

WELL POSEDNESS AND THE GLOBAL ATTRACTOR OF SOME QUASI-LINEAR PARABOLIC EQUATIONS WITH NONLINEAR DYNAMIC BOUNDARY CONDITIONS

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Abstract. We consider a quasi-linear parabolic equation with nonlinear dynamic boundary conditions occurring as generalizations of semilinear reaction-diffusion equations with dynamic boundary conditions and various other phase-field models, such as the isothermal Allen-Cahn equation with dynamic boundary conditions. We thus formulate a class of initial and boundary-value problems whose global existence and uniqueness is proven by means of an appropriate Faedo-Galerkin approximation scheme developed for problems with dynamic boundary conditions. We analyze the asymptotic behavior of the solutions within the theory of infinite-dimensional dynamical systems. In particular, we demonstrate the existence of the global attractor.

1. INTRODUCTION

A well-known mathematical model which describes the behavior of the phases in the absence of temperature variations and mechanical stresses is given by the Allen-Cahn equation (see, e.g., [1]), which is given by

$$\partial_t \phi - \Delta \phi + f_1(\phi) = g_1(x), \text{ in } \Omega \times (0, +\infty), \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^3 . Here, ϕ is the order parameter, g_1 is an external force and f_1 is the derivative of a potential function F_1 , which accounts for the presence of different phases. For instance, F_1 can be a logarithmic potential which is usually approximated by a double well potential, i.e., $F_1(s) = (s^2 - 1)^2$.

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While equation (1.1) has been extensively studied in the literature, the following quasi-linear reaction-diffusion equation:

$$\partial_t \phi - \Delta_p \phi + f_1(\phi) = g_1(x), \text{ in } \Omega \times (0, +\infty), \quad (1.2)$$

has received less attention and it has only recently captured the attention of mathematicians (see, e.g., [2, 7, 8, 9, 11, 37, 45, 50] and the references therein). Above, in (1.2), the operator Δ_p denotes the p -Laplace operator, which is defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $p > 1$. It is obvious that (1.2) reduces to (1.1) when $p = 2$. Regarding suitable boundary conditions for equations (1.1) or (1.2), the usual ones considered in the literature are Dirichlet or Neumann. These standard boundary conditions together with (1.2) result in the fact that the following energy functional is decreasing:

$$\mathcal{E}_\Omega(\phi) := \int_\Omega \left[\frac{1}{p} |\nabla \phi|^p + F_1(\phi) - g_1(x) \phi \right] dx, \quad (1.3)$$

where, for simplicity, we take $F_1(r) = \int_0^r f_1(\zeta) d\zeta$. This can be easily checked by differentiating the expression in (1.3) with respect to t , then using (1.2) and the boundary conditions (as long as a smooth solution exists). The existence and the asymptotic behavior of strong solutions for (1.2), subject to Dirichlet or Neumann boundary conditions, were already studied by [4], [9], [34], [40] (and the references therein) in the reflexive Banach space framework. However, not much seems to be known about the existence, regularity and long term dynamics of (1.2), despite some classical results and recent results from [5, 6, 7, 14, 36] (and their references).

In this article, motivated by many applications in the theory of phase transitions phenomena, we plan to investigate (1.2), subject to dynamic boundary conditions. It has been recently discovered by physicists that, as far as the Allen-Cahn equation (and some Cahn-Hilliard equations, cf. e.g., [10, 24, 25, 27, 28, 29, 43] and their references) is concerned, for certain materials a dynamical interaction with the walls (i.e., with $\Gamma := \partial\Omega$) must be taken into account (see, e.g., [20, 21, 22, 46] and their references). In other words, the following energy functional should be added to (1.3):

$$\mathcal{E}_\Gamma(\phi) := \int_\Gamma [F_2(\phi) - g_2(x) \phi] dS, \quad (1.4)$$

to form a total energy functional

$$\mathcal{E}(\phi) := \mathcal{E}_\Omega(\phi) + \mathcal{E}_\Gamma(\phi), \quad (1.5)$$

where in (1.4) we neglect potential forces that account for surface diffusion; see however, Remark 3.13. Here, F_2 is a nonlinear function such that $F_2(r) = \int_0^r f_2(\zeta) d\zeta$. We need to derive the physically correct boundary condition for

(1.2) such that the function $\mathcal{E}(\phi)$ is decreasing for all time $t \geq 0$. Using integration by parts, we proceed by formally calculating the time derivative of $\mathcal{E}(\phi)$, for a smooth solution ϕ , as follows:

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\phi(t)) &= \int_{\Omega} [|\nabla\phi(t)|^{p-2}\nabla\phi(t) \cdot \nabla\partial_t\phi(t) + f_1(\phi(t))\partial_t\phi(t) - g_1\partial_t\phi(t)] dx \\ &\quad + \int_{\Gamma} [f_2(\phi(t))\partial_t\phi(t) - g_2\partial_t\phi(t)] dS \\ &= \int_{\Omega} (-\Delta_p\phi(t) + f_1(\phi(t)) - g_1)\partial_t\phi(t) dx \\ &\quad + \int_{\Gamma} (|\nabla\phi(t)|^{p-2}\partial_{\mathbf{n}}\phi(t) + f_2(\phi(t)) - g_2)\partial_t\phi(t) dS \\ &= - \int_{\Omega} |\partial_t\phi(t)|^2 dx - \int_{\Gamma} |\partial_t\phi(t)|^2 dS \leq 0, \text{ for all } t \geq 0. \end{aligned} \tag{1.6}$$

As a consequence, one deduces a dynamic boundary condition of the form

$$\partial_t\phi + |\nabla\phi|^{p-2}\partial_{\mathbf{n}}\phi + f_2(\phi) = g_2(x), \text{ on } \Gamma \times (0, +\infty). \tag{1.7}$$

Phenomenologically speaking, the boundary condition (1.7) means that the density of the binary mixture at the surface relaxes towards equilibrium with a rate proportional to the driving force given by the Frechét derivative of the free energy functional \mathcal{E}_{Γ} . The term $|\nabla\phi|^{p-2}\partial_{\mathbf{n}}\phi$ is due to the contribution coming from the variation of the free energy functional \mathcal{E}_{Ω} . Here, $\partial_{\mathbf{n}}\phi$ denotes the normal derivative of the function ϕ in the direction of the outer normal vector \mathbf{n} . A physical interpretation of the dynamic boundary condition (1.7), in the case $p = 2$, for linear heat equations was given in [32]. We point out that these types of boundary conditions are also used for modelling various physical situations including fluid diffusion within a (semi)permeable boundary (see, e.g., [12, 35, 38]) or several situations when the heat flow inside the domain Ω is subject to nonlinear heating or cooling at the boundary (see, e.g., [17, 18]).

This paper is concerned with the analysis of the system (1.2), (1.7), subject to the initial condition

$$\phi|_{t=0} = \phi_0. \tag{1.8}$$

Problems such as (1.2), (1.7)-(1.8) have already been partially examined in [10, 27, 28] within the theory of phase transitions, [13, 19, 23, 48, 49] and their references, always in the case $p = 2$, assuming that the nonlinearities f_1 and f_2 satisfy suitable assumptions. Such systems have also been investigated for general p in a number of papers (see, for instance, [15, 16, 47] and their references), where these contributions are mainly concerned with existence and uniqueness issues. Let us point out some of the main difficulties in

dealing with the boundary conditions (1.7). It is worth mentioning that the issue of well posedness for problem (1.2), (1.7)-(1.8), even in the case $p = 2$, seems to be hard in general since standard techniques based on the use of fractional power operators or monotone operator theory cannot be exploited. Indeed, our functions f_1 and f_2 are not monotone in general, even though the perturbation theory of maximal monotone operators can be employed to deal with globally bounded perturbations of monotone operators, but this always requires additional assumptions on the nonlinearities (see, e.g., [47]; cf. also [6] for standard boundary conditions). In particular, assuming $f_1 = g_1 = g_2 \equiv 0$, f_2 is a maximal monotone graph with $f_2(0) = 0$ and by taking slightly regular initial data ϕ_0 , the author of [47] proves that problem (1.1), (1.7)-(1.8) possesses a unique (strong) solution in a suitable sense (cf. [47, Theorem 2.1 and Corollary 3.3]). However, in view of applications to problems in phase separation and heat flow, it is desirable to work in less regular phase spaces (typically, in $L^2(\Omega)$). This represents the main goal of the paper. By showing that the principal part of equation (1.2) is monotone in an appropriate Hilbert space setting and by making general hypotheses on p , f_1 and f_2 (see Definition 2.1 below), so that the growth of the nonlinearities f_1, f_2 is dominated by a suitable version of the p -Laplacian, we first prove the well posedness of the problem (see Theorem 2.6). Then we want to establish the existence of the global attractor (see Corollary 3.11). The key step in the proofs is the construction of a new approximation scheme that can be employed when dealing with parabolic equations with dynamic boundary conditions. Such a scheme was already successfully applied to a coupled system of parabolic equations involving a Cahn-Hilliard equation (see, e.g., [25]). However, with respect to the standard results, our problem (1.2), (1.7)- (1.8) has different features arising from the nature of the boundary conditions which are now dynamic for ϕ . In particular, making use of the above scheme, we can rigorously now prove for the first time the existence of (unique) weak solutions to our system in a suitable Hilbert space, under minimal assumptions on the structural parameters of the problem (compare with [47]).

We outline the plan of the paper, as follows. In Section 2, we introduce some notation and preliminary facts, then we show the existence and uniqueness of solutions to our system (1.2), (1.7)- (1.8), using a suitable Faedo-Galerkin approximation scheme. Section 3 is devoted to the existence of a bounded absorbing set and, then, of the global attractor. Some regularity properties of the weak solutions and hence of the global attractor will be also derived.

2. WELL POSEDNESS

To establish the well posedness of problem (1.2), (1.7)-(1.8) we need to introduce some preliminary results. Using sufficiently strong global a priori estimates, we will be able to prove well posedness of our problem in a suitable Sobolev space setup. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with boundary $\Gamma := \partial\Omega$. For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $p \geq 1$, we define the Sobolev spaces $W^{k,p}(\Omega)$ and $W^{k,p}(\Gamma)$ to be respectively the completion of $C^k(\bar{\Omega})$ and $C^k(\Gamma)$, with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \left(\int_{\Omega} |\nabla^{\alpha} u|^p dx \right)^{1/p},$$

and

$$\|u\|_{W^{k,p}(\Gamma)} := \sum_{j=0}^k \left(\int_{\Gamma} |\nabla_{\Gamma}^j u|^p dS \right)^{1/p}.$$

Here, dx denotes the Lebesgue measure on Ω and dS denotes the natural surface measure on Γ . For $p \in (1, \infty)$ we define the fractional order Sobolev space

$$W^{1-\frac{1}{p},p}(\Gamma) := \left\{ u \in L^p(\Gamma) : \int_{\Gamma} \int_{\Gamma} \left(\frac{|u(x) - u(y)|}{|x - y|^{1-\frac{1}{p}}} \right)^p \frac{1}{|x - y|^{N-1}} dS_x dS_y < \infty \right\}.$$

Since Ω has a smooth boundary, it is well known that

$$W^{1,p}(\Omega) \hookrightarrow L^{p_s}(\Omega), \tag{2.1}$$

where $p_s := Np/(N - p)$ if $p < N$ and $1 \leq p_s < \infty$ if $p = N$. If $p > N$, one has that $W^{1,p}(\Omega)$ is continuously embedded into $C^{0,1-\frac{N}{p}}(\bar{\Omega})$. Moreover, one has the following continuous embedding:

$$W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow L^{q_s}(\Gamma), \tag{2.2}$$

where $q_s = (N - 1)p/(N - p)$ if $p < N$ and $1 \leq q_s < \infty$ if $N = p$. From now on, we denote by $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|_{W^{k,q}(\Gamma)}$ the norms on $W^{k,p}(\Omega)$ and $W^{k,q}(\Gamma)$, respectively. Also, $\langle \cdot, \cdot \rangle_s$ and $\langle \cdot, \cdot \rangle_{s,\Gamma}$ stand for the usual scalar product in $L^s(\Omega)$ and $L^s(\Gamma)$, respectively.

The natural space for problem (1.2), (1.7)-(1.8) turns out to be

$$\mathbb{X}^s := L^s(\Omega) \oplus L^s(\Gamma) = \{F = (f, g) : f \in L^s(\Omega), g \in L^s(\Gamma)\}, \quad s \in [1, +\infty],$$

endowed with the norm defined through

$$\|F\|_{\mathbb{X}^s}^s = \int_{\Omega} |f(x)|^s dx + \int_{\Gamma} |g(x)|^s dS_x, \tag{2.3}$$

if $s \in [1, \infty)$, and

$$\|F\|_{\mathbb{X}^\infty} := \max\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Gamma)}\}.$$

Identifying each function $u \in W^{1,p}(\Omega)$ with the vector $U := (u, u|_\Gamma)$, it is easy to see that $W^{1,p}(\Omega)$ is a dense subspace of \mathbb{X}^s for $s \in [1, \infty)$. Moreover, one has that $\mathbb{X}^s = L^s(\bar{\Omega}, d\mu)$, $s \in [1, +\infty]$, where the measure $d\mu = dx|_\Omega \oplus dS|_\Gamma$ on $\bar{\Omega}$ is defined for any measurable set $A \subset \bar{\Omega}$ by $\mu(A) = |A \cap \Omega| + S(A \cap \Gamma)$. Identifying each function $\theta \in C(\bar{\Omega})$ with the vector $\Theta = (\theta|_\Omega, \theta|_\Gamma) \in C(\bar{\Omega}) \times C(\Gamma)$, one also has that $C(\bar{\Omega})$ is a dense subspace of \mathbb{X}^s for every $s \in [1, \infty)$ and a closed subspace of \mathbb{X}^∞ . Next, for each $p > 1$, we let $\mathbb{V}^p = \{U := (u, u|_\Gamma) : u \in W^{1,p}(\Omega)\}$ and endow it with the norm $\|\cdot\|_{\mathbb{V}^p}$ given by

$$\|(u, u|_\Gamma)\|_{\mathbb{V}^p} = \|u\|_{W^{1,p}(\Omega)} + \|u|_\Gamma\|_{W^{1-1/p,p}(\Gamma)}.$$

It is easy to see that we can identify \mathbb{V}^p with $W^{1,p}(\Omega) \oplus W^{1-1/p,p}(\Gamma)$ under this norm. Moreover, since $W^{1,p}(\Omega) \hookrightarrow W^{1-1/p,p}(\Gamma)$, one has that the norms on \mathbb{V}^p and $W^{1,p}(\Omega)$ are equivalent. It is also immediate that \mathbb{V}^p is compactly embedded into \mathbb{X}^2 , for any $p > 2N/(N+2)$. For this, note that $\bar{\Omega}$ is a smooth, compact N -dimensional Riemannian manifold whose boundary Γ is an $(N-1)$ -dimensional Riemannian manifold (see, e.g., [33, Chapter 2]). Other authors have worked in a similar framework but only if $p = 2$. For instance, [3] deals with a general class of inhomogeneous parabolic initial and boundary-value problems with linear dynamic boundary conditions. A simpler and more general approach based on semigroup theory can be found in [17], while the abstract approach used in [41] allows us to consider dynamic boundary conditions involving surface diffusion represented by a Laplace-Beltrami operator (see also Remark 3.13).

Finally, we need to specify more rigorously which class of problems we want to solve. Following [24, 29, 39], it is more convenient, however, to introduce the unknown function $\psi(t) := \phi(t)|_\Gamma$, defined on the boundary Γ , and to interpret (1.7) as an additional parabolic equation, acting now on the boundary Γ . Throughout the remainder of this article, for functions depending on $x \in \bar{\Omega}$ and time t , we will sometimes omit the dependence on x in our notation. We formulate the following.

Problem (P). Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with a smooth boundary $\Gamma := \partial\Omega$ (e.g., of class C^2). Let $p \in [p_0, +\infty) \cap (1, +\infty)$, where $p_0 := 2N/(N+2)$. For any given pair of initial data $(\phi_0, \psi_0) \in \mathbb{X}^2$, find

$(\phi(t), \psi(t))$ with

$$\begin{cases} (\phi, \psi) \in C([0, +\infty); \mathbb{X}^2) \cap L^\infty((0, +\infty); \mathbb{V}^p), \\ (\phi, \psi) \in W_{\text{loc}}^{1,2}((0, \infty); \mathbb{X}^2), \quad \phi \in L_{\text{loc}}^p([0, +\infty); W^{1,p}(\Omega)), \\ \psi \in L_{\text{loc}}^p([0, +\infty); W^{1-1/p,p}(\Gamma)), \end{cases} \quad (2.4)$$

such that $(\phi(0), \psi(0)) = (\phi_0, \psi_0)$, and for almost all $t \geq 0$, $(\phi(t), \psi(t))$ satisfies the following partial differential equations:

$$\begin{cases} \partial_t \phi - \Delta_p \phi + f_1(\phi) = g_1, \text{ in } \Omega \times (0, +\infty), \\ \partial_t \psi + |\nabla \psi|^{p-2} \partial_{\mathbf{n}} \phi + f_2(\psi) = g_2, \text{ on } \Gamma \times (0, +\infty), \\ \psi(t) := \phi(t)|_{\Gamma}. \end{cases} \quad (2.5)$$

Regarding the assumptions on the nonlinearities $f_i, i = 1, 2$, we assume that $f_i \in C^1(\mathbb{R}), i = 1, 2$, satisfy the conditions

$$\liminf_{|y| \rightarrow \infty} f'_i(y) > 0, \quad i = 1, 2, \quad (2.6)$$

and we also require that

$$\begin{cases} \eta_1 |y|^{r_1} - \nu'_1 \leq f_1(y) y \leq \nu_1 |y|^{r_1} + \nu'_1, \\ -\nu'_2 \leq f_2(y) y \leq \nu_2 |y|^{r_2} + \nu'_2, \end{cases} \quad (2.7)$$

for some $\eta_1, \nu_i > 0, \nu'_i \geq 0$ and $r_1, r_2 > 1$. It is easy to realize that the derivative of the typical double well potential F_1 satisfies both conditions (2.6) and (2.7).

Now, let us define the admissible set of nonlinear functions $f_i, i = 1, 2$, for our problem **(P)**.

Definition 2.1. We say that $f_1 \in \mathcal{N}_1(p, r_1)$, if the following conditions on f_1 are satisfied: f_1 fulfills assumptions (2.6)-(2.7)₁ with

$$r_1 \in \begin{cases} (p, pN/(N-p)], & \text{if } p \in (p_0, N), \\ (p, +\infty), & \text{if } p = N, \\ +\infty, & \text{if } p > N. \end{cases}$$

Analogously, we say that $f_2 \in \mathcal{N}_2(p, r_2)$, if the following conditions on f_2 are satisfied: f_2 fulfills assumptions (2.6)-(2.7)₂ with

$$r_2 \in \begin{cases} [1, (N-1)p/(N-p)], & \text{if } p \in (p_0, N), \\ [1, +\infty), & \text{if } p = N, \\ +\infty, & \text{if } p > N \text{ or } N = 1. \end{cases}$$

If $r_1, r_2 = +\infty$, then instead of the estimates (2.7) for f_1 and f_2 above, we actually assume $f_i(y), i = 1, 2$, to be bounded if $|y| \leq y_0$, for all $y_0 \in \mathbb{R}_+$,

i. e.,

$$\sup_{|y| \leq y_0} |f_i(y)| < \infty, \quad \forall y_0 \in \mathbb{R}_+,$$

and

$$f_1(y)y \geq \eta_1 |y|^p - \nu'_1, \quad f_2(y)y \geq -\nu'_2, \quad \forall y \in \mathbb{R}.$$

The following is a direct consequence of Definition 2.1.

Remark 2.2. Suppose that $p \in [p_0, +\infty) \cap (1, +\infty)$ and $(f_1, f_2) \in \mathcal{N}_1(p, r_1) \times \mathcal{N}_2(p, r_2)$. Let q, r'_1 and r'_2 be the conjugate exponents of p, r_1 and r_2 , respectively. By known Sobolev inequalities [33, Chapter 2] (see (2.1)-(2.2)), we have the following useful continuous inclusions: $\mathbb{V}^p \hookrightarrow W^{1,p}(\Omega) \hookrightarrow L^{r_1}(\Omega) \hookrightarrow L^p(\Omega)$, $\mathbb{V}^p \hookrightarrow W^{1-1/p,p}(\Gamma) \hookrightarrow L^{r_2}(\Gamma)$, and $L^q(\Omega) \hookrightarrow L^{r'_1}(\Omega) \hookrightarrow (W^{1,p}(\Omega))^* \hookrightarrow (\mathbb{V}^p)^*$, where $(W^{1,p}(\Omega))^*$ and $(\mathbb{V}^p)^*$ denote the dual of $W^{1,p}(\Omega)$ and \mathbb{V}^p , respectively. Note that $(W^{1,p}(\Omega))^*$ is a subspace of $W^{-1,q}(\Omega)$, where $W^{-1,q}(\Omega)$ denotes the dual of the Sobolev space $W_0^{1,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)}$.

Here, we introduce the rigorous variational (weak) formulation of **(P)**.

Definition 2.3. The pair $\Phi(t) = (\phi(t), \psi(t))$ is said to be a solution of (2.5), if it has the regularity (2.4),

$$\partial_t \Phi \in L^s_{loc}([0, +\infty); (\mathbb{V}^p)^*), \quad s = \min(q, r'_1, r'_2), \quad (2.8)$$

$\psi(t) := \phi(t)|_\Gamma$ almost everywhere in $(0, +\infty)$, and for all $\sigma \in W^{1,p}(\Omega)$ (hence, for all $\Xi = (\sigma, \sigma|_\Gamma) \in \mathbb{V}^p$) and for almost every $t \in (0, +\infty)$, the following relation holds:

$$\begin{aligned} & \langle \partial_t \Phi, \Xi \rangle_{\mathbb{X}^2} + \left\langle |\nabla \phi|^{p-2} \nabla \phi, \nabla \sigma \right\rangle_2 + \langle f_1(\phi), \sigma \rangle_2 + \langle f_2(\psi), \sigma|_\Gamma \rangle_{2,\Gamma} \\ & = \langle g_1, \sigma \rangle_2 + \langle g_2, \sigma|_\Gamma \rangle_{2,\Gamma}. \end{aligned} \quad (2.9)$$

Moreover, we have, in the space \mathbb{X}^2 , $\Phi|_{t=0} = (\phi_0, \psi_0)$, where $\phi|_{t=0} = \phi_0$ almost everywhere in Ω , $\psi|_{t=0} = \psi_0$ almost everywhere in Γ .

Remark 2.4. Note that, in the above framework of (2.9), it is not clear whether we can identify $\partial_t \Phi$ as a pair of functionals related to the time derivative of ϕ and ψ , respectively. More precisely, we are not able to write $\partial_t \Phi = (\partial_t \phi, \partial_t \psi)$, because the regularity in (2.8) does not necessarily imply that $\partial_t \Phi|_\Omega = \partial_t \phi$ and $\partial_t \Phi|_\Gamma = \partial_t \psi$ (see, however, Remark 2.9).

Before showing the existence of solutions to problem **(P)**, we begin by deriving several estimates. In a first step, we obtain a global a priori estimate for the solutions in the space \mathbb{X}^2 .

Proposition 2.5. *Assume that $f_i \in \mathcal{N}_i(p, r_i)$, $i = 1, 2$, and $g_1 \in L^{r'_1}(\Omega)$, $g_2 \in L^{r'_2}(\Gamma)$, where r'_i are conjugate to r_i . Let $\Phi(t) = (\phi(t), \psi(t))$ be a smooth solution of (2.5) (or, more precisely the weak formulation (2.9)). Then the following estimate holds:*

$$\begin{aligned} & \|\Phi(t)\|_{\mathbb{X}^2}^2 + \int_0^t \left(\|\phi(s)\|_{W^{1,p}(\Omega)}^p + \|\psi(s)\|_{W^{1-1/p,p}(\Gamma)}^p \right) ds \quad (2.10) \\ & \leq c \|\Phi(0)\|_{\mathbb{X}^2}^2 + ct \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \end{aligned}$$

for all $t \geq 0$, where $q := p/(p-1)$ and the positive constant c is independent of time and initial data.

Proof. We prove (2.10) when $p \leq N$. The case $p > N$ follows easily with minor modifications. The following estimates will be deduced by a formal argument. This can be justified by means of the approximation procedure devised in the proof of Theorem 2.6 below (where, by the way, we have, at least, $\partial_t \Phi \in L^2_{\text{loc}}([0, +\infty); \mathbb{X}^2)$, in which case $\partial_t \Phi$ can be viewed as a pair $(\partial_t \phi, \partial_t \psi)$, because of this higher regularity). Choose $\Xi = \Phi(t)$, $\sigma = \phi(t)$ and $\sigma|_{\Gamma} = \psi(t)$ in (2.9). After standard transformations, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Phi(t)\|_{\mathbb{X}^2}^2 + \|\nabla \phi(t)\|_{L^p(\Omega)}^p + \langle f_1(\phi(t)), \phi(t) \rangle_2 + \langle f_2(\psi(t)), \psi(t) \rangle_{2,\Gamma} \\ & = \langle g_1, \phi(t) \rangle_2 + \langle g_2, \psi(t) \rangle_{2,\Gamma}. \quad (2.11) \end{aligned}$$

Using now the assumptions (cf. Definition 2.1) on the nonlinearities f_i in order to control the last two terms on the left-hand side of (2.11), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Phi(t)\|_{\mathbb{X}^2}^2 + \|\nabla \phi(t)\|_{L^p(\Omega)}^p + c \|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} \quad (2.12) \\ & \leq \langle g_1, \phi(t) \rangle_2 + \langle g_2, \psi(t) \rangle_{2,\Gamma} + c. \end{aligned}$$

Note that c is a positive constant that is independent of time and initial data, which only depends on the other structural parameters of the problem, that is, $|\Omega|$, $S(\Gamma)$, ν_i , η_1 , ν'_i and p . From now on c will denote a positive constant of this kind. Such a constant may vary even from line to line. Applying now suitable Hölder and Young inequalities on the right-hand side of (2.12) and recalling the sequence of continuous embeddings $W^{1,p}(\Omega) \hookrightarrow L^{r_1}(\Omega) \hookrightarrow L^p(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^{r_2}(\Gamma)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Phi(t)\|_{\mathbb{X}^2}^2 + \frac{c}{2} \left(\|\phi(t)\|_{W^{1,p}(\Omega)}^p + \|\psi(t)\|_{W^{1-1/p,p}(\Gamma)}^p \right) + \frac{c}{2} \|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} \\ & \leq c \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \quad (2.13) \end{aligned}$$

for almost every $t \geq 0$. Integrating (2.13) from 0 to t yields (2.10) and this completes the proof. \square

Now, we are ready to state and prove the main result of this section.

Theorem 2.6. *Let the assumptions of Proposition 2.5 be satisfied. Let $T > 0$ be fixed. For any initial data $(\phi_0, \psi_0) \in \mathbb{X}^2$, problem **(P)** has a unique weak solution $(\phi(t), \psi(t)) \in C([0, T]; \mathbb{X}^2)$ which satisfies (2.10). In addition to the regularity stated in (2.4), we also have that*

$$\begin{aligned} \phi &\in L^{r_1}([0, T] \times \Omega), \text{ if } p \leq N, \\ \phi &\in L^p([0, T]; C(\overline{\Omega})), \text{ if } p > N. \end{aligned}$$

Furthermore, this problem defines a (nonlinear) continuous semigroup \mathcal{S}_t on the phase space \mathbb{X}^2 , $\mathcal{S}_t : \mathbb{X}^2 \rightarrow \mathbb{X}^2$, given by

$$\mathcal{S}_t(\phi_0, \psi_0) = (\phi(t), \psi(t)), \quad (2.14)$$

where the function $(\phi(t), \psi(t))$ solves (2.9).

Proof. We need to verify the existence of solutions. We make use of a Faedo–Galerkin approximating scheme to project on a finite-dimensional space (using the scheme developed in [25] for the case $p = 2$), where we can locally solve a Cauchy problem for a system of ordinary differential equations (\mathbf{P}_n) having a local solution. Then, thanks to the a priori estimates (2.10), the local solution will become a global one. The a priori estimates will be also used to extract a subsequence which converges weakly to a certain vector-valued function, a candidate to be the solution to the problem. Due to the presence of the nonlinearities, before passing to the limit, we need some extra (strong) convergence, in order to prove that the mentioned weak limit is indeed the solution. To apply such arguments to a problem with a dynamic boundary condition, we need to construct suitable self-adjoint operators acting on \mathbb{X}^2 . Such constructions have already appeared in [26]. To this end, let us consider the operator \mathcal{B}_0 given by

$$\mathcal{B}_0(\phi, \phi|_\Gamma) = (-\Delta\phi, -(\Delta\phi)|_\Gamma), \quad (2.15)$$

for functions $\phi \in C^2(\overline{\Omega})$ that satisfy the Wentzell boundary conditions

$$\Delta\phi + \partial_{\mathbf{n}}\phi + \phi|_\Gamma = 0 \text{ on } \Gamma. \quad (2.16)$$

Here $(\Delta\phi)|_\Gamma$ stands for the restriction of the function $\Delta\phi$ on the boundary Γ . The domain of \mathcal{B}_0 is $D(\mathcal{B}_0) = \{\Theta = (\phi, \phi|_\Gamma) : \phi \in C^2(\overline{\Omega}), (2.16) \text{ holds}\}$. Now consider a function $f \in C(\overline{\Omega})$ and let $F = (f_1, f_2)$ with $f_1 := f|_\Omega$ and

$f_2 := f|_\Gamma$. By the equality $\mathcal{B}_0\Theta = F$, we mean the following boundary-value problem:

$$\begin{cases} -\Delta\phi = f_1 & \text{in } \Omega, \\ -\Delta\phi = f_2 & \text{on } \Gamma. \end{cases} \tag{2.17}$$

Using the Wentzell boundary conditions (2.16) and replacing f_2 by $f|_\Gamma$, the condition on Γ in (2.17) becomes

$$\partial_n\phi + \phi|_\Gamma = f|_\Gamma \text{ on } \Gamma. \tag{2.18}$$

Let \mathcal{B} be the closure of \mathcal{B}_0 in \mathbb{X}^2 . Then, \mathcal{B} is the self-adjoint operator associated with the closed symmetric form $\mathcal{A} : \mathbb{V}^2 \times \mathbb{V}^2 \rightarrow \mathbb{R}$ on \mathbb{X}^2 , defined by

$$\mathcal{A}(\Theta, \Phi) = \int_\Omega \nabla\phi\nabla\varphi dx + \int_\Gamma \phi\varphi dS,$$

where $\Theta := (\phi, \phi|_\Gamma)$ and $\Phi := (\varphi, \varphi|_\Gamma)$. Since Ω is assumed to be smooth, using standard elliptic regularity results, we also have the following characterization of \mathcal{B} :

$$D(\mathcal{B}) = \{ \Theta := (\phi, \phi|_\Gamma) : \phi \in W^{2,2}(\Omega) \}, \quad \mathcal{B}\Theta = (-\Delta\phi, \partial_n\phi + \phi|_\Gamma).$$

Note that since Γ is smooth, if $\phi \in W^{2,2}(\Omega)$ then $\phi|_\Gamma \in H^{3/2}(\Gamma) = W^{\frac{3}{2},2}(\Gamma)$. Moreover, since the embedding $\mathbb{V}^2 \hookrightarrow \mathbb{X}^2$ is compact, it follows that \mathcal{B} has a compact resolvent.

Therefore, for $i \in \mathbb{N}$, we take a complete system of eigenfunctions $\Phi_i = (\phi_i, \psi_i)$ of the problem $\mathcal{B}\Phi_i = \lambda_i\Phi_i$ in \mathbb{X}^2 with $\Phi_i \in D(\mathcal{B}) \cap C^2(\overline{\Omega})$. According to the general spectral theory, the eigenvalues $\lambda_i \in (0, +\infty)$ can be increasingly ordered and counted according to their multiplicities in order to form a real divergent sequence. Moreover, the respective eigenvectors turn out to form an orthogonal basis both in \mathbb{V}^2 and \mathbb{X}^2 and may be assumed to be normalized in the norm of \mathbb{X}^2 . At this point, we set the spaces

$$K_n = \text{span} \{ \Phi_1, \Phi_2, \dots, \Phi_n \}, \quad K_\infty = \bigcup_{n=1}^\infty K_n.$$

Clearly, K_∞ is a dense subspace of both \mathbb{V}^2 and $D(\mathcal{B})$. Let $\text{Pr}_n : \mathbb{X}^2 \rightarrow K_n$ be an orthogonal projection. For any $n \in \mathbb{N}$, we look for functions of the form

$$\Phi = \Phi_n = \sum_{i=1}^n d_i(t) \Phi_i \tag{2.19}$$

solving the approximate problem (\mathbf{P}_n) that we will introduce below. Note that in the definition of Φ_n , $d_i(t)$ are sought to be suitably regular real-valued functions (i.e., $d_i \in C^2(0, T)$, $i = 1, \dots, n$). Note that, from (2.19), we also have

$$\phi = \phi_n = \sum_{i=1}^n d_i(t) \phi_i, \quad \psi = \psi_n = \sum_{i=1}^n d_i(t) \psi_i.$$

As approximations for the initial data $\Phi_0 = (\phi_0, \psi_0)$, $\psi_0 = \phi_0|_\Gamma$, we take $\Phi_{n0} \in \mathbb{V}^p$, such that

$$\lim_{n \rightarrow \infty} \Phi_{n0} = \Phi_0 \text{ in } \mathbb{X}^2,$$

since \mathbb{V}^p is dense in \mathbb{X}^2 . The problem that we must solve is given, for any $n \geq 1$, by (\mathbf{P}_n) : find Φ_n such that, for all $\bar{\Phi} = (\bar{\phi}, \bar{\psi}) \in K_n$,

$$\langle \partial_t \Phi_n, \bar{\Phi} \rangle_{\mathbb{X}^2} + \langle \text{Pr}_n \mathcal{B}_p \Phi_n, \bar{\Phi} \rangle_{\mathbb{X}^2} + \langle \text{Pr}_n \mathcal{F}(\Phi_n), \bar{\Phi} \rangle_{\mathbb{X}^2} = \langle \text{Pr}_n \mathcal{G}, \bar{\Phi} \rangle_{\mathbb{X}^2}, \tag{2.20}$$

and

$$\langle \Phi_n(0), \bar{\Phi} \rangle_{\mathbb{X}^2} = \langle \Phi_{n0}, \bar{\Phi} \rangle_{\mathbb{X}^2}. \tag{2.21}$$

Here, $\mathcal{G} = (g_1, g_2)$ and the operators $\mathcal{B}_p : D(\mathcal{B}_p) \rightarrow \mathbb{X}^2$, $\mathcal{F} : D(\mathcal{F}) \subset \mathbb{X}^2 \rightarrow \mathbb{X}^2$ are given, formally, by

$$\begin{aligned} \mathcal{B}_p \begin{pmatrix} \phi \\ \psi \end{pmatrix} &= \begin{pmatrix} -\Delta_p \phi + |\phi|^{p-2} \phi \\ |\nabla \psi|^{p-2} \partial_n \phi \end{pmatrix}, \\ \mathcal{F} \begin{pmatrix} \phi \\ \psi \end{pmatrix} &= \begin{pmatrix} f_1(\phi) - |\phi|^{p-2} \phi \\ f_2(\psi) \end{pmatrix}. \end{aligned} \tag{2.22}$$

We give the complete definition and some useful results about \mathcal{B}_p which are taken from [30, Proposition 3.1 and Remark 3.2] and will be used in the sequel. Consider the functional $\mathcal{J}_p : \mathbb{X}^2 \rightarrow [0, \infty]$ defined by

$$\mathcal{J}_p(\Phi) = \begin{cases} \frac{1}{p} \int_{\Omega} (|\nabla \phi|^p + |\phi|^p) \, dx, & \text{if } \Phi = (\phi, \phi|_\Gamma) \in D(\mathcal{J}_p), \\ +\infty, & \text{if } \Phi \in \mathbb{X}^2 \setminus D(\mathcal{J}_p), \end{cases}$$

where the effective domain is given by $D(\mathcal{J}_p) = \mathbb{V}^p$. Note that, since $p > 2N/(N + 2)$, one obtains that, in any case, $D(\mathcal{J}_p)$ is always a subspace of \mathbb{X}^2 . The functional \mathcal{J}_p is proper, convex and lower semicontinuous on \mathbb{X}^2 (cf. [30]). Using standard arguments of monotone operator theory (see [30]), it follows that the nonlinear operator \mathcal{B}_p coincides with the subdifferential of \mathcal{J}_p on \mathbb{X}^2 . More precisely, $\mathcal{B}_p \Phi = \partial \mathcal{J}_p(\Phi)$, for $\Phi \in D(\mathcal{B}_p) = D(\partial \mathcal{J}_p)$. In particular, one has that the operator \mathcal{B}_p is maximal monotone and coercive.

In order to prove the existence of at least one solution to (2.20), we aim to apply the standard existence theorems for ODE's. For this purpose, if n is

fixed, let us choose $\bar{\Phi} = \Phi_j$, $1 \leq j \leq n$, and substitute the expressions (2.19) to the unknowns Φ_n in (2.20). Using the fact that $f_k \in C^1(\mathbb{R})$, $k = 1, 2$, then applying Cauchy's theorem for ODE's, we find a small time $t_n \in (0, T)$ such that $d_i \in C^2(0, t_n)$, $1 \leq i \leq n$, and (2.20) holds (in the classical sense) for all $t \in [0, t_n]$. This gives the desired local solution Φ_n to our problem (2.20). Now, based on the uniform a priori estimates (2.10) with respect to t , derived for the solution Φ of problem **(P)** (see (2.9)), we obtain, in particular, that any local solution is actually a global solution that is defined on the whole interval $[0, T]$. It remains then to pass to the limit as $n \rightarrow \infty$ and obtain a weak solution which is unique by Lemma 2.7. According to the a priori estimates derived in the beginning of Section 2, we learn that the sequence $\Phi_n(t)$ is uniformly bounded in norm in various Banach spaces by a positive constant C that depends only on $|\Omega|$, $|\Gamma|$, T , p , r_i , ϕ_0 , ψ_0 , but is independent of n and t . In detail, we have

$$\begin{cases} \|\Phi_n(t)\|_{L^\infty([0,T];\mathbb{X}^2)} & \leq C, \\ \|\Phi_n(t)\|_{L^p([0,T];\mathbb{V}^p)} & \leq C, \\ \|\phi_n(t)\|_{L^{r_1}([0,T];L^{r_1}(\Omega))} & \leq C, \text{ if } p \leq N, \end{cases} \tag{2.23}$$

and

$$\|\phi_n(t)\|_{L^p([0,T];C(\bar{\Omega}))} \leq C, \text{ if } p > N.$$

On account of the uniform bounds in (2.23), we can get further uniform bounds for $\text{Pr}_n \mathcal{B}_p \Phi_n$ and $\text{Pr}_n \mathcal{F}(\Phi_n)$. Indeed, by the boundedness of the projection Pr_n , we immediately deduce that

$$\|\text{Pr}_n \mathcal{B}_p \Phi_n(t)\|_{(\mathbb{V}^p)^*} = \sup_{\|\bar{\Phi}\|_{\mathbb{X}^2} \leq 1} |\langle \mathcal{B}_p \Phi_n(t), \text{Pr}_n \bar{\Phi} \rangle_{\mathbb{X}^2}| \leq \|\Phi_n(t)\|_{\mathbb{V}^p}^{p-1},$$

so that

$$\|\text{Pr}_n \mathcal{B}_p \Phi_n(t)\|_{L^q([0,T];(\mathbb{V}^p)^*)} \leq C, \tag{2.24}$$

where we may identify the dual space of \mathbb{V}^p , $(\mathbb{V}^p)^*$ with a closed subspace of $W^{-1,q}(\Omega) \oplus W^{1/p-1,q}(\Gamma)$, where $q = p/(p-1)$ is the conjugate exponent of p .

Let us now focus on the case $p \leq N$. On account of assumptions (2.7) and the last two uniform bounds of (2.23), it is not difficult to show that $f_1(\phi_n)$, $f_2(\psi_n)$ are uniformly bounded (with respect to n) in $L^{r'_1}([0, T]; L^{r'_1}(\Omega))$ and $L^{r'_2}([0, T]; L^{r'_2}(\Gamma))$, respectively, where r'_i are conjugate to r_i . Since $L^{r'_i}([0, T]; L^{r'_i}) \hookrightarrow L^{r'_i}([0, T]; (\mathbb{V}^p)^*)$, it follows that $f_1(\phi_n)$ and $f_2(\psi_n)$ are uniformly bounded in $L^{r'_1}([0, T]; (\mathbb{V}^p)^*)$ and $L^{r'_2}([0, T]; (\mathbb{V}^p)^*)$, respectively.

When $p > N$, the same is true except that we need to take $r'_i = q$, for all $i = 1, 2$. Finally, we need a uniform bound on the derivatives. By comparison, from equation (2.20), we have that $\partial_t \Phi_n$ is uniformly bounded in $L^s([0, T]; (\mathbb{V}^p)^*)$, with $s = \min(q, r'_1, r'_2) > 1$. It is now obvious that

$$\partial_t \Phi_n + \text{Pr}_n \mathcal{B}_p \Phi_n + \text{Pr}_n \mathcal{F}(\Phi_n) = \text{Pr}_n \mathcal{G}$$

holds as an equality in $L^s([0, T]; (\mathbb{V}^p)^*)$.

From this point on, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabelled. Thus, we observe that weak and weak star compactness results applied to the above sequences $\Phi_n = (\phi_n, \psi_n)$ entail that there exists a function $\Phi = (\phi, \psi)$ such that, as $n \rightarrow +\infty$, the following properties hold:

$$\Phi_n \rightarrow \Phi \text{ weakly star in } L^\infty([0, T]; \mathbb{X}^2), \quad (2.25)$$

$$\Phi_n \rightarrow \Phi \text{ weakly in } L^p([0, T]; \mathbb{V}^p),$$

$$(\partial_t \phi_n, \partial_t \psi_n) \rightarrow \partial_t \Phi \text{ weakly in } L^s([0, T]; (\mathbb{V}^p)^*). \quad (2.26)$$

Using the fact that $\mathbb{V}^p \hookrightarrow \mathbb{X}^2 = (\mathbb{X}^2)^* \hookrightarrow (\mathbb{V}^p)^*$, then, standard interpolation and compact embedding results for vector-valued functions (see, e.g., [24, Lemma 8]), ensure that

$$\Phi_n \rightarrow \Phi \text{ strongly in } C([0, T]; ((\mathbb{V}^p)^*, \mathbb{X}^2)_\kappa), \quad (2.27)$$

$$\Phi_n \rightarrow \Phi \text{ strongly in } L^p([0, T]; \mathbb{X}^2), \quad (2.28)$$

where $(\cdot, \cdot)_\kappa$ denotes the interpolation space. Clearly, $\Phi \in C([0, T]; (\mathbb{V}^p)^*)$. Also, by refining in (2.28), ϕ_n converges to ϕ almost everywhere in $\Omega \times (0, T)$ and ψ_n converges to ψ almost everywhere in $\Gamma \times (0, T)$, respectively. Then, by means of known results of measure theory (see, e.g., [42, Lemma 8.3]), the continuity of f_k , $k = 1, 2$, and the convergence of (2.28) easily imply that $f_1(\phi_n)$ converges weakly to $f_1(\phi)$ in $L^{r'_1}(\Omega \times (0, T))$, and thus, weakly star in $L^s([0, T]; (\mathbb{V}^p)^*)$. Moreover, $f_2(\psi_n)$ converges weakly to $f_2(\psi)$ in $L^{r'_2}(\Gamma \times (0, T))$, and thus, weakly star in $L^s([0, T]; (\mathbb{V}^p)^*)$. Since the orthogonal projection $L_n := (I - \text{Pr}_n)$ to Pr_n has the known property that $L_n \Phi_i \rightarrow 0$ in \mathbb{V}^p (as $n \rightarrow \infty$) for each $1 \leq i \leq n$, it is now an easy task to show that

$$\text{Pr}_n \mathcal{F}(\Phi_n) \rightarrow \mathcal{F}(\Phi) \text{ weakly star in } L^s([0, T]; (\mathbb{V}^p)^*). \quad (2.29)$$

On the other hand, on account of (2.23), (2.27)-(2.28) and exploiting suitable monotonicity arguments, we show that $\text{Pr}_n \mathcal{B}_p \Phi_n$ converges weakly star to $\mathcal{B}_p \Phi$ in $L^s([0, T]; (\mathbb{V}^p)^*)$. Indeed, since $\mathcal{B}_p = L_n \mathcal{B}_p + \text{Pr}_n \mathcal{B}_p$, it only suffices to show that $\mathcal{B}_p \Phi_n$ converges weakly star to $\mathcal{B}_p \Phi$ in $L^s([0, T]; (\mathbb{V}^p)^*)$. We argue as follows. By (2.24), the function $\mathcal{B}_p \Phi_n$ is uniformly bounded in

$L^q([0, T]; (\mathbb{V}^p)^*)$. Hence, we can choose a subsequence of Φ_n which we also denote by Φ_n such that

$$\mathcal{B}_p \Phi_n \rightarrow \Xi \text{ weakly star in } L^q([0, T]; (\mathbb{V}^p)^*), \tag{2.30}$$

and thus weakly star in $L^s([0, T]; (\mathbb{V}^p)^*)$, since $s \leq q$. Exploiting the above convergence properties (2.26), (2.29), (2.30), and passing to the limit in equation (2.20), we have that

$$\partial_t \Phi + \Xi + \mathcal{F}(\Phi) = \mathcal{G} \tag{2.31}$$

holds as an equality in $L^s([0, T]; (\mathbb{V}^p)^*)$. Let us now prove the equality $\Xi = \mathcal{B}_p \Phi$. To this end, consider the integral

$$\int_0^T \langle \mathcal{B}_p \Phi_n - \mathcal{B}_p \Psi, \Phi_n - \Psi \rangle_{\mathbb{X}^2} dt =: \delta_n, \tag{2.32}$$

where $\Phi_n, \Psi \in L^p([0, T]; \mathbb{V}^p)$. Obviously, $\delta_n \geq 0$ since \mathcal{B}_p is maximal monotone on \mathbb{X}^2 . Moreover, the following integration by parts formula holds:

$$\int_0^t \left(\langle \partial_t \Phi_1, \Phi_2 \rangle_{\mathbb{X}^2} + \langle \Phi_1, \partial_t \Phi_2 \rangle_{\mathbb{X}^2} \right) ds = \langle \Phi_1(t), \Phi_2(t) \rangle_{\mathbb{X}^2} - \langle \Phi_1(0), \Phi_2(0) \rangle_{\mathbb{X}^2}, \tag{2.33}$$

for all $t \in [0, T]$, and $\Phi_k \in L^p([0, T]; \mathbb{V}^p)$, $\partial_t \Phi_k \in L^q([0, T]; (\mathbb{V}^p)^*)$, $k = 1, 2$. Such formulas can be easily obtained by approximating Φ_k by a sequence of mollifiers (in time) $(\Phi_k)_h \in C^1([0, T]; \mathbb{V}^p)$, followed by passage to the limit in h . Take $\Phi_1 = \Phi_2 = \Phi_n$ in (2.33) (note that, within the approximation scheme (2.20), the functions Φ_n already belong to $C^1([0, T]; \mathbb{V}^p)$ since $d_i \in C^1(0, T)$, for all i). We have

$$\begin{aligned} \int_0^t \langle \mathcal{B}_p \Phi_n, \Phi_n \rangle_{\mathbb{X}^2} ds &= - \int_0^t [\langle \mathcal{F}(\Phi_n), \Phi_n \rangle_{\mathbb{X}^2} - \langle \mathcal{G}, \Phi_n \rangle_{\mathbb{X}^2}] ds \\ &+ \frac{1}{2} \|\Phi_n(0)\|_{\mathbb{X}^2}^2 - \frac{1}{2} \|\Phi_n(t)\|_{\mathbb{X}^2}^2. \end{aligned} \tag{2.34}$$

Therefore, by (2.32) and (2.34),

$$\begin{aligned} \delta_n &= \frac{1}{2} \|\Phi_n(0)\|_{\mathbb{X}^2}^2 - \frac{1}{2} \|\Phi_n(t)\|_{\mathbb{X}^2}^2 - \int_0^t [\langle \mathcal{F}(\Phi_n), \Phi_n \rangle_{\mathbb{X}^2} - \langle \mathcal{G}, \Phi_n \rangle_{\mathbb{X}^2}] ds \\ &- \int_0^t [\langle \mathcal{B}_p \Phi_n, \Psi \rangle_{\mathbb{X}^2} + \langle \mathcal{B}_p \Psi, \Phi_n - \Psi \rangle_{\mathbb{X}^2}] ds. \end{aligned} \tag{2.35}$$

Following known results of measure theory, (2.27), (2.28) also yield that $\Phi_n(t) \rightarrow \Phi(t)$ in the weak topology of \mathbb{X}^2 and $\Phi(t)$ is a weakly continuous

function from $[0, T]$ to \mathbb{X}^2 . Besides,

$$\|\Phi(t)\|_{\mathbb{X}^2}^2 \leq \liminf_n \|\Phi_n(t)\|_{\mathbb{X}^2}^2 \quad (2.36)$$

and $\Phi_n(0) \rightarrow \Phi(0)$ strongly in \mathbb{X}^2 (which follows by construction). Exploiting (2.36) and the above convergence properties for $\mathcal{F}(\Phi_n)$, $\mathcal{B}_p\Phi_n$ (see (2.25)-(2.30)), and passing to the limit as $n \rightarrow +\infty$ in (2.35), we get

$$\begin{aligned} & -\frac{1}{2}\|\Phi(0)\|_{\mathbb{X}^2}^2 + \frac{1}{2}\|\Phi(t)\|_{\mathbb{X}^2}^2 + \int_0^t [\langle \mathcal{F}(\Phi), \Phi \rangle_{\mathbb{X}^2} - \langle \mathcal{G}, \Phi \rangle_{\mathbb{X}^2}] ds \\ & \leq -\int_0^t [\langle \mathcal{B}_p\Psi, \Phi - \Psi \rangle_{\mathbb{X}^2} + \langle \Xi, \Psi \rangle_{\mathbb{X}^2}] ds. \end{aligned} \quad (2.37)$$

Now, let us take $\Phi_1 = \Phi_2 = \Phi$ in (2.33) and exploit the equality (2.31) once again. We deduce that

$$\begin{aligned} -\int_0^t \langle \Xi, \Phi \rangle_{\mathbb{X}^2} ds &= \int_0^t [\langle \mathcal{F}(\Phi), \Phi \rangle_{\mathbb{X}^2} - \langle \mathcal{G}, \Phi \rangle_{\mathbb{X}^2}] ds \\ &+ \frac{1}{2}\|\Phi(t)\|_{\mathbb{X}^2}^2 - \frac{1}{2}\|\Phi(0)\|_{\mathbb{X}^2}^2. \end{aligned} \quad (2.38)$$

Adding the relations (2.37) and (2.38) together, it is not difficult to see that we obtain the following inequality:

$$\int_0^t \langle \Xi - \mathcal{B}_p\Psi, \Phi - \Psi \rangle_{\mathbb{X}^2} ds \geq 0, \quad (2.39)$$

for all $\Psi \in L^p([0, T]; \mathbb{V}^p)$. Finally, by setting $\Psi = \Phi + \varepsilon\Theta$, with $\varepsilon \in (0, 1]$, $\Theta \in L^p([0, T]; \mathbb{V}^p)$ in (2.39) and dividing by ε , and arguing in a standard way (we employ Lebesgue's theorem and pass to the limit as $\varepsilon \rightarrow 0$), it is not difficult to realize that

$$\int_0^t \langle \Xi - \mathcal{B}_p\Phi, \Theta \rangle_{\mathbb{X}^2} ds = 0, \quad \forall \Theta \in L^p([0, T]; \mathbb{V}^p).$$

This proves that $\Xi = \mathcal{B}_p\Phi$ in $L^q([0, T]; (\mathbb{V}^p)^*)$, and since weak limits are unique,

$$\mathcal{B}_p\Phi_n \rightarrow \mathcal{B}_p\Phi \text{ weakly star in } L^q([0, T]; (\mathbb{V}^p)^*). \quad (2.40)$$

By means of the above convergence properties (2.26), (2.29) and (2.40), we can readily pass to the limit in (2.20) and find that $(\phi(t), \psi(t))$ solves (2.9).

Finally, it is left to show that $\psi(0) = \psi_0$ and $\phi(0) = \phi_0$. To this end, choose a test function $\Pi \in C^1([0, T]; \mathbb{V}^p)$ with $\Pi(T) = 0$, and compare the results of taking $\bar{\Phi} = \Pi$ in (2.20) and (2.31), respectively. Passing to the limit once again by using the above convergence properties of the solutions (ϕ_n, ψ_n) , we easily obtain our desired conclusion. The uniqueness of the

weak solution follows from Lemma 2.7 below. The proof of the theorem is finished. \square

It remains to verify the uniqueness of the solution and the Lipschitz continuity with respect to the initial data $(\phi_0, \psi_0) \in \mathbb{X}^2$.

Lemma 2.7. *Let the assumptions above hold and let the functions (ϕ_1, ψ_1) and (ϕ_2, ψ_2) be two solutions of problem (2.9). Set*

$$(\bar{\phi}(t), \bar{\psi}(t)) := (\phi_1(t) - \phi_2(t), \psi_1(t) - \psi_2(t)).$$

Then, the following estimate holds:

$$\|\bar{\phi}(t)\|_{L^2(\Omega)}^2 + \|\bar{\psi}(t)\|_{L^2(\Gamma)}^2 \leq L e^{Lt} \left(\|\bar{\phi}(0)\|_{L^2(\Omega)}^2 + \|\bar{\psi}(0)\|_{L^2(\Gamma)}^2 \right), \quad (2.41)$$

for all $t \geq 0$, where the positive constant L is independent of t , but depends on the norm of the initial data $(\phi_i(0), \psi_i(0))$, $i = 1, 2$, in \mathbb{X}^2 .

Proof. We work within the Galerkindiscretization scheme introduced above. With reference to the proof of Theorem 2.6, let $\Phi_n(t) = (\phi_n(t), \psi_n(t))$ and $\Phi_m(t) = (\phi_m(t), \psi_m(t))$ be two solutions of problem \mathbf{P}_n (see (2.20)-(2.21)) and \mathbf{P}_m , corresponding to the initial data $\text{Pr}_n \Phi_1(0)$ and $\text{Pr}_m \Phi_2(0)$, and the same source term \mathcal{G} , respectively. First notice that $\Phi_n(t), \Phi_m(t) \in C^1((0, T); \mathbb{V}^p)$, for all $m, n \geq 1$; set $\Phi_{mn}(t) := \Phi_m(t) - \Phi_n(t)$. Then, the following equation is satisfied:

$$\begin{aligned} & \langle \partial_t \phi_{mn}, \bar{\phi} \rangle_2 + \langle \partial_t \psi_{mn}, \bar{\psi} \rangle_2 + \langle \mathcal{B}_p \Phi_m - \mathcal{B}_p \Phi_n, \bar{\Phi} \rangle_{\mathbb{X}^2} \\ & + \langle \mathcal{F}(\Phi_m) - \mathcal{F}(\Phi_n), \bar{\Phi} \rangle_{\mathbb{X}^2} = 0, \end{aligned} \quad (2.42)$$

for all $\bar{\Phi} = (\bar{\phi}, \bar{\psi}) \in K_m \cap K_n$, with the initial data $\Phi_{mn}(0) = \text{Pr}_m \Phi_2(0) - \text{Pr}_n \Phi_1(0)$.

Taking $\bar{\Phi} = \Phi_{mn}(t)$, i.e., $\bar{\phi} = \phi_{mn}(t)$ and $\bar{\psi} = \psi_{mn}$, respectively, the equality (2.42) then yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|\phi_{mn}(t)\|_{L^2(\Omega)}^2 + \|\psi_{mn}(t)\|_{L^2(\Gamma)}^2 \right] \\ & + \int_{\Omega} \left(|\nabla \phi_m|^{p-2} \nabla \phi_m - |\nabla \phi_n|^{p-2} \nabla \phi_n \right) \cdot \nabla \phi_{mn} dx \\ & = - \langle f_1(\phi_m) - f_1(\phi_n), \phi_{mn}(t) \rangle_2 - \langle f_2(\psi_m) - f_2(\psi_n), \psi_{mn}(t) \rangle_{2,\Gamma}. \end{aligned} \quad (2.43)$$

Now, it is readily seen that assumption (2.6) also implies that

$$f'_i(y) \geq -M_i, \forall y \in \mathbb{R}, \quad (2.44)$$

for some $M_i > 0$, $i = 1, 2$. Using now the obvious inequality

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)(a - b) \geq v \left(|a|^{p-2} + |b|^{p-2}\right) |a - b|^2,$$

which holds for some positive constant v , and for all $a, b \in \mathbb{R}^N$ and $p \in [p_0, +\infty) \cap (1, +\infty)$ (see, e.g., [30, Lemma 2.13]), we can control the second term on the left-hand side of (2.43). Using (2.44) to control the right-hand side, we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|(\phi_{mn}(t), \psi_{mn}(t))\|_{\mathbb{X}^2}^2 \right) + v \int_{\Omega} \left(|\nabla \phi_m|^{p-2} + |\nabla \phi_n|^{p-2} \right) |\nabla \phi_{mn}|^2 dx \\ & \leq c \|(\phi_{mn}(t), \psi_{mn}(t))\|_{\mathbb{X}^2}^2, \quad \forall t \geq 0. \end{aligned} \tag{2.45}$$

Integrating this inequality from 0 to t and passing to the limit as $m, n \rightarrow \infty$, we immediately obtain the claim (2.41), owing to Gronwall’s inequality. Let us choose $\Phi_2(0) = \Phi_1(0)$. It is clear then, from (2.45), that $\{(\phi_n, \psi_n)\}$ is also a Cauchy sequence in \mathbb{X}^2 and, therefore, $(\phi(t), \psi(t)) \in C([0, T]; \mathbb{X}^2)$. The proof of Lemma 2.7 is now finished. \square

Theorem 2.8. *Let the assumptions of Theorem 2.6 be satisfied. The semi-group \mathcal{S}_t is $(\mathbb{X}^2, \mathbb{V}^p)$ -bounded for $t > 0$; i.e., for any (ϕ_0, ψ_0) bounded in \mathbb{X}^2 , the solution $(\phi(t), \psi(t)) = \mathcal{S}_t(\phi_0, \psi_0)$ is bounded in \mathbb{V}^p , for all $t > 0$. More precisely, the following estimate holds:*

$$\begin{aligned} & \|(\phi(t), \psi(t))\|_{\mathbb{V}^p}^p + \|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} \\ & \leq c_0 t^{-1} \|(\phi(0), \psi(0))\|_{\mathbb{X}^2}^2 + c_0 \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \end{aligned} \tag{2.46}$$

for all $t > 0$, where $c_0 > 0$ is independent of time and initial data. In particular, the set $\{(\phi(t), \psi(t)) \in \mathbb{V}^p : t \geq 1\}$ is precompact in \mathbb{X}^2 .

Proof. Take $\bar{\Phi} = \partial_t \Phi_n = (\partial_t \phi_n, \partial_t \psi_n)$ in (2.20) and recall (1.6). We have

$$\|\partial_t \phi_n(t)\|_{L^2(\Omega)}^2 + \|\partial_t \psi_n(t)\|_{L^2(\Gamma)}^2 + \frac{d}{dt} [\mathcal{E}_{\Omega}(\phi_n(t)) + \mathcal{E}_{\Gamma}(\psi_n(t))] = 0, \tag{2.47}$$

where the energy functionals $\mathcal{E}_{\Omega}(\phi)$, $\mathcal{E}_{\Gamma}(\psi)$ are defined in (1.3)-(1.4). Recall that $F'_i = f_i$, $i = 1, 2$. From the assumptions (2.6)-(2.7), it is easy to check that these functions also satisfy

$$\tilde{\eta}_1 |y|^{r_1} - \tilde{\nu}'_1 \leq F_1(y) \leq \tilde{\nu}_1 |y|^{r_1} + \tilde{\nu}'_1, \quad -\tilde{\nu}'_2 \leq F_2(y) \leq \tilde{\nu}_2 |y|^{r_2} + \tilde{\nu}'_2, \tag{2.48}$$

while this fact implies that, for all $t \geq 0$,

$$c \left(\|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} - |\Omega| \tilde{\nu}'_1 - S(\Gamma) \tilde{\nu}'_2 \right) \leq \int_{\Omega} F_1(\phi(t)) dx + \int_{\Omega} F_2(\psi(t)) dS$$

$$\leq c \left(\|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} + \|\psi(t)\|_{L^{r_2}(\Gamma)}^{r_2} \right) + c \left(\tilde{\nu}'_1 |\Omega| + \tilde{\nu}'_2 S(\Gamma) \right). \tag{2.49}$$

Let us now multiply (2.47) by t and integrate the resulting relation over $(0, t)$. We deduce

$$\begin{aligned} & t [\mathcal{E}_\Omega(\phi_n(t)) + \mathcal{E}_\Gamma(\psi_n(t))] + \int_0^t s \left(\|\partial_t \phi_n(s)\|_{L^2(\Omega)}^2 + \|\partial_t \psi_n(s)\|_{L^2(\Gamma)}^2 \right) ds \\ &= \int_0^t (\mathcal{E}_\Omega(\phi_n(s)) + \mathcal{E}_\Gamma(\psi_n(s))) ds, \end{aligned} \tag{2.50}$$

for all $t \geq 0$. Using estimates (2.10), (2.49) and exploiting the embeddings $W^{1,p}(\Omega) \hookrightarrow L^{r_1}(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^{r_2}(\Gamma)$, we conclude that the right-hand side of (2.50) is bounded from above; that is,

$$\begin{aligned} & \int_0^t (\mathcal{E}_\Omega(\phi_n(s)) + \mathcal{E}_\Gamma(\psi_n(s))) ds \\ & \leq c \|(\phi(0), \psi(0))\|_{\mathbb{X}^2}^2 + ct \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \end{aligned} \tag{2.51}$$

for all $t \geq 0$. Combining (2.50) with (2.51), we obtain the following inequality:

$$\begin{aligned} & \mathcal{E}_\Omega(\phi_n(t)) + \mathcal{E}_\Gamma(\psi_n(t)) + t^{-1} \int_0^t s \left(\|\partial_t \phi_n(s)\|_{L^2(\Omega)}^2 + \|\partial_t \psi_n(s)\|_{L^2(\Gamma)}^2 \right) ds \\ & \leq ct^{-1} \|(\phi(0), \psi(0))\|_{\mathbb{X}^2}^2 + c \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \end{aligned} \tag{2.52}$$

for all $t > 0$. From (2.52), it is readily seen, on account of (2.49), that

$$\begin{aligned} & \|(\phi_n(t), \psi_n(t))\|_{\mathbb{V}^p}^p + \|\phi_n(t)\|_{L^{r_1}(\Omega)}^{r_1} \\ & \leq ct^{-1} \|(\phi(0), \psi(0))\|_{\mathbb{X}^2}^2 + c \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \end{aligned} \tag{2.53}$$

for all $t > 0$. Passing to the limit as $n \rightarrow +\infty$ over a subsequence $\Phi_n(t) = (\phi_n(t), \psi_n(t))$ which is weakly star convergent in $L^\infty([\varepsilon, T]; \mathbb{V}^p \cap L^{r_1}(\Omega))$ and weakly in $W^{1,2}([\varepsilon, T]; \mathbb{X}^2)$, for any $\varepsilon > 0$, the claim (2.46) follows immediately from (2.53), owing to the lower semicontinuity of the norms. Moreover, we have

$$\partial_t \Phi \in W^{1,2}([\varepsilon, T]; \mathbb{X}^2), \tag{2.54}$$

for any $\varepsilon > 0$. The relative compactness of any trajectory $(\phi(t), \psi(t))$, $t \geq 1$ is a straightforward consequence of (2.46) and the compact embedding of \mathbb{V}^p into \mathbb{X}^2 . The proof is finished. \square

Remark 2.9. Because of the regularity in (2.54), the function $\partial_t \Phi(t)$, $t \in [\varepsilon, T]$ (for any $\varepsilon > 0$), can now be viewed as a pair and, effectively, the solution $(\phi(t), \psi(t))$ of problem **P** satisfies the following “improved” weak formulation:

$$\begin{aligned} & \langle \partial_t \phi(t), \sigma \rangle_2 + \langle \partial_t \psi(t), \sigma \rangle_{2,\Gamma} + \left\langle |\nabla \phi(t)|^{p-2} \nabla \phi(t), \nabla \sigma \right\rangle_2 \\ & + \langle f_1(\phi(t)), \sigma \rangle_2 + \langle f_2(\psi(t)), \sigma|_\Gamma \rangle_{2,\Gamma} = \langle g_1, \sigma \rangle_2 + \langle g_2, \sigma|_\Gamma \rangle_{2,\Gamma}, \end{aligned}$$

for all $t \geq \varepsilon > 0$ and all $\Xi = (\sigma, \sigma|_\Gamma) \in \mathbb{V}^p$.

Remark 2.10. Theorem 2.6 still remains true if the quasilinear operators $\Delta_p \phi$ and $|\nabla \phi|^{p-2} \partial_{\mathbf{n}} \phi$ from (1.2) and (1.7) are replaced by the operators $\operatorname{div}(a(|\nabla \phi|) \nabla \phi)$ and $b(x) a(|\nabla \phi|) \partial_{\mathbf{n}} \phi$, where $b \in L^\infty(\Gamma)$, $b \geq b_0 > 0$ and $a \in C^1(\mathbb{R}^N, \mathbb{R})$ is a monotone nondecreasing function that satisfies the following assumptions:

$$|a(y)| \leq c_1(1 + |y|^{p-2}), \quad \forall y \in \mathbb{R}^N, \quad a(y)|y|^2 \geq c_2|y|^p, \quad \forall y \in \mathbb{R}^N,$$

for some positive constants c_1, c_2 .

3. GLOBAL ATTRACTORS

Before we state our main results of this section, we recall some basic definitions about global attractors.

Definition 3.1. Let $\{\mathcal{S}_t\}_{t \geq 0}$ be a semigroup on a Banach space X and Z be a metric space. A set $\mathcal{A} \subset X \cap Z$, which is invariant (that is, $\mathcal{S}_t \mathcal{A} = \mathcal{A}$, for all $t \geq 0$), closed in X , compact in Z and attracts bounded subsets of X in the topology of Z is called an (X, Z) -attractor. More precisely, for any bounded subset $B \subset X$, we have

$$\lim_{t \rightarrow +\infty} \operatorname{dist}_Z(\mathcal{S}_t B, \mathcal{A}) = 0, \quad (3.1)$$

where dist_Z denotes the Hausdorff semidistance with respect to the metric of Z , which is defined as

$$\operatorname{dist}_Z(X, Y) := \sup_{x \in X} \inf_{y \in Y} \|x - y\|_Z.$$

Definition 3.2. A bounded subset $B_0 \subset Z$ is called an (X, Z) -bounded absorbing set, if, for any bounded subset $B \subset X$, there exists a time $t^\# = t^\#(B)$, such that $\mathcal{S}_t B \subset B_0$, for any $t \geq t^\#$.

Next, we derive stronger global a priori estimates for the solutions of problem **(P)** in the space \mathbb{X}^2 (compare with Proposition 2.5). These a priori

estimates are necessary for the study of the asymptotic behavior of our problem.

Proposition 3.3. *Suppose that $f_i \in \mathcal{N}_i(p, r_i)$ if $p \geq 2$ and $f_i \in \tilde{\mathcal{N}}_i(p, r_i)$ if $p \in (p_0, 2) \cap (1, +\infty)$, for each $i = 1, 2$, where by $f_i \in \tilde{\mathcal{N}}_i(p, r_i)$, we mean the following: f_i satisfies (1.4) and*

$$\begin{cases} \eta_1 |y|^2 - \nu'_1 \leq f_1(y) y \leq \nu_1 |y|^{r_1} + \nu'_1, \\ \eta_2 |y|^2 - \nu'_2 \leq f_2(y) y \leq \nu_2 |y|^{r_2} + \nu'_2, \end{cases} \quad (3.2)$$

for some $\eta_i, \nu_i > 0, \nu'_i \geq 0$, with r_1, r_2 as in Definition 2.1. Let $g_1 \in L^{r'_1}(\Omega), g_2 \in L^{r'_2}(\Gamma)$, where r'_i are conjugate to r_i . Let $(\phi(t), \psi(t))$ be a weak solution of (2.5) (i.e., satisfies (2.9)). Then, the following estimate holds:

$$\begin{aligned} & \|(\phi(t), \psi(t))\|_{\mathbb{X}^2}^2 + \int_0^t \left(\|\phi(s)\|_{W^{1,p}(\Omega)}^p + \|\psi(s)\|_{W^{1-1/p,p}(\Gamma)}^p \right) ds \quad (3.3) \\ & \leq c \left(\|(\phi(0), \psi(0))\|_{\mathbb{X}^2}^2 \right) e^{-\rho t} + c \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \end{aligned}$$

for all $t \geq 0$, where the positive constants c, ρ are independent of time and initial data.

Proof. We divide the proof of (3.3) into two parts. First, we show (3.3) for $2 \leq p \leq N$. The case $p > N$ follows similarly with minor modifications. Using the facts that $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ and $W^{1-1/p,p}(\Gamma) \hookrightarrow L^2(\Gamma)$ for $p \in [p_0, +\infty) \cap (1, +\infty)$, we can rewrite (2.13) as follows:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|\phi(t)\|_{L^2(\Omega)}^2 + \|\psi(t)\|_{L^2(\Gamma)}^2 \right] + \frac{c}{4} \left(\|\phi(t)\|_{L^2(\Omega)}^p + \|\psi(t)\|_{L^2(\Gamma)}^p \right) \quad (3.4) \\ & + \frac{c}{2} \|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} + \frac{c}{4} \left(\|\phi(t)\|_{W^{1,p}(\Omega)}^p + \|\psi(t)\|_{W^{1-1/p,p}(\Gamma)}^p \right) \\ & \leq c \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right). \end{aligned}$$

First, from Young's inequality, we notice that

$$\|\phi(t)\|_{L^2(\Omega)}^2 \leq c \left(1 + \|\phi(t)\|_{L^2(\Omega)}^p \right) \quad \text{and} \quad \|\psi(t)\|_{L^2(\Gamma)}^2 \leq c \left(1 + \|\psi(t)\|_{L^2(\Gamma)}^p \right). \quad (3.5)$$

Using (3.5), from (3.4), we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|\phi(t)\|_{L^2(\Omega)}^2 + \|\psi(t)\|_{L^2(\Gamma)}^2 \right] + \frac{\rho}{2} \left(\|\phi(t)\|_{L^2(\Omega)}^2 + \|\psi(t)\|_{L^2(\Gamma)}^2 \right) \quad (3.6) \\ & + \frac{c}{2} \|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} + \frac{c}{4} \left(\|\phi(t)\|_{W^{1,p}(\Omega)}^p + \|\psi(t)\|_{W^{1-1/p,p}(\Gamma)}^p \right) \end{aligned}$$

$$\leq c \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right),$$

for some positive constant ρ that depends only on p , c , Ω and Γ . Applying a suitable version of Gronwall's inequality (see e.g., [31, Lemma 2.5]) to (3.6), we easily deduce estimate (3.3). In particular, we also have that

$$\begin{aligned} & \int_0^t \left(\|(\phi(s), \psi(s))\|_{\mathbb{V}^p}^p + \|\phi(s)\|_{L^{r_1}(\Omega)}^{r_1} \right) ds \\ & \leq c \left(\|(\phi(0), \psi(0))\|_{\mathbb{X}^2}^2 \right) e^{-\rho t} + c \left(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q \right), \end{aligned} \quad (3.7)$$

for all $t \geq 0$.

Let us now show (3.3) when $p \in (p_0, 2) \cap (1, +\infty)$. To this end, recall (2.11). Exploiting the new assumptions (3.2), from (2.11), we deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|\phi(t)\|_{L^2(\Omega)}^2 + \|\psi(t)\|_{L^2(\Gamma)}^2 \right] \\ & \quad + \|\nabla \phi(t)\|_{L^p(\Omega)}^p + \eta_1 \|\phi(t)\|_{L^2(\Omega)}^2 + \eta_2 \|\psi(t)\|_{L^2(\Gamma)}^2 \\ & \leq \langle g_1, \phi(t) \rangle_2 + \langle g_2, \psi(t) \rangle_{2,\Gamma} + c. \end{aligned} \quad (3.8)$$

Setting $\rho = \min(\eta_1, \eta_2) > 0$, then using the embeddings $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, $L^2(\Omega) \hookrightarrow L^p(\Omega)$ (which, by Young's inequality also imply that $|\cdot|_{L^p(\Omega)}^p \leq c(1 + |\cdot|_{L^p(\Omega)}^2)$, for some $c > 0$), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|\phi(t)\|_{L^2(\Omega)}^2 + \|\psi(t)\|_{L^2(\Gamma)}^2 \right] + c \|\phi(t)\|_{W^{1,p}(\Omega)}^p \\ & + \frac{\rho}{2} \left(\|\phi(t)\|_{L^2(\Omega)}^2 + \|\psi(t)\|_{L^2(\Gamma)}^2 \right) \leq \langle g_1, \phi(t) \rangle_2 + \langle g_2, \psi(t) \rangle_{2,\Gamma} + c, \end{aligned} \quad (3.9)$$

for all $t \geq 0$. Finally, exploiting suitable Young inequalities in order to control the terms on the right-hand side of (3.9) (see, e.g., (2.13)) and arguing as in (3.6)-(3.7), we obtain the desired inequality (3.3) for $p < 2$. The proof is finished. \square

Remark 3.4. Obviously, the result of Theorem 2.6 is still valid if $f_i \in \tilde{\mathcal{N}}_i(p, r_i)$, $i = 1, 2$, when $p \in (p_0, 2) \cap (1, +\infty)$; that is, we still have a unique weak solution under the new assumptions. Indeed, the global a priori estimate (2.10) follows easily by integrating the inequality (3.9) over $(0, t)$.

Theorem 3.5. *Let the assumptions of Proposition 3.3 be satisfied. Endow the Banach space $\mathbb{Z}_{r_1}^p := \mathbb{V}^p \cap L^{r_1}(\Omega)$ with the metric topology of \mathbb{V}^p . Then the solution semigroup $\{\mathcal{S}_t\}_{t \geq 0}$ has an $(\mathbb{X}^2, \mathbb{Z}_{r_1}^p)$ -bounded absorbing set. More*

precisely, there is a positive constant C , depending only on the physical parameters of the problem, such that for any bounded subset $B \subset \mathbb{X}^2$, there exists a positive constant $t^\# = t^\#(\|B\|_{\mathbb{X}^2})$ such that

$$\|(\phi(t), \psi(t))\|_{\mathbb{V}^p} + \|\phi(t)\|_{L^{r_1}(\Omega)} \leq C, \tag{3.10}$$

for all $t \geq t^\#$.

Proof. First, we note that, setting $R_0 := 2c(1 + \|g_1\|_{L^{r'_1}(\Omega)}^q + \|g_2\|_{L^{r'_2}(\Gamma)}^q)$ and $t^\# = \frac{1}{\rho} \ln(2C\|(\phi(0), \psi(0))\|_{\mathbb{X}^2}^2/R_0)$ in (3.3), we obtain that

$$\|(\phi(t), \psi(t))\|_{\mathbb{X}^2} \leq R_0, \text{ for all } t \geq t^\#. \tag{3.11}$$

Next, we show that

$$\begin{aligned} & \int_t^{t+1} \left(\|(\phi(s), \psi(s))\|_{\mathbb{V}^p}^p + \|\phi(s)\|_{L^{r_1}(\Omega)}^{r_1} \right) ds \\ & - \int_t^{t+1} \left(\langle g_1, \phi(s) \rangle_2 + \langle g_2, \psi(s) \rangle_{2,\Gamma} \right) ds \leq C(R_0), \end{aligned} \tag{3.12}$$

for all $t \geq t^\#$, where the positive constant $C(R_0)$ depends only on R_0 , but is independent of t and initial data. Indeed, integrating (2.12) from t to $t + 1$ yields

$$\begin{aligned} & \|(\phi(t+1), \psi(t+1))\|_{\mathbb{X}^2}^2 - \|(\phi(t), \psi(t))\|_{\mathbb{X}^2}^2 \\ & + c \left(\int_t^{t+1} \|\nabla \phi(s)\|_{L^p(\Omega)}^p + \|\phi(s)\|_{L^{r_1}(\Omega)}^{r_1} \right) ds \\ & \leq \int_t^{t+1} \left(\langle g_1, \phi(s) \rangle_2 + \langle g_2, \psi(s) \rangle_{2,\Gamma} + c \right) ds. \end{aligned} \tag{3.13}$$

Using the fact that $L^{r_1}(\Omega) \hookrightarrow L^p(\Omega)$ and Young's inequality, we have, for every $\varepsilon > 0$, that

$$\|\phi(t)\|_{L^p(\Omega)}^p \leq c\|\phi(t)\|_{L^{r_1}(\Omega)}^p \leq c\varepsilon^{-\frac{p}{r_1}}\|\phi(t)\|_{L^{r_1}(\Omega)}^{r_1} + \varepsilon^{\frac{r_1-p}{r_1}}. \tag{3.14}$$

Using (3.14) and the fact that the norms $\|u\|_{W^{1,p}(\Omega)}$, $\|(u, u_\Gamma)\|_{\mathbb{V}^p}$ are equivalent and recalling (3.11), from (3.13), we obtain

$$\begin{aligned} & \int_t^{t+1} \left(\|(\phi(s), \psi(s))\|_{\mathbb{V}^p}^p + \|\phi(s)\|_{L^{r_1}(\Omega)}^{r_1} \right) ds \\ & - \int_t^{t+1} \left(\langle g_1, \phi(s) \rangle_2 + \langle g_2, \psi(s) \rangle_{2,\Gamma} \right) ds \leq c + \|(\phi(t), \psi(t))\|_{\mathbb{X}^2}^2 \leq C(R_0), \end{aligned}$$

for all $t \geq t^\#$. This completes the proof of (3.12).

Take now $\sigma = \partial_t \phi(t)$ and $\sigma_\Gamma = \partial_t \psi(t)$ in (2.9); using (1.6) and recalling Remark 2.9, we deduce that

$$\|\partial_t \phi(t)\|_{L^2(\Omega)}^2 + \|\partial_t \psi(t)\|_{L^2(\Gamma)}^2 + \frac{d}{dt} [\mathcal{E}_\Omega(\phi(t)) + \mathcal{E}_\Gamma(\psi(t))] = 0, \quad (3.15)$$

for all $t \geq t^\#$. Recall that $F'_i = f_i$, $i = 1, 2$, satisfy (2.48) and estimate (2.49) holds for any weak solution $(\phi(t), \psi(t))$ of **(P)**. Applying now the uniform Gronwall’s lemma (see e.g., [44, Chapter 3, Lemma 1.1]) to (3.15), and employing the estimates (3.12), (2.49) once more, we can find another time $t^* \geq t^\# \geq 1$, such that the following inequality holds:

$$\mathcal{E}_\Omega(\phi(t)) + \mathcal{E}_\Gamma(\psi(t)) \leq \tilde{C}(R_0), \quad \forall t \geq t^*, \quad (3.16)$$

where the positive constant $\tilde{C}(R_0)$ depends on $C(R_0)$, but is independent of time and initial data. Estimate (3.10) follows now immediately, thanks to (2.49)-(3.16), since $g_1 \in L^{r'_1}(\Omega)$, $g_2 \in L^{r'_2}(\Gamma)$. The proof is finished. \square

Remark 3.6. It follows from (3.10) that the semigroup $\{\mathcal{S}_t\}_{t \geq 0}$ always possesses an $(\mathbb{X}^2, C(\bar{\Omega}))$ -bounded absorbing set if $p > N$ or $N = 1$, since then $\mathbb{V}^p \hookrightarrow C(\bar{\Omega})$, where we identify each function $u \in C(\bar{\Omega})$ with $U = (u, u|_\Gamma)$. However, if $p \leq N$, this is not always so. We will show that if we impose additional assumptions on the nonlinearities f_i , $i = 1, 2$, then we obtain that the semigroup $\{\mathcal{S}_t\}_{t \geq 0}$ also possesses an $(\mathbb{X}^2, \mathbb{X}^\infty)$ -bounded absorbing set if $p \leq N$. This issue is investigated below.

Assuming that the external forces g_i are bounded, we have the following result which may be of interest on its own.

Theorem 3.7. *Let $p \in [p_0, +\infty) \cap (1, +\infty)$ and assume now that $g_1 \in L^\infty(\Omega)$ and $g_2 \in L^\infty(\Gamma)$. Suppose that*

$$f_i(y)y \geq C_i |y|^{2+\beta} - C'_i |y|, \quad \forall y \in \mathbb{R}, \quad (3.17)$$

for some $\beta > 0$, $C_i > 0$, $C'_i \geq 0$, $i = 1, 2$. The following estimate holds for the (weak) solutions of problem **(P)** :

$$\|(\phi(t), \psi(t))\|_{\mathbb{X}^\infty} \leq c_\star + c_0 t^{-1/\beta}, \quad \text{for all } t > 0, \quad (3.18)$$

where the constants $c_\star, c_0 > 0$ are independent of time and initial data, and can be computed explicitly in terms of the structural parameters of the problem.

Proof. Estimate (3.18) can be justified by means of the approximation procedure devised in the proof of Theorem 2.6. Therefore, we proceed formally (note that $d_i \in C^2(0, T)$ in (2.19), therefore we can differentiate the equation with respect to time). We employ the weak formulation (2.9) of problem (2.5) once again. Let $s > 1$ and define the function

$$\Psi_s(t) = \left(|\phi(t)|^{s-2} \phi(t), |\psi(t)|^{s-2} \psi(t) \right). \tag{3.19}$$

Taking $\sigma = |\phi|^{s-2} \phi$, $\sigma_{|\Gamma} = |\psi|^{s-2} \psi$ in (2.9), and setting $\Theta(t) := (\phi(t), \psi(t))$ and $\mathcal{G} := (g_1, g_2)$, we obtain

$$\begin{aligned} & \frac{1}{s} \frac{d}{dt} \|\Theta(t)\|_{\mathbb{X}^s}^s + (s-1) \int_{\Omega} |\phi(t)|^{s-2} |\nabla \phi(t)|^p dx \\ & + \int_{\Omega} |\phi(t)|^{s-2} f_1(\phi) \phi(t) dx + \int_{\Gamma} |\psi(t)|^{s-2} f_2(\psi) \psi(t) dS = \langle \mathcal{G}, \Psi_s(t) \rangle_{\mathbb{X}^2}. \end{aligned} \tag{3.20}$$

By (3.17), we can control the last two integral terms on the left-hand side of (3.20) as follows:

$$\begin{aligned} & \int_{\Omega} |\phi(t)|^{s-2} f_1(\phi) \phi(t) dx + \int_{\Gamma} |\psi(t)|^{s-2} f_2(\psi) \psi(t) dS \\ & \geq \left(C_1 \|\phi(t)\|_{L^{s+\beta}(\Omega)}^{s+\beta} + C_2 \|\psi(t)\|_{L^{s+\beta}(\Gamma)}^{s+\beta} \right) \\ & - \left(C'_1 \|\phi(t)\|_{L^{s-1}(\Omega)}^{s-1} + C'_2 \|\psi(t)\|_{L^{s-1}(\Gamma)}^{s-1} \right). \end{aligned} \tag{3.21}$$

Using now Hölder's inequality with exponents $(\infty, 1)$ in order to estimate the term on the right-hand side of (3.20) and employing (3.21), we deduce that

$$\frac{1}{s} \frac{d}{dt} \|\Theta(t)\|_{\mathbb{X}^s}^s + c \|\Theta(t)\|_{\mathbb{X}^{s+\beta}}^{s+\beta} \leq (\|\mathcal{G}\|_{\mathbb{X}^\infty} + c) \|\Theta(t)\|_{\mathbb{X}^{s-1}}^{s-1} \tag{3.22}$$

where, from now on, c will denote a positive constant, independent of time, initial data and s , and which may take on different values, sometimes even in the same line. Let us now estimate the \mathbb{X}^s and \mathbb{X}^{s-1} -norms of the solution $\Theta(t)$, using Hölder's inequality with exponents $((s+\beta)/s, (s+\beta)/\beta)$ and $(s/(s-1), s)$, respectively. We get

$$\|\Theta(t)\|_{\mathbb{X}^{s-1}}^{s-1} \leq \|\Theta(t)\|_{\mathbb{X}^s}^{s-1} (\mu(\overline{\Omega}))^{1/s}$$

and

$$\|\Theta(t)\|_{\mathbb{X}^s}^s \leq \|\Theta(t)\|_{\mathbb{X}^{s+\beta}}^s (\mu(\overline{\Omega}))^{\beta/(s+\beta)},$$

where

$$\mu(\overline{\Omega}) = \int_{\Omega} dx + \int_{\Gamma} dS = |\Omega| + S(\Gamma).$$

Inserting the above inequalities in (3.22), and observing that

$$\frac{1}{s} \frac{d}{dt} \|\Theta(t)\|_{\mathbb{X}^s}^s = \|\Theta(t)\|_{\mathbb{X}^{s-1}}^{s-1} \frac{d}{dt} \|\Theta(t)\|_{\mathbb{X}^s},$$

from (3.22), we deduce that

$$\frac{d}{dt} \|\Theta(t)\|_{\mathbb{X}^s} + c (\mu(\overline{\Omega}))^{-\beta/s} \|\Theta(t)\|_{\mathbb{X}^s}^{1+\beta} \leq (\|\mathcal{G}(x)\|_{\mathbb{X}^\infty} + c) (\mu(\overline{\Omega}))^{1/s}. \tag{3.23}$$

Setting now $\zeta_1 := c \inf_{s \geq 2} (\mu(\overline{\Omega}))^{-\beta/s} > 0$ and

$$\zeta_2 := \sup_{s \geq 2} (\|\mathcal{G}(x)\|_{\mathbb{X}^\infty} + c) (\mu(\overline{\Omega}))^{1/s},$$

and observing that ζ_2 is finite, we have

$$\frac{d}{dt} \|\Theta(t)\|_{\mathbb{X}^s} + \zeta_1 \|\Theta(t)\|_{\mathbb{X}^s}^{1+\beta} \leq \zeta_2. \tag{3.24}$$

Applying now a suitable version of Gronwall’s inequality (see, e.g., [44, Chapter 3, Lemma 5.1]) to (3.24), there are positive constants ζ_3 and ζ_4 depending only on ζ_1, ζ_2 (but, independent of time, initial data and s), such that

$$\|\Theta(t)\|_{\mathbb{X}^s} \leq \zeta_3 + \zeta_4 t^{-1/\beta}, \quad \forall t > 0. \tag{3.25}$$

Fixing now $t > 0$, and then passing to the limit in (3.25) as $s \rightarrow +\infty$, we readily obtain the assertion (3.18). The proof is finished. \square

Remark 3.8. Note that, for instance, if $f_1(y) = y^2(y - 1)$, then condition (3.17) is satisfied for some $\beta > 0$.

As a consequence of Theorem 3.7, we also obtain the following result.

Theorem 3.9. *Let the assumptions of Theorem 3.7 be satisfied. If $p \leq N$, then the solution semigroup $\{\mathcal{S}_t\}_{t \geq 0}$ has a $(\mathbb{X}^2, \mathbb{X}^\infty)$ -bounded absorbing set; that is, there is a positive constant C (independent of initial data and time), such that, for any bounded subset $B \subset \mathbb{X}^2$, there exists a positive constant $t^* = t^*(\|B\|_{\mathbb{X}^2})$ such that*

$$\|(\phi(t), \psi(t))\|_{\mathbb{X}^\infty} \leq C, \text{ for all } t \geq t^*. \tag{3.26}$$

We continue with a smoothing estimate for $(\partial_t \phi, \partial_t \psi)$ of the solutions of problem **(P)**.

Theorem 3.10. *Let $p \geq 2$ and suppose that $f_i \in \mathcal{N}_i(p, r_i)$, $i = 1, 2$. For any set $B \subset \mathbb{X}^2$ of bounded initial data $(\phi_0, \psi_0) \in \mathbb{X}^2$, there exists a positive constant $t^\# = t^\#(B) > 0$ such that*

$$\|\partial_t \phi(t)\|_{L^2(\Omega)}^2 + \|\partial_t \psi(t)\|_{L^2(\Gamma)}^2 \leq c_0, \quad \forall t \geq t^\#, \quad \forall (\phi_0, \psi_0) \in B, \tag{3.27}$$

where $c_0 > 0$ is independent of time and initial data. In particular,

$$\|\partial_t \phi(t)\|_{L^2(\Omega)}^2 + \|\partial_t \psi(t)\|_{L^2(\Gamma)}^2 \leq c_0 (1 + t^{-1}), \quad \forall t > 0, \quad \forall (\phi_0, \psi_0) \in B.$$

Proof. We will show (3.27) by a formal argument, since we can always employ the approximation procedure in Theorem 2.6. By differentiating the equations of (2.9), formally, with respect to time, and denoting $u(t) = \partial_t \phi(t)$, $v(t) = \partial_t \psi(t)$, we get

$$\begin{aligned} & \langle \partial_t u, \sigma \rangle_2 + \langle \partial_t v, \sigma|_{\Gamma} \rangle_{2,\Gamma} + \left\langle |\nabla \phi|^{p-2} \nabla u, \nabla \sigma \right\rangle_2 \\ & + (p-2) \left\langle |\nabla \phi|^{p-4} (\nabla \phi \cdot \nabla u) \nabla \phi, \nabla \sigma \right\rangle_2 \\ & + \left\langle f'_1(\phi) u, \sigma \right\rangle_2 + \left\langle f'_2(\psi) v, \sigma|_{\Gamma} \right\rangle_{2,\Gamma} = 0, \end{aligned} \tag{3.28}$$

for all $\sigma \in \mathbb{V}^p$, almost everywhere in $(0, +\infty)$, where $v(t) := u(t)|_{\Gamma}$. Taking $\sigma = u(t)$, $\sigma|_{\Gamma} = v(t)$ in (3.28), and using (2.44), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\|u(t)\|_{L^2(\Omega)}^2 + \|v(t)\|_{L^2(\Gamma)}^2 \right] + (p-1) \int_{\Omega} |\nabla \phi|^{p-2} |\nabla u|^2 \, dx \\ & \leq M_1 \|u(t)\|_{L^2(\Omega)}^2 + M_2 \|v(t)\|_{L^2(\Gamma)}^2, \quad \forall t \geq 0. \end{aligned} \tag{3.29}$$

Integrating this inequality over $(s, t+1)$, $t < s < t+1$, and setting $M := \max(2M_1, 2M_2)$, we deduce

$$\begin{aligned} & \|u(t+1)\|_{L^2(\Omega)}^2 + \|v(t+1)\|_{L^2(\Gamma)}^2 \\ & \leq \|u(s)\|_{L^2(\Omega)}^2 + \|v(s)\|_{L^2(\Gamma)}^2 + M \int_s^{t+1} \left[\|u(\tau)\|_{L^2(\Omega)}^2 + \|v(\tau)\|_{L^2(\Gamma)}^2 \right] d\tau, \end{aligned}$$

and then integrate again between t and $t+1$, so that

$$\begin{aligned} & \|u(t+1)\|_{L^2(\Omega)}^2 + \|v(t+1)\|_{L^2(\Gamma)}^2 \\ & \leq (1+M) \int_t^{t+1} \left[\|u(\tau)\|_{L^2(\Omega)}^2 + \|v(\tau)\|_{L^2(\Gamma)}^2 \right] d\tau, \quad \forall t > 0. \end{aligned} \tag{3.30}$$

On the other hand, integrating (3.15) over $(t, t+1)$, and employing (3.16) once more, it is not difficult to see that

$$\int_t^{t+1} \left[\|u(\tau)\|_{L^2(\Omega)}^2 + \|v(\tau)\|_{L^2(\Gamma)}^2 \right] d\tau \leq 2\tilde{C}(R_0), \quad \forall t \geq t^*, \tag{3.31}$$

where $\tilde{C}(R_0)$ and t^* are the constants defined in the proof of Theorem 3.5. The claim (3.27) follows now from (3.30) and (3.31). The proof is finished. \square

From Theorems 3.4, 3.9, and the compactness of the embeddings $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, $W^{1-1/p,p}(\Gamma) \hookrightarrow L^2(\Gamma)$, we state the main result of this paper.

Corollary 3.11. *Let the assumptions of Proposition 3.3 be satisfied. Then the dynamical system $(\mathcal{S}_t, \mathbb{X}^2)$ generated by the problem (P) has an $(\mathbb{X}^2, \mathbb{V}_w^p)$ -global attractor \mathcal{A}_p ; that is, \mathcal{A}_p is a compact subset of \mathbb{X}^2 (and weakly compact in \mathbb{V}^p), which attracts all bounded subsets B of \mathbb{X}^2 with respect to the strong and weak topologies of \mathbb{X}^2 and \mathbb{V}_w^p , respectively. More precisely,*

$$\lim_{t \rightarrow +\infty} \text{dist}_{\mathbb{Y}}(\mathcal{S}_t B, \mathcal{A}_p) = 0, \tag{3.32}$$

where \mathbb{Y} is either \mathbb{V}_w^p or \mathbb{X}^2 endowed with the corresponding topology.

Proof. According to standard results on the existence of the global attractor (see, e.g., [7]), we need to check that the semigroup \mathcal{S}_t is continuous in the \mathbb{X}^2 -topology for every bounded subset of \mathbb{X}^2 and that \mathcal{S}_t possesses a bounded in \mathbb{V}^p , compact in \mathbb{X}^2 and weakly compact in \mathbb{V}^p , absorbing set. The first is an immediate consequence of (2.41), whereas it follows from estimates (3.10), (3.26), that the C -ball in the space \mathbb{V}^p is an absorbing set for the semigroup \mathcal{S}_t if C is large enough. Since this ball is obviously compact in the \mathbb{X}^2 -topology and weakly compact in \mathbb{V}^p , the existence of the $(\mathbb{X}^2, \mathbb{V}_w^p)$ -global attractor \mathcal{A}_p follows from [7, 42, 44], while the convergence estimate (3.32) holds in the metrics of \mathbb{X}^2 and \mathbb{V}_w^p , respectively. \square

Remark 3.12. Theorems 3.7, 3.9 and Corollary 3.11 also imply the following smoothing property:

$$\mathcal{S}_t : \mathbb{X}^2 \rightarrow \mathbb{V}^p \cap \mathbb{X}^\infty, \quad \forall t > 0. \tag{3.33}$$

The global attractor \mathcal{A}_p is a bounded subset of $\mathbb{V}^p \cap \mathbb{X}^\infty$.

Remark 3.13. Concerning the quasi-linear parabolic equations (1.2), we can also handle dynamic boundary conditions that involve surface diffusion:

$$\partial_t \phi - \Delta_{m,\Gamma} \phi + |\nabla \phi|^{p-2} \partial_{\mathbf{n}} \phi + f_2(\phi) = g_2, \quad \text{on } \Gamma \times (0, +\infty), \tag{3.34}$$

where $\Delta_{m,\Gamma}$ is defined as the generalized m -Laplace-Beltrami operator on Γ ; that is,

$$\Delta_{m,\Gamma} \phi = \text{div}_\Gamma \left(|\nabla_\Gamma \phi|^{m-2} \nabla_\Gamma \phi \right), \quad m \in (1, +\infty).$$

In particular, $\Delta_{2,\Gamma} = \Delta_\Gamma$ is the well-known Laplace-Beltrami operator on Γ . Here, for any real-valued function ϕ ,

$$\text{div}_\Gamma \phi = \sum_{i=1}^{N-1} \partial_{\tau_i} \phi,$$

where $\partial_{\tau_i}\phi$ denotes the directional derivative of ϕ along the tangential directions τ_i at each point on the boundary, whereas $\nabla_{\Gamma}\phi = (\partial_{\tau_1}\phi, \dots, \partial_{\tau_{N-1}}\phi)$ is the tangential gradient at the boundary Γ . Under suitable assumptions on the nonlinearities f_k , $k = 1, 2$, and the other parameters of the problem, we can also show that such problems as (1.2), (3.34), (1.8) are well posed in suitable Banach spaces. They also possess global attractors and the solutions have properties similar to the ones obtained above for problem **(P)**.

Remark 3.14. Let us now recall the following functional $\mathcal{E} : \mathbb{V}^p \rightarrow \mathbb{R}$:

$$\mathcal{E}(\Phi) = \int_{\Omega} \left[\frac{1}{p} |\nabla\phi|^p + F_1(\phi) - g_1(x)\phi \right] dx + \int_{\Gamma} [F_2(\psi) - g_2(x)\psi] dS, \tag{3.35}$$

for all $\Phi = (\phi, \psi) \in \mathbb{V}^p$. If $(\phi, \psi) \in L^\infty((0, +\infty); \mathbb{V}^p) \cap C([0, +\infty); \mathbb{X}^2)$ is a weak solution of **(P)**, then from (1.6) we have

$$\frac{d}{dt} \mathcal{E}((\phi(t), \psi(t))) = - \|\partial_t\phi(t)\|_{L^2(\Omega)}^2 - \|\partial_t\psi(t)\|_{L^2(\Gamma)}^2, \text{ a.e. } t > 0. \tag{3.36}$$

It is worth mentioning that it would be interesting to show that the problem **(P)** has a gradient structure, which would require proving that \mathcal{E} is a continuous Lyapunov functional for the semigroup \mathcal{S}_t . Note, from Theorem 3.5, that the weak solution $(\phi(t), \psi(t))$ of (2.5) is only a weak continuous function from $(0, +\infty)$ with values in \mathbb{V}^p . However, we cannot show that $(\phi(t), \psi(t))$ is norm continuous from $(0, +\infty)$ to \mathbb{V}^p , unless we require additional assumptions on the nonlinearities f_1, f_2 , such as, $f_i = f_{i1} + f_{i2}$, $i = 1, 2$, with f_{i1} being monotone, satisfying similar growth conditions to f_i and with f_{i2} being a globally bounded Lipschitz perturbation. Thus, using monotone operator arguments (see, e.g., [47]; cf. also [4, Remark 3.6], [6]), we should be able to show that the connected global attractor \mathcal{A}_p , associated with the dynamical system $(\mathcal{S}_t, \mathbb{X}^2)$, coincides with the unstable manifold of the set Σ_p of stationary points of (2.5); that is, $\mathcal{A}_p = \mathcal{W}^u(\Sigma_p)$.

Let us also mention that it would also be interesting to investigate whether any bounded trajectory of problem **(P)** converges to a single equilibrium that is a stationary solution of (2.5). If f_i , $i = 1, 2$, are monotone, it should be possible to prove such a convergence via a suitable version of the Łojasiewicz-Simon inequality, proven in [9] for the quasilinear parabolic equations (1.2) with Dirichlet boundary conditions. We will come back to these issues in a forthcoming article.

Remark 3.15. In contrast to the case of non-degenerate parabolic PDE's, not much seems to be known about the long-term behavior of quasilinear

parabolic equations in general. Indeed, even though we are dealing with a parabolic problem in a bounded domain, the global attractor \mathcal{A}_p seems to have infinite fractal dimension when $p > 2$. Indeed, it was observed in [14] for the parabolic equation with p -Laplacian (1.2), subject to Dirichlet boundary conditions, that the ε -Kolmogorov entropy of the attractor behaves as a polynomial of $1/\varepsilon$ as $\varepsilon \rightarrow 0$, thus resulting in an infinite-dimensional attractor.

However, since the parabolic equation (1.2) possesses the finite speed propagation property, we can employ similar arguments to show that the global attractor associated with problem **(P)** can also be infinite dimensional when $p > 2$. A natural question arises: Under what conditions on the nonlinearities f, g is the global attractor finite dimensional? All these issues will be the topic of future investigation.

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