\ &= \alpha_u \int_M T|_{z=0} [\frac{\partial T|_{z=0}}{\partial t} + ((w+Z)\cdot\nabla)T|_{z=0} - \frac{1}{Rt_1}\triangle T|_{z=0} - Q_1|_{z=0}] \\ &= \alpha_u (\frac{1}{2}\frac{d|T(z=0)|_2^2}{dt} + \frac{1}{Rt_1}|\nabla T(z=0)|_2^2) \end{aligned}$$

$$+ \alpha_u \int_M T|_{z=0}[((w+Z)\cdot\nabla)T|_{z=0} - Q_1|_{z=0}].$$

By integration by parts, we have

$$\begin{aligned}
&- \alpha_u \int_M T|_{z=0}[((w+Z)\cdot\nabla)T|_{z=0} - Q_1|_{z=0}] \qquad (4.26)\\
&= -\frac{\alpha_u}{2}\int_M [((w+Z)\cdot\nabla)T^2]|_{z=0} + \alpha_u \int_M T|_{z=0}Q_1|_{z=0}\\
&= \frac{\alpha_u}{2}\int_M T^2|_{z=0}\mathrm{div}(w+Z)|_{z=0} + \alpha_u \int_M T|_{z=0}Q_1|_{z=0}\\
&= \frac{\alpha_u}{2}\int_M T^2|_{z=0}(\int_z^0 \mathrm{div}(w_z+Z_z)dz' + \mathrm{div}(w+Z)) + \alpha_u \int_M T|_{z=0}Q_1|_{z=0}\\
&\le c|T|_{z=0}|_4^4 + c\|w_z\|^2 + c\|Z\|_2^2 + c\|w\|^2 + c|T|_{z=0}|_2^2 + c|Q_1|_{z=0}|_2^2.
\end{aligned}$$

Taking the inner product of equation (4.22) with T_z in $L^2(\Omega)$, by integration by parts and the Hölder inequality, choosing ε small enough, we derive from (4.23)-(4.26)

$$\begin{aligned}
&\frac{d(|T_z|_2^2 + \alpha_u|T|_{z=0}|_2^2)}{dt} + \frac{1}{Rt_1}\int_\Omega |\nabla T_z|^2 + \frac{1}{Rt_2}\int_\Omega |T_{zz}|^2 + \frac{\alpha_u}{R_{t1}}|\nabla T(z=0)|_2^2\\
&\le c(1 + |T|_4^8 + |Z|_4^8 + |\tilde{w}|_4^8 + \|\bar{w}\|_{L^4(M)}^4)|T_z|_2^2 + c\|w_z\|^2 + c\|w\|^2 + c|T|_4^8|w_z|_2^2\\
&+c\|Z\|_2^2 + c\|Z\|_2^2|T|_4^2 + c|T|_{z=0}|_4^4 + c|T|_{z=0}|_2^2 + c|Q_1|_{z=0}|_2^2 + c|Q_{1z}|_2^2. \qquad (4.27)
\end{aligned}$$

By the Gronwall inequality, Lemma 3.2, (4.7), (4.9), (4.12), (4.14), (4.16) and (4.21), for $\mathcal{T} > t_0$ given, there exists $C_7(\mathcal{T}, \mathcal{U}_{t_0}, Q_1, Q_2, Z_{t_0})$ such that

$$\int_{t_0}^{\mathcal{T}} \|T_z\|^2 + |T_z(t)|_2^2 \le C_7, \text{ for any } \ t \in [t_0, \mathcal{T}). \qquad (4.28)$$

In order to obtain the above inequality, we have used

$$|T(t)|_4^4 + \int_t^{\mathcal{T}} |T|_{z=0}|_4^4 \le C,$$

which can be proved by taking the inner product of equation (3.5) with $|T|^2T$. Similarly to (4.28), we know that there exists $C_8(\mathcal{T}, \mathcal{U}_{t_0}, Q_1, Q_2, Z_{t_0})$ such that

$$\int_{t_0}^{\mathcal{T}} \|S_z\|^2 + |S_z(t)|_2^2 \le C_8, \text{ for any } \ t \in [t_0, \mathcal{T}). \qquad (4.29)$$

Here, we have used $|S(t)|_4^4 \le C$.

4.2. **Uniqueness of weakly strong solutions.**
Proof of the uniqueness. Let (w_1, T_1, S_1) and (w_2, T_2, S_2) be two weakly strong solutions of (3.4)-(3.11) on the time interval $[t_0, \mathcal{T}]$ with p'_{b_2}, p''_{b_2} and initial data $((w_{t_0})_1, (T_{t_0})_1, (S_{t_0})_1)$, $((w_{t_0})_2, (T_{t_0})_2, (S_{t_0})_2)$ respectively.

Define $w = w_1 - w_2$, $T = T_1 - T_2$, $S = S_1 - S_2$, $p_{b_2} = p'_{b_2} - p''_{b_2}$. Then w, T, S, p_{b_2} satisfy

$$\frac{\partial w}{\partial t} - \frac{1}{Re_1}\triangle w - \frac{1}{Re_2}\frac{\partial^2 w}{\partial z^2} + [(w_1+Z)\cdot\nabla]w + (w\cdot\nabla)(w_2+Z) + \Phi(w_1+Z)\frac{\partial w}{\partial z} + \Phi(w)\frac{\partial(w_2+Z)}{\partial z} + fk\times w + \nabla p_{b_2} - \int_{-1}^{z}\nabla(\nu_1 T - \nu_2 S)dz' = 0, \tag{4.30}$$

$$\frac{\partial T}{\partial t} - \frac{1}{Rt_1}\triangle T - \frac{1}{Rt_2}\frac{\partial^2 T}{\partial z^2} + [(w_1+Z)\cdot\nabla]T + (w\cdot\nabla)T_2 + \Phi(w_1+Z)\frac{\partial T}{\partial z} + \Phi(w)\frac{\partial T_2}{\partial z} = 0, \tag{4.31}$$

$$\frac{\partial S}{\partial t} - \frac{1}{Rs_1}\triangle S - \frac{1}{Rs_2}\frac{\partial^2 S}{\partial z^2} + [(w_1+Z)\cdot\nabla]S + (w\cdot\nabla)S_2 + \Phi(w_1+Z)\frac{\partial S}{\partial z} + \Phi(w)\frac{\partial S_2}{\partial z} = 0, \tag{4.32}$$

$$\int_{-1}^{0}\nabla\cdot w dz = 0, \tag{4.33}$$

$$w_{t_0} = (w_{t_0})_1 - (w_{t_0})_2,\ T_{t_0} = (T_{t_0})_1 - (T_{t_0})_2,\ S_{t_0} = (S_{t_0})_1 - (S_{t_0})_2, \tag{4.34}$$

and

$$(w, T, S) \text{ satisfies the boundary-value conditions (3.8)-(3.10).} \tag{4.35}$$

We take the inner product of (4.30) with w in $L^2(\Omega)\times L^2(\Omega)$ and obtain

$$\frac{1}{2}\frac{d|w|_2^2}{dt} + \frac{1}{Re_1}\int_\Omega|\nabla w|^2 + \frac{1}{Re_2}\int_\Omega|w_z|^2 = -\int_\Omega\{[(w_1+Z)\cdot\nabla]w + \Phi(w_1+Z)\frac{\partial w}{\partial z}\}\cdot w - \int_\Omega(fk\times w + \nabla p_{b_2})\cdot w - \int_\Omega[(w\cdot\nabla)(w_2+Z) + \Phi(w)\frac{\partial(w_2+Z)}{\partial z}]\cdot w + \int_\Omega[\int_{-1}^{z}\nabla(\nu_1 T - \nu_2 S)dz']\cdot w. \tag{4.36}$$

By integration by parts and Lemma 4.1,

$$|\int_\Omega[(w\cdot\nabla)(w_2+Z)]\cdot w|$$

$$\leq \varepsilon \int_\Omega (|\nabla w|^2 + |w_z|^2) + c(|\bar{w}|_2^2 \|\bar{w}\|^2 + |Z|_4^4 + |Z|_4^8 + |\tilde{w_2}|_4^8)|w|_2^2,$$

$$|\int_\Omega \Phi(w) \frac{\partial(w_2 + Z)}{\partial z} \cdot w|$$

$$\leq \varepsilon \int_\Omega |\nabla w|^2 + c(|w_{2z}|_2^2 |\nabla w_{2z}|_2^2 + |w_{2z}|_2^4 + |Z_z|_2^2 |\nabla Z_z|_2^2 + |Z_z|_2^4)|w|_2^2. \quad (4.37)$$

By integration by parts and the Hölder and Young inequalities, we derive from (4.36) and (4.37) that

$$\frac{1}{2}\frac{d|w|_2^2}{dt} + \frac{1}{Re_1}\int_\Omega |\nabla w|^2 + \frac{1}{Re_2}\int_\Omega |w_z|^2 \quad (4.38)$$

$$\leq 2\varepsilon \int_\Omega (|\nabla w|^2 + |w_z|^2) + \varepsilon |\nabla T|_2^2 + c(1 + |\bar{w}|_2^2 \|\bar{w}\|^2$$

$$+ |Z|_4^8 + |\tilde{w_2}|_4^8 + |w_{2z}|_2^2 |\nabla w_{2z}|_2^2 + |w_{2z}|_2^4 + |Z_z|_2^2 |\nabla Z_z|_2^2 + |Z_z|_2^4)|w|_2^2.$$

Similarly to (4.38), we get

$$\frac{1}{2}\frac{d|T|_2^2}{dt} + \frac{1}{Rt_1}\int_\Omega |\nabla T|^2 + \frac{1}{Rt_2}\int_\Omega |T_z|^2 + \frac{\alpha_u}{Rt_2}|T|_{z=0}|_2^2 \quad (4.39)$$

$$\leq \varepsilon \int_\Omega (|\nabla w|^2 + |w_z|^2) + \varepsilon \int_\Omega (|\nabla T|^2 + |T_z|^2)$$

$$+ c(|T_2|_4^8 + |T_{2z}|_2^2 |\nabla T_{2z}|_2^2 + |T_{2z}|_2^4)(|w|_2^2 + |T|_2^2),$$

$$\frac{1}{2}\frac{d|S|_2^2}{dt} + \frac{1}{Rs_1}\int_\Omega |\nabla S|^2 + \frac{1}{Rs_2}\int_\Omega |S_z|^2 \quad (4.40)$$

$$\leq \varepsilon \int_\Omega (|\nabla w|^2 + |w_z|^2) + \varepsilon \int_\Omega (|\nabla S|^2 + |S_z|^2)$$

$$+ c(|S_2|_4^8 + |S_{2z}|_2^2 |\nabla S_{2z}|_2^2 + |S_{2z}|_2^4)(|w|_2^2 + |S|_2^2).$$

From (4.38)-(4.40), and choosing ε small enough, using the Gronwall inequality and the result of *a priori* estimates in Subsection 4.1, we prove the uniqueness.

5. The existence of random attractors

5.1. **Preliminaries for random attractors.** We recall some definitions and results from the theory of random dynamical systems, see, e.g., [1, 2, 3, 24]. Let (X, d) be a complete separable metric space (a Polish space), $(\mathbf{\Omega}, \mathcal{F}, P)$ a complete probability space, $\{\vartheta_t : \mathbf{\Omega} \to \mathbf{\Omega} : t \in \mathbb{R}\}$ a family of measure-preserving transformation such that $\vartheta_0 = \mathrm{id}_{\mathbf{\Omega}}$ and $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all $t,\ s \in \mathbb{R}$. $\{\vartheta_t\}$ is called a metric dynamical system on $\mathbf{\Omega}$, which represents the noise driving a random dynamical system. Here, we assume ϑ_t is ergodic under P.

Definition 5.1. (random dynamical system) *A measurable map* $\psi : \mathbb{R}^+ \times \mathbf{\Omega} \times X \to X, (t, \omega, \mathcal{U}) \mapsto \psi(t,\omega)\mathcal{U}$ *is called a random dynamical system if* ψ *satisfies the cocycle property:* $\psi(0,\omega) = id_X, \ \psi(t+s,\omega) = \psi(t, \vartheta_s\omega)\psi(s,\omega)$ *for all* $t, \ s \in \mathbb{R}^+$ *and* $P-a.s.$ $\omega \in \mathbf{\Omega}$. *If* $\psi(t,\omega) : X \longrightarrow X$ *is continuous, then* ψ *is called a continuous random dynamical system.*

Random dynamical systems with continuous time are generated by infinite dimensional stochastic evolution equations under an additive noise with a unique global solution, as well as by differential equations with random coefficients or stochastic differential equations.

Definition 5.2. (random compact set) *Let* $K : \mathbf{\Omega} \to 2^X$, *where* 2^X *is the set of all subsets of* X. K *is called a random compact set if* $K(\omega)$ *is compact* $P-a.s.$ *and the map* $\omega \to d(\mathcal{U}, K(\omega))$ *is measurable for any* $\mathcal{U} \in X$, *where* $d(\mathcal{U}, K(\omega)) = \inf_{\mathcal{U}_1 \in K(\omega)} d(\mathcal{U}, \mathcal{U}_1)$.

Definition 5.3. *Let* $A(\omega), \ B(\omega)$ *be two random sets.*

i) $A(\omega)$ *attracts* $B(\omega)$ *if*

$$\lim_{t \to +\infty} d(\psi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), A(\omega)) = 0, \ P-a.s.$$

ii) $A(\omega)$ *absorbs* $B(\omega)$ *if there exists* $t_B(\omega)$ *such that, for all* $t \geq t_B(\omega)$,

$$\psi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subset A(\omega), \ P-a.s.$$

Definition 5.4. (random attractor) *A random set* $\mathcal{A}(\omega)$ *is said to be a random attractor for the random dynamical system* ψ *if,* $P-a.s.$, *the following hold.*

i) $\mathcal{A}(\omega)$ *is a random compact set.*

ii) $\mathcal{A}(\omega)$ *is invariant; that is,* $\psi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\vartheta_t\omega)$, *for* $t \geq 0$.

iii) $\mathcal{A}(\omega)$ *attracts all deterministic bound sets* $B \subset X$*; i.e.,*

$$\lim_{t \to +\infty} d(\psi(t, \vartheta_{-t}\omega)B, A(\omega)) = 0, \ P-a.s.$$

Remark 5.5. $\psi(t, \vartheta_{-t}\omega)\mathcal{U}$ can be interpreted as the position at $t = 0$ of the trajectory which was $\mathcal{U}$ at time $-t$; that is, when t is moving, the trajectory $\psi(t, \vartheta_{-t}\omega)\mathcal{U}$ is always at the position at $t = 0$. Therefore, the random attractor is also called the *random pull-back attractor*.

Theorem 5.6. (cf. [2, 3]) *If there exists a random compact set* $K(\omega)$ *absorbing every bounded non-random set* $B \subset X$, *the continuous random* *dynamical system ψ possesses a random pull-back attractor $\mathcal{A}(\omega)$, where*

$$\mathcal{A}(\omega) = \overline{\cup_{B\subset X}\Lambda_B(\omega)}, \quad \Lambda_B(\omega) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\psi(t,\vartheta_{-t}\omega)B}.$$

Moreover, $\mathcal{A}(\omega) \subset K(\omega)$, $\mathcal{A}(\omega)$ is unique.

5.2. **Proof of Theorem 1.2.** At first, we give the definition of strong solutions and a result of the global existence of strong solutions to (3.4)-(3.11).

Definition 5.7. *If Z is defined in Section 3, $w_{t_0} \in V_1$, $T_{t_0} \in V_2$, $S_{t_0} \in V_3$, let $\mathcal{T} > t_0$ be a fixed time. For P-a.e. $\omega \in \mathbf{\Omega}$, (w, T, S) is called a strong solution of the system (3.4)-(3.11) on the time interval $[t_0, \mathcal{T}]$ if it satisfies (3.4)-(3.7) in a weak sense such that*

$$\begin{aligned}
&w \in L^\infty(t_0, \mathcal{T}; V_1) \cap L^2(t_0, \mathcal{T}; (H^2(\Omega))^2),\\
&T \in L^\infty(t_0, \mathcal{T}; V_2) \cap L^2(t_0, \mathcal{T}; H^2(\Omega)),\\
&S \in L^\infty(t_0, \mathcal{T}; V_3) \cap L^2(t_0, \mathcal{T}; H^2(\Omega)),\\
&\frac{\partial w}{\partial t} \in L^2(t_0, \mathcal{T}; (L^2(\Omega))^2),\ \frac{\partial T}{\partial t}, \frac{\partial S}{\partial t} \in L^2(t_0, \mathcal{T}; L^2(\Omega)).
\end{aligned}$$

Theorem 5.8. (Global existence of strong solutions of (3.4)-(3.11)) *If $Q_1, Q_2 \in H^1(\Omega)$ and $\mathcal{U}_{t_0} = (w_{t_0}, T_{t_0}, S_{t_0}) \in V$, then, for any given $\mathcal{T} > t_0$, there exists a unique strong solution U of the system (3.4)-(3.11) on the interval $[t_0, \mathcal{T}]$. Moreover, the strong solution $\mathcal{U}$ is dependent continuously on the initial data.*

Proof. According to the results in Section 4, in order to prove Theorem 5.8, we only need to make uniform *a priori* L^2 estimates about ∇w, ∇T, ∇S.

L^2 **estimates about** ∇w**.** Similarly to Lemma 4.1, by (4.4), we have

$$\begin{aligned}
&\Big|\int_\Omega \{[(w+Z)\cdot\nabla](w+Z)\}\cdot\triangle w\Big| \leq \varepsilon(|\triangle w|_2^2 + |\nabla w_z|_2^2) \qquad (5.1)\\
&+ c(|\bar{w}|_2^2\|\bar{w}\|_2^2 + |Z|_4^4 + |\tilde{w}|_4^8 + |Z|_4^8)(|\nabla w|_2^2 + |\nabla Z|_2^2) + c\|Z\|_2^2,\\
&\Big|\int_\Omega [\Phi(w+Z)(w_z+Z_z)]\cdot\triangle w\Big| \qquad (5.2)\\
&\leq \varepsilon|\triangle w|_2^2 + c\|Z\|_2^2 + c(|w_z|_2^2 + |\nabla w_z|_2^2 + |w_z|_2^2|\nabla w_z|_2^2)(|\nabla w|_2^2 + |\nabla Z|_2^2)\\
&+ c(|Z_z|_2^2 + |\nabla Z_z|_2^2 + |Z_z|_2^2|\nabla Z_z|_2^2)(|\nabla w|_2^2 + |\nabla Z|_2^2).
\end{aligned}$$

Taking the inner product of equation (3.4) with $-\triangle w$ in $L^2(\Omega)\times L^2(\Omega)$, by the Hölder inequality, $(fk\times w)\cdot\triangle w = 0$, $\int_\Omega \nabla p_{b_2}\cdot\triangle w = 0$, (5.1), (5.2), and, choosing ε small enough, we get

$$\frac{d|\nabla w|_2^2}{dt} + \frac{1}{Re_1}\int_\Omega |\triangle w|^2 + \frac{1}{Re_2}\int_\Omega |\nabla w_z|^2 \qquad (5.3)$$

$$\leq c(|w|_4^8 + |Z|_4^8 + |w_z|_2^2 + |\nabla w_z|_2^2 + |w_z|_2^2|\nabla w_z|_2^2 + |Z_z|_2^2 + |Z_z|_2^2|\nabla Z_z|_2^2 + |\nabla Z_z|_2^2)|\nabla w|_2^2 + c|\nabla T|_2^2 + c|\nabla S|_2^2 + c(|\bar{w}|_2^2\|\bar{w}\|_2^2 + |Z|_4^4 + |\tilde{w}|_4^8 + |Z|_4^8 + |w_z|_2^2 + |\nabla w_z|_2^2 + |w_z|_2^2|\nabla w_z|_2^2 + |Z_z|_2^2 + |\nabla Z_z|_2^2 + |Z_z|_2^2|\nabla Z_z|_2^2)|\nabla Z|_2^2 + c\|Z\|_2^2.$$

By the Gronwall inequality, Lemma 3.2, (4.7), (4.9), (4.16) and (4.21), for $\mathcal{T} > t_0$ given, there exists a constant $C_9(\mathcal{T}, \mathcal{U}_{t_0}, Q_1, Q_2, Z_{t_0})$ such that

$$\int_{t_0}^{\mathcal{T}} |\triangle w|_2^2 + |\nabla w(t)|_2^2 \leq C_9, \text{ for any } \ t \in [t_0, \mathcal{T}). \tag{5.4}$$

L^2 **estimates about** $\nabla T, \nabla S$**.** Similarly to Lemma 4.1, by (4.4),

$$\Big|\int_\Omega [(w+Z)\cdot\nabla]T(\triangle T)\Big| \leq c(|\bar{w}|_2^2\|\bar{w}\|_2^2 + |Z|_4^4 + |\tilde{w}|_4^8 + |Z|_4^8)|\nabla T|_2^2 + \varepsilon(|\triangle T|_2^2 + |\nabla T_z|_2^2), \tag{5.5}$$

$$\Big|\int_\Omega \Phi(w+Z)T_z(\triangle T)\Big| \leq \varepsilon(|\triangle T|_2^2 + |\nabla T_z|_2^2) + c[(|\nabla w|_2^4 + |\nabla w|_2^2|\triangle w|_2^2) + c(|\nabla Z|_2^4 + |\nabla Z|_2^2|\triangle Z|_2^2)]|T_z|_2^2. \tag{5.6}$$

Taking the inner product of (3.5) with $-\triangle T$ in $L^2(\Omega)$, by the Hölder and Young inequalities, (5.5), (5.6), and choosing ε small enough, we reach

$$\frac{d|\nabla T|_2^2}{dt} + \frac{1}{Rt_1}|\triangle T|_2^2 + \frac{1}{Rt_2}|\nabla T_z|_2^2 + \frac{\alpha_u}{Rt_2}|\nabla T|_{z=0}|_2^2 \leq c(|\bar{w}|_2^2\|\bar{w}\|_2^2 + |Z|_4^4 + |\tilde{w}|_4^8 + |Z|_4^8)|\nabla T|_2^2 + c[(|\nabla w|_2^4 + |\nabla w|_2^2|\triangle w|_2^2) + c(\|Z\|^4 + \|Z\|_2^4)]|T_z|_2^2 + c|Q_1|_2^2. \tag{5.7}$$

By the Gronwall inequality, Lemma 3.2, (4.7), (4.13), (4.16), (4.21), and (5.4), for $\mathcal{T} > t_0$ given, there exists a constant $C_{10}(\mathcal{T}, \mathcal{U}_{t_0}, Q_1, Q_2, Z_{t_0})$ such that

$$|\nabla T(t)|_2^2 \leq C_{10}, \text{ for any } \ t \in [t_0, \mathcal{T}). \tag{5.8}$$

Similarly, we get

$$\frac{d|\nabla S|_2^2}{dt} + \frac{1}{Rs_1}|\triangle S|_2^2 + \frac{1}{Rs_2}|\nabla S_z|_2^2 \leq c(|\bar{w}|_2^2\|\bar{w}\|_2^2 + |Z|_4^4 + |\tilde{w}|_4^8 + |Z|_4^8)|\nabla S|_2^2 + c[(|\nabla w|_2^4 + |\nabla w|_2^2|\triangle w|_2^2) + c(|\nabla Z|_2^4 + |\nabla Z|_2^2|\triangle Z|_2^2)]|S_z|_2^2 + c|Q_2|_2^2, \tag{5.9}$$

which implies that there exists $C_{11}(\mathcal{T},\mathcal{U}_{t_0},Q_1,Q_2,Z_{t_0})$ such that

$$|\nabla S(t)|_2^2 \le C_{11}, \text{ for any } \ t\in[t_0,\mathcal{T}). \tag{5.10}$$

Now, we construct a random dynamical system modeling the boundary-value problem of 3D stochastic primitive equations (3.4)-(3.10). Let $\mathbf{\Omega}=\{\omega:\omega\in C(\mathbb{R},l^2),\ \omega(0)=0\}$, $\mathcal{F}$ the Borel sigma-algebra induced by the compact open-topology of $\mathbf{\Omega}$, P a Wiener measure on $(\mathbf{\Omega},\mathcal{F})$. Write $(\omega_1(t,\omega),\ldots,\omega_k(t,\omega),\ldots)=\omega(t)$. Define

$$\vartheta_t\omega(s)=\omega(t+s)-\omega(t). \tag{5.11}$$

Then ϑ_t satisfies $\vartheta_{t+s}=\vartheta_t\circ\vartheta_s$ for all $t,\ s\in\mathbb{R}$, and ϑ_t is ergodic under P. By Theorem 5.8, let $\mathcal{U}(t,\omega)=\phi(t,s;\omega)\mathcal{U}_s$, where $\mathcal{U}(t,\omega)=(w(t,\omega),T(t,\omega),S(t,\omega))$ is a strong solution to (3.4)-(3.10) on $[s,t]$ with the initial data $\mathcal{U}(s)=\mathcal{U}_s=(w_s,T_s,S_s)$. Then, for $s\le r\le t$, we have $\phi(t,s;\omega)=\phi(t,r;\omega)\phi(r,s;\omega)$. Due to (5.11), for any $s,t\in\mathbb{R}^+$, $\mathcal{U}_0\in V$, we have $P-a.s.$

$$\phi(t+s,0;\omega)\mathcal{U}_0=\phi(t,0;\vartheta_s\omega)\phi(s,0;\omega)\mathcal{U}_0.$$

Define $\psi:\mathbb{R}^+\times\mathbf{\Omega}\times V\to V,\ \psi(t,\omega)\mathcal{U}_0=\phi(t,0;\omega)\mathcal{U}_0$. The following Proposition 5.10 will show that ψ is a continuous random dynamical system on V with weak topology over $(\mathbf{\Omega},\mathcal{F},P,\{\vartheta_t\}_{t\in\mathbb{R}})$ and models the random dynamical system generated by (3.4)-(3.10).

In the course of proving Theorem 1.2 with Theorem 5.6, the key step is the proof of the existence of a bounded absorbing set $K(\omega)$, which is compact in V with weak topology.

Proposition 5.9. (Existence of bounded absorbing sets for the random dynamical system (3.4)-(3.10)) *If $Q_1,\ Q_2\in H^1(\Omega)$ and $B_\rho=\{\mathcal{U}:\|\mathcal{U}\|\le\rho,\mathcal{U}\in V\}$, then there exist $r_0(\omega,\|Q_1\|_1,\|Q_2\|_1)$ and $t(\omega,\rho)\le -1$ such that, for any $t_0\le t(\omega,\rho),\ \mathcal{U}_{t_0}\in B_\rho$,*

$$\|\phi(0,t_0;\omega)\mathcal{U}_{t_0}\|\le r_0(\omega).$$

By Remark 5.5, *for any bounded set $B\subset V$, there exists $-t_0(B)>0$ big enough such that*

$$\psi(-s,\vartheta_s\omega)B=\phi(0,s;\omega)B\subset B_{r_0(\omega)},\ \textit{for any } s\le t_0.$$

Proof. From the ergodicity of the process Z with values in $D(A_1^{1+\gamma})$, we have

$$\begin{aligned}\lim_{s\to-\infty}\frac{1}{-s}\int_s^0(|Z(\tau)|_4^8+\|Z(\tau)\|^4+\|Z(\tau)\|^2\|Z(\tau)\|_2^2)d\tau\\ =E(|Z(0)|_4^8+\|Z(0)\|^4+\|Z(0)\|^2\|Z(0)\|_2^2).\end{aligned}$$

By the condition (C) in Subsection 3.2, i.e.,

$$-\lambda_1 + cE(|Z(0)|_4^8 + \|Z(0)\|^4 + \|Z(0)\|^2\|Z(0)\|_2^2) < 0,$$

there exists a positive constant c_0 such that

$$\lim_{s\to-\infty}\frac{1}{-s}\int_s^0[-\lambda_1 + c(|Z(\tau)|_4^8 + \|Z(\tau)\|^4 + \|Z(\tau)\|^2\|Z(\tau)\|_2^2)]d\tau \le -c_0,$$

which implies the existence of $s_0(\omega)$ such that $s < s_0(\omega)$ and

$$\int_s^0[-\lambda_1 + c(|Z(\tau)|_4^8 + \|Z(\tau)\|^4 + \|Z(\tau)\|^2\|Z(\tau)\|_2^2)]d\tau \le -\frac{c_0}{2}(-s). \quad (5.12)$$

By the definition of Z, (4.7) and a similar argument as in [7], $\|Z(\tau)\|^2 + |T(\tau)|_2^2 + |Z(\tau)|_2^2$ has at most polynomial growth as $\tau \to -\infty$. (4.11) and (5.12) imply the existence of $t_0(\rho,\omega)$ and an almost surely finite random variable $R_0(\omega)$ such that almost surely

$$|w(t)|_2^2 \le R_0(\omega), \text{ for any } s \le t \le 0, \quad (5.13)$$

where $s \le t_0$, (w,T,S) is the strong solution of (3.4)-(3.10) with initial data (w_s,T_s,S_s) and $(w_s,T_s,S_s) \in B_\rho$. Integrating (4.10) from t to $t+1$ for $s \le t \le -1$, by (5.13), we know there exists an almost surely finite random variable $R_1(\omega)$ such that

$$c_2\int_t^{t+1}[\int_\Omega(|\nabla w|^2 + |\frac{\partial w}{\partial z}|^2 + |w|^2)] \le R_1(\omega). \quad (5.14)$$

By the uniform Gronwall lemma, the fact that $|\tilde{w}|_3^3 \le |\tilde{w}|_2^{\frac{3}{2}}\|\tilde{w}\|_2^{\frac{3}{2}}$, (5.13), (5.14) and Lemma 3.2, we derive from the inequality before (4.15) that

$$|\tilde{w}(t+1)|_3^3 \le R_2(\omega), \quad (5.15)$$

where $R_2(\omega)$ is an almost surely finite random variable and $t \in [s,-1]$.

By the uniform Gronwall lemma, (5.13)-(5.15), Lemma 3.2 and the fact that $|\tilde{w}|_4^4 \le |\tilde{w}|_3^2\|\tilde{w}\|^2$, from the inequality before (4.16), we get

$$|\tilde{w}(t+2)|_4^4 \le R_3(\omega), \quad (5.16)$$

where $R_3(\omega)$ is an almost surely finite random variable and $t \in [s,-2]$.

Similarly, there exist almost surely finite random variables $R_4(\omega)$, $R_5(\omega)$, $R_6(\omega)$ and $R_7(\omega)$ such that

$$|w_z(t+3)|_2^2 \le R_4(\omega), \text{ for any } t \in [s,-3], \quad (5.17)$$

$$|T_z(t+4)|_2^2 + |S_z(t+4)|_2^2 \le R_5(\omega), \text{ for any } t \in [s,-4], \quad (5.18)$$

$$|\nabla w(t+5)|_2^2 \le R_6(\omega), \text{ for any } t \in [s,-5], \quad (5.19)$$

$$|\nabla T(t+6)|_2^2 + |\nabla S(t+6)|_2^2 \le R_7(\omega), \text{ for any } t \in [s, -6]. \tag{5.20}$$

From (4.6), (4.8), (5.13), (5.17)-(5.20), we know that there then exist $r_0(\omega)$ and $t(\omega, \rho) \le -1$ such that, for any $t_0 \le t(\omega, \rho)$, $\mathcal{U}_{t_0} \in B_\rho$,

$$\|\phi(0, t_0; \omega)\mathcal{U}_{t_0}\| \le r_0(\omega). \qquad \square$$

In order to prove Theorem 1.2, we need the following property about the family of mappings $\{\phi(t, s; \omega)\}_{t \ge s}$.

Proposition 5.10. *For any $t \ge s$, the mapping $\phi(t, s; \omega)$ is weakly continuous from V to V.*

Proof. Let $\{\mathcal{U}_n\}$ be a sequence in V such that $\mathcal{U}_n \to \mathcal{U}^1$ weakly in V. Then $\{\mathcal{U}_n\}$ is bounded in V. According to the *a priori* estimates in Section 4 and the proof of Theorem 5.8 and Proposition 5.9, we know that, for any $t \ge s$, $\{\phi(t, s; \omega)\mathcal{U}_n\}$ is bounded in V. So we extract a subsequence $\{\phi(t, s; \omega)\mathcal{U}_{n_k}\}$ such that $\phi(t, s; \omega)\mathcal{U}_{n_k} \to \mathcal{U}$ weakly in V. Since the embedding $V \hookrightarrow H$ is compact, $\mathcal{U}_{n_k} \to \mathcal{U}^1$ strongly in H. By (4.38)-(4.40), we obtain that $\phi(t, s; \omega)\mathcal{U}_{n_k} \to \phi(t, s; \omega)\mathcal{U}^1$ strongly in H. Thus $\mathcal{U} = \phi(t, s; \omega)\mathcal{U}^1$. Therefore, the sequence $\{\phi(t, s; \omega)\mathcal{U}_n\}$ has a subsequence $\{\phi(t, s; \omega)\mathcal{U}_{n_k}\}$ such that $\phi(t, s; \omega)\mathcal{U}_{n_k} \to \phi(t, s; \omega)\mathcal{U}^1$ weakly in V. $\square$

Proof of Theorem 1.2. Applying Theorem 5.6, by Proposition 5.9 and Proposition 5.10, we prove Theorem 1.2. $\square$

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