Differential and Integral Equations Volume 23, Numbers 3-4 (2010), 303–326

MARTINGALE AND STATIONARY SOLUTIONS FOR STOCHASTIC NON-NEWTONIAN FLUIDS

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(Submitted by: Giuseppe Da Prato)

Abstract. Stochastic non-Newtonian fluids with multiplicative noise are studied under the case of shear thinning and shear thickening and the existence of martingale solutions and stationary solutions is achieved for the first time.

1. Introduction

As is well known, the Navier-Stokes model of fluid restricts the linear relation between the stress tensor and the velocity gradient (see [24, 25]); fluids satisfying such constitutive relationships are called Newtonian fluids, e.g., water and air. However, for many fluid materials, such as molten plastics, synthetic fibers, biological fluids, paints and greases, etc., their flow behavior cannot be characterized by Newtonian relationships in real life. By relaxing the constraints of the Stokes hypothesis, the mathematical theory of viscous non-Newtonian fluids generalizes the usual Stokes model in three important respects: nonlinear constitutive relations between the viscous part of the stress tensor and velocity gradients; dependence of the viscous stress on velocity gradients of order two or higher; constitutive relations for higher-order stress tensors which must be present in the balance of energy equation as soon as higher-order velocity gradients are admitted into the theory ([5, 15]).

Accepted for publication: July 2009.
AMS Subject Classifications: 60H15, 76A05.

The viscous non-Newtonian fluids which allow for nonlinearity in the constitutive theory have been extensively studied, see for example Ladyzhenskaya [18, 19], Du and Gunzburger [13]. J.L. Lions proposed to replace the negative Laplacian $-\Delta$ by the term $-\Delta + k(-\Delta)^{\frac{l}{2}}, l > 2, k > 0$ (in a way that is reminiscent of Taylor expansions) and considered such a modified problem in [20].

In fact, when considering the propagation of weakly nonlinear waves in random media (plasma, atmosphere, fluid etc.), a random forcing term has to be added to the equation (see [17, 23]). Many authors have proposed the study of stochastic equations, and there has been much work related to stochastic equations and dynamical systems, such as the stochastic Navier-Stokes equation, KdV equation, Burgers equation, Schrödinger equation etc., see [6]-[12] for these topics and the progress in these fields.

Up to now, to the best of our knowledge, most of the well-known results of the non-Newtonian fluids are deterministic; the non-trivial effects of random factors are not included among all the former literature. In this paper, we study stochastic isothermal nonlinear incompressible bipolar non-Newtonian fluids in a bounded domain, and are interested in the following case when the forcing term is of multiplicative noise type.

Let $n = 2$ or 3, $D \subset R^n$ be a bounded open domain,

$$du + \big(u \cdot \nabla u - \nabla \cdot \tau(e(u)) + \nabla \pi\big)dt = f(x)dt + G(u)dW(t) \qquad (1.1)$$
$$x \in D,\ t \in [0, T],$$

$$u(x, 0) = u_0(x), \quad x \in D, \qquad (1.2)$$

$$\nabla \cdot u = 0, \qquad (1.3)$$

subject to the boundary condition

$$u(x, t) = 0, \quad \tau_{ijl}n_j n_l = 0, \qquad x \in \partial D,\, t \in [0, T], \qquad (1.4)$$

where $\tau_{ijl} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_l}$ $(n = 2, i, j, l = 1, 2; n = 3, i, j, l = 1, 2, 3)$, and $n = (n_1, n_2)(n = (n_1, n_2, n_3))$ denotes the exterior unit normal to the boundary ∂D. The first condition in (1.4) represents the usual no-slip condition associated with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on ∂D, a direct consequence of the principle of virtual work. The unknown vector function u denotes the velocity of the fluid, f is a deterministic forcing term, and the scalar function π represents the pressure, $\tau(e(u))$ is a symmetric stress tensor. There are many fluid materials, for example, liquid foams, polymeric fluids such as oil in water, blood, etc., whose viscous stress tensors are represented by the form

$$\tau(e(u)) = 2\mu_0(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e(u) - 2\mu_1 \Delta e(u), \quad \epsilon > 0, \tag{1.5}$$

$$e(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \quad |e(u)|^2 = \sum_{i,j=1}^{n} |e_{ij}(u)|^2, \quad n = 2 \text{ or } 3,$$

where $\mu_0 > 0$ and $\mu_1 > 0$ are constants.

Ladyzhenskaya [18, 19] considered the existence and uniqueness of weak solutions for the monopolar fluids model (in the case of $\mu_1 = 0$). Bleustein and Green [4] presented an early model for bipolar fluids based on continuum thermodynamics. Necãs and Šilhavy [21] developed a theory of constitutive equations for multipolar viscous fluids based on the general ideas of Green and Rivlin [16]; the theory is compatible with the principle of material-frame indifference and the second law of thermodynamics as expressed by the Clausius-Duhem inequality. Bellout et al. in [1] expanded upon some of the consequences of the multipolar fluid model; they first placed the emphasis on the nonlinear isothermal incompressible bipolar case, and proposed the constitutive form of (1.5). After that, Pokorny [22] and Bellout et al. [3] investigated the Cauchy problem and the initial-boundary-value problem for incompressible viscous non-Newtonian fluids respectively. Zhao and Zhou [26] studied the pullback attractors for the incompressible non-Newtonian fluid with the constitutive form (1.5). One can refer to [2, 5] and the references therein for more details.

We use $W(t)$ to describe the cylindrical Wiener process for white noise in a Hilbert space H defined on the stochastic probability basis $(\Omega, \mathcal{F}, \mathbb{P})$; further assumptions on $G(u)$ will be given below. Obviously, for $p = 2$, $\mu_1 = G(u) = 0$, Equation (1.1) is a deterministic equation and reduces to the Navier-Stokes equation. For $\mu_0 = \mu_1 = G(u) = 0$, it is the Euler equation. The fluids are shear thinning when $1 < p < 2$ and shear thickening when $p > 2$.

Our purpose in this paper is to study the existence of solutions to (1.1)-(1.4). We use the method developed by Flandoli and Gatarek [14] to construct martingale solutions of stochastic partial differential equations. The following three steps are key to proving the existence of martingale solutions.

First, the existence of martingale solutions to approximating finite-dimensional problems whose nonlinear terms satisfy linear growth and continuity condition is obtained; in particular, we employ the method of truncated functions and Lipschitz continuous properties of nonlinear terms for u in the proof.

Second, in order to prove the family of laws is tight, we need some norm estimates with expectation of solutions to approximating finite-dimensional problems.

Third, taking the limit of approximating finite-dimensional problems, the martingale solutions of equations (1.1)-(1.4) can be obtained with the aid of a representation theorem for martingales.

In particular, in the shear thinning case $1 < p < 2$ ($n = 2, 3$) and the shear thickening case $2 < p \leq 2 + \frac{2}{n+2}$ ($n = 2, 3$), the nonlinear term including p is dealt with distinct methods in the steps above and the existence of martingale solutions is obtained; further details can be found in section 3.

Referring to stationary solutions, in this paper we show that a stationary martingale solution can be constructed as the limit of stationary solutions of approximating finite-dimensional problem. With this approach it is sufficient to show that the family of laws

$$\{\mathcal{L}(u_m(t)) : t \geq 0, m \geq 1\}$$

is bounded in probability (see Theorem 11.29 P_{338} [10]).

For the sake of convenience in the following contexts, first we set some notation.

$L^q(D)$ will be the Lebesgue space with norm $\|\cdot\|_{L^q}$; in particular, $\|\cdot\|_{L^2} = \|\cdot\|$, and $\|u\|_{L^\infty} = ess\sup_{x\in D} |u(x)|$, for $q = \infty$; $W^{\sigma,q}(D)$ will be the Sobolev space $\{u \in L^q(D) : D^k u \in L^q(D), \quad k \leq \sigma\}$ with norm $\|\cdot\|_{\sigma,q}$, $\mathcal{C}(I, X)$ the space of continuous functions from the interval I to X, $L^q(0, T; X)$ the space of all measurable functions $u : [0, T] \mapsto X$, with the norm

$$\|u\|^q_{L^q(0,T;X)} = \int_0^T \|u(t)\|^q_X dt,$$

and when $q = \infty$, $\|u\|_{L^\infty(0,T;X)} = ess\sup_{t\in[0,T]} \|u(t)\|_X$.

Define a space of smooth functions with compact support strictly contained in D, and satisfying the divergence free condition

$$\mathcal{V} = \{u \in C_c^\infty(D) : \nabla \cdot u = 0\}.$$

H is the closure of $\mathcal{V}$ in $L^2(D)$ with norm $\|\cdot\|$, and $(\cdot, \cdot)$ is the inner product of H, $H^\sigma(D)$ is the closure of $\mathcal{V}$ in $W^{\sigma,2}(D)$ with norm $\|\cdot\|_\sigma$, and $H^{-\sigma}(D)$ is the dual space ($\sigma \geq 1$). In particular, when $\sigma = 2$, $V = H^2(D)$ and V' is the dual space of V.

We define similarly as above the spaces $L^q(\Omega, X)$ and $\mathcal{L}(u(t))$ denotes the family of laws of $u(t)$.

Denote by $L_2(X,Y)$ the space of Hilbert-Schmidt operators from X to Y, then $L_2^{0,\sigma}$ denotes the space of Hilbert-Schmidt operators from H to $H^\sigma(D)$. In particular, $L_2^{0,0}$ denotes the space of Hilbert-Schmidt operators from H to H.

If G is an operator acting on solutions u, we assume that

(G_1): $G : H \to L_2^{0,0}$ is continuous,

(G_2): $\|G(u)\|^2_{L_2^{0,0}} \leq \lambda_0 \|u\|^2 + \rho, \quad u \in H,$

where $\lambda_0,\ \rho > 0$ are constants. For further details about the cylindrical Wiener process and the Hilbert-Schmidt operator, one can refer to [10].

The paper is organized as follows. In section 2, we recall some definitions and present some lemmas as an introduction and preparation. In section 3, we prove the existence of martingale solutions to stochastic non-Newtonian fluids. In section 4, we prove the existence of stationary martingale solutions to stochastic non-Newtonian fluids.

For notational simplicity, C is a generic constant and may assume various values from line to line.

2. Preliminaries

We introduce the linear operator A as follows: consider the positive definite V-elliptic symmetric bilinear form $a(\cdot,\cdot) : V \times V \to R$ given by

$$a(u,v) = \int_D \frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(v)}{\partial x_k} dx\,, \quad (u,\, v \in V).$$

As a consequence of the Lax-Milgram lemma, we obtain

$$< Au, v >_{V' \times V} = a(u,v) = < f, v >_{V' \times V}\,, \quad \forall v \in V,$$

where V' is the dual space of V, and the domain of A is

$$D(A) = \{u \in V\ : a(u,v) = (f,v), \quad f \in H \subset V', \quad \forall v \in V\}.$$

In fact $A = P\Delta^2$ and P is the projection from $L^2(D)$ to H.

According to the Rellich theorem, A^{-1} is compact in H, thus

$$A\phi_n = \lambda_n \phi_n\ ,\ \ \phi_n \in D(A), \tag{2.1}$$

where $\{\phi_n\}_{n=1}^\infty$ are the eigenfunctions and also are a basis of V, $\lambda_n > 0, \lambda_n \to \infty$, when $n \to \infty$.

In place of equations (1.1)-(1.4) we shall consider the following abstract problems in H:

$$du + [\mu_1 Au + 2\mu_0 A_p u + B(u,u)]dt = f dt + G(u) dW(t)\,, \quad t > 0\,, \tag{2.2}$$

$$u(0) = u_0\,, \tag{2.3}$$

where $A = P\Delta^2$, and $B(u,u) = P(u \cdot \nabla u)$.

We also define a continuous bilinear operator $B(\cdot,\cdot) : H^1(D) \times H^1(D) \to H^{-1}(D)$ as follows:

$$(B(u,\upsilon),\varpi) = \int_D u_i \frac{\partial \upsilon_j}{\partial x_i} \varpi_j dx\,, \quad u,\upsilon,\varpi \in H^1(D),$$

which has the properties

$$(B(u,\upsilon),\varpi) = -(B(u,\varpi),\upsilon), \quad \text{and} \quad (B(u,\upsilon),\upsilon) = 0.$$

For $u \in V$, the operator $A_p(\cdot) : V \to V'$ is defined by

$$\big(A_p(u),\upsilon\big) = \int_D \gamma(u) e_{ij}(u) e_{ij}(\upsilon) dx, \quad u,\upsilon \in V,$$

where $\gamma(u) = (\epsilon + |e(u)|^2)^{\frac{p-2}{2}}$.

Some properties of the operator A_p are presented as follows.

Lemma 2.1. *For $1 < p < 2$ $(n = 2,3)$, and $2 < p \leq 2 + \frac{2}{n+2}$ $(n = 2,3)$, $A_p(\cdot) : V \to V'$ is locally Lipschitz continuous. Namely,*

$$\|A_p(u) - A_p(\upsilon)\|_{V'} \leq C(\|u\|_V, \|\upsilon\|_V, p, \epsilon)\|u - \upsilon\|_V \quad \forall\, u,\upsilon \in V.$$

Proof. Case 1: for all $\varpi \in V$, for $n = 2,3$, $1 < p < 2$,

$$\begin{aligned}
&|(A_p(u),\varpi) - (A_p(\upsilon),\varpi)| \\
=&\Big| \int_D [(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e_{ij}(u) - (\epsilon + |e(\upsilon)|^2)^{\frac{p-2}{2}} e_{ij}(\upsilon)] e_{ij}(\varpi) dx \Big| \\
\leq&\Big| \int_D (\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e_{ij}(u-\upsilon) e_{ij}(\varpi) dx \Big| \\
&+ \Big| \int_D [(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} - (\epsilon + |e(\upsilon)|^2)^{\frac{p-2}{2}}] e_{ij}(\upsilon) e_{ij}(\varpi) dx \Big| \\
=&\mathcal{A}_1 + \mathcal{A}_2.
\end{aligned} \tag{2.4}$$

Obviously,

$$\mathcal{A}_1 \leq \epsilon^{\frac{p-2}{2}} \|e(u-\upsilon)\| \|e(\varpi)\| \leq C\|u-\upsilon\|_1 \|\varpi\|_1 \leq C\|u-\upsilon\|_2 \|\varpi\|_2. \tag{2.5}$$

As to the estimate of $\mathcal{A}_2$, if $|e(u)| < |e(\upsilon)|$, by the mean value theorem, there exists m such that $|e(u)| < m < |e(\upsilon)|$, and

$$\begin{aligned}
&(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} - (\epsilon + |e(\upsilon)|^2)^{\frac{p-2}{2}} = \frac{p-2}{2}(\epsilon + m^2)^{\frac{p-4}{2}} (|e(u)|^2 - |e(\upsilon)|^2) \\
&\quad \leq \frac{p-2}{2}(\epsilon + |e(\upsilon)|^2)^{\frac{p-4}{2}} (2|e(\upsilon)|)(|e(u)| - |e(\upsilon)|)
\end{aligned}$$

$$\leq (p-2)(\epsilon + |e(\upsilon)|^2)^{\frac{p-4}{2}} |e(\upsilon)||e(u) - e(\upsilon)|, \tag{2.6}$$

thus

$$\begin{aligned} \mathcal{A}_2 &\leq (p-2) \int_D (\epsilon + |e(\upsilon)|^2)^{\frac{p-2}{2}} |e(u-\upsilon)||e(\varpi)| dx \\ &\leq (p-2)\epsilon^{\frac{p-2}{2}} \|e(u-\upsilon)\| \|e(\varpi)\| \leq C\|u-\upsilon\|_1 \|\varpi\|_1 \leq C\|u-\upsilon\|_2 \|\varpi\|_2 \,. \end{aligned} \tag{2.7}$$

If $|e(u)| > |e(\upsilon)|$, the result can be obtained similarly.

Case 2: For all $\varpi \in V$, for $n = 2,3$, $2 < p \leq 2 + \frac{2}{n+2}$,

$$\begin{aligned} &|(A_p(u), \varpi) - (A_p(\upsilon), \varpi)| \leq |\int_D (\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e_{ij}(u-\upsilon) e_{ij}(\varpi) dx| \\ &+ |\int_D [(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} - (\epsilon + |e(\upsilon)|^2)^{\frac{p-2}{2}}] e_{ij}(\upsilon) e_{ij}(\varpi) dx| \\ =& I_1 + I_2; \end{aligned}$$

using the Hölder inequality and the Sobolev embedding $H^1(D) \hookrightarrow L^6(D)$, for $n = 2,3$, it follows that

$$I_1 \leq C[\int_D (\epsilon + |e(u)|^2)^{\frac{3(p-2)}{4}} dx]^{\frac{2}{3}} \|u-\upsilon\|_2 \|\varpi\|_2.$$

Noticing that $2 < p \leq \frac{5}{2}$ (respectively $2 < p \leq \frac{12}{5}$) and $(a+b)^m \leq a^m + b^m$, for $0 < m < 1$, it follows that

$$\int_D (\epsilon + |e(u)|^2)^{\frac{3(p-2)}{4}} dx \leq \int_D (\epsilon^{\frac{3(p-2)}{4}} + |e(u)|^{\frac{3(p-2)}{2}}) dx;$$

we therefore obtain

$$\begin{aligned} I_1 &\leq C[\epsilon^{\frac{p-2}{2}} |D|^{\frac{2}{3}} + |D|^{\frac{10-3p}{6}} (\int_D |e(u)|^2 dx)^{\frac{p-2}{2}})] \|u-\upsilon\|_2 \|\varpi\|_2 \\ &\leq C(\epsilon^{\frac{p-2}{2}} |D|^{\frac{2}{3}} + |D|^{\frac{10-3p}{6}} \|u\|_2^{p-2}) \|u-\upsilon\|_2 \|\varpi\|_2, \end{aligned}$$

where we have used the Hölder inequality.

By the mean value theorem, if $|e(u)| < |e(\upsilon)|$, there exists ζ such that $|e(u)| < \zeta < |e(\upsilon)|$, thus

$$(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} - (\epsilon + |e(\upsilon)|^2)^{\frac{p-2}{2}} \leq (p-2)(\epsilon + |e(\upsilon)|^2)^{\frac{p-4}{2}} |e(\upsilon)|(|e(u) - e(\upsilon)|) \tag{2.8}$$

and

$$\begin{aligned} I_2 &\leq (p-2) \int_D (\epsilon + |e(\upsilon)|^2)^{\frac{p-2}{2}} |e(u-\upsilon)||e(\varpi)| dx \\ &\leq C(\epsilon^{\frac{p-2}{2}} |D|^{\frac{2}{3}} + |D|^{\frac{10-3p}{6}} \|\upsilon\|_2^{p-2}) \|u-\upsilon\|_2 \|\varpi\|_2. \end{aligned} \tag{2.9}$$

If $|e(u)| > |e(v)|$, the result can be obtained similarly.

From the estimates of the two cases, we have

$$|(A_p(u), \varpi) - (A_p(v), \varpi)| \leq C\|u - v\|_2 \|\varpi\|_2,$$

thus, $A_p(\cdot)$ is locally Lipschitz continuous from V to V'. □

Lemma 2.2. *Assume that u belongs to $L^2(0, T; V) \bigcap L^\infty(0, T; H)$. Then $A_p(u)$ belongs to $L^2(0, T; V')$.*

Proof. Case 1: For all $v \in V$, and $n = 2\,, 3$, $1 < p < 2$,

$$\begin{aligned} |(A_p(u), v)| &= |(\nabla \cdot [(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e_{ij}(u)], v)| \\ &\leq \int_D |(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e_{ij}(u) e_{ij}(v)| dx \\ &\leq \epsilon^{\frac{p-2}{2}} \int_D |e(u)||e(v)| dx \leq C\|u\|_1 \|v\|_1 \leq C\|u\|_2 \|v\|_2. \quad (2.10) \end{aligned}$$

Therefore,

$$\int_0^T \|A_p(u(t))\|_{V'}^2 dt \leq C \int_0^T \|u(t)\|_2^2 dt < \infty. \quad (2.11)$$

Case 2: For all $v \in V$, and $n = 2$, $2 < p \leq \frac{5}{2}$,

$$\begin{aligned} |(A_p(u), v)| &= |\int_D (\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e_{ij}(u) e_{ij}(v) dx| \\ &\leq \epsilon^{\frac{p-2}{2}} \int_D |e(u)||e(v)| dx + \int_D |e(u)|^{p-1} |e(v)| dx \\ &\leq C\|u\|_1 \|v\|_1 + \|v\|_1 (\int_D |e(u)|^{2p-2} dx)^{\frac{1}{2}}, \\ &\leq C(\|u\|_2 + (\int_D |e(u)|^{2p-2} dx)^{\frac{1}{2}}) \|v\|_2. \quad (2.12) \end{aligned}$$

Thus,

$$\int_0^T \|A_p(u(s))\|_{V'}^2 ds \leq C(\int_0^T \|u(s)\|_2^2 ds + \int_0^T \|\nabla u\|_{L^{2p-2}}^{2p-2} ds);$$

using the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^{2p-2}}^{2p-2} \leq C\|u\| \; \|u\|_2^{2p-3}, \quad n = 2,$$

$$\int_0^T \|\nabla u(s)\|_{L^{2p-2}}^{2p-2} ds \leq C \sup_{0 \leq s \leq T} \|u(s)\| \int_0^T \|u(s)\|_2^{2p-3} ds, \quad (2.13)$$

noticing $2 < p \le \frac{5}{2}$, so that $1 < 2p-3 \le 2$, then using the Hölder inequality,

$$\int_0^T \|u(s)\|_2^{2p-3} ds \le CT^{\frac{5-2p}{2}} (\int_0^T \|u(s)\|_2^2 ds)^{\frac{2p-3}{2}}.$$

Therefore,

$$\int_0^T \|A_p(u(s))\|_{V'}^2 ds \le C(\int_0^T \|u(s)\|_2^2 ds + (\int_0^T \|u(s)\|_2^2 ds)^{\frac{2p-3}{2}}) < \infty.$$

Case 3: For all $v \in V$, and $n = 3$, $2 < p \le \frac{12}{5}$, we have the following Gagliardo-Nirenberg inequality:

$$\|\nabla u\|_{L^{2p-2}}^{2p-2} \le C\|u\|^{\frac{4-p}{2}} \|u\|_2^{\frac{5p-8}{2}}, \quad n = 3,$$

$$\int_0^T \|\nabla u\|_{L^{2p-2}}^{2p-2} ds \le C \sup_{0\le s\le T} \|u(s)\|^{\frac{4-p}{2}} \int_0^T \|u(s)\|_2^{\frac{5p-8}{2}} ds; \tag{2.14}$$

noticing $2 < p \le \frac{12}{5}$, so that $1 < \frac{5p-8}{2} \le 2$, then using the Hölder inequality,

$$\int_0^T \|u(s)\|_2^{\frac{5p-8}{2}} ds \le CT^{\frac{12-5p}{4}} (\int_0^T \|u(s)\|_2^2 ds)^{\frac{5p-8}{4}}.$$

Therefore,

$$\int_0^T \|A_p(u(s))\|_{V'}^2 ds \le C(\int_0^T \|u(s)\|_2^2 ds + (\int_0^T \|u(s)\|_2^2 ds)^{\frac{5p-8}{4}}) < \infty.$$

□

Definition 2.1. *Let H be a separable Hilbert space; given $1 < q < \infty$, $\alpha \in (0,1)$, let $W^{\alpha,q}(0,T;H)$ be the Sobolev space of all $u \in L^q(0,T;H)$ such that*

$$\int_0^T \int_0^T \frac{\|u(t)-u(s)\|^q}{|t-s|^{1+\alpha q}} dt\, ds < \infty,$$

endowed with the norm

$$\|u\|_{W^{\alpha,q}(0,T;H)}^q = \int_0^T \|u(t)\|^q dt + \int_0^T \int_0^T \frac{\|u(t)-u(s)\|^q}{|t-s|^{1+\alpha q}} dt\, ds.$$

Definition 2.2. *For any progressively measurable process $g \in L^2(\Omega; L^2(0,T; L_2(X,Y)))$ denoted by $I(g)$, the Itô integral is defined as*

$$I(g)(t) = \int_0^t g(s) dW(s), \qquad t \in [0,T].$$

Clearly, $I(g)$ is a progressively measurable process in $L^2(\Omega; L^2(0,T;Y))$.

Definition 2.3. *We say that there exists a Martingale solution of the equations* (2.2)-(2.3) *if there exist a stochastic basis* $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in[0,T]}, \mathbb{P})$, *a cylindrical Wiener process on the space* H *and a progressively measurable process* $u : [0,T] \times \Omega \to H$, *with* $\mathbb{P}$*-a.e. paths*

$$u(\cdot, \omega) \in \mathcal{C}(0,T; H^{-3}(D)) \cap L^{\infty}(0,T;H) \cap L^2(0,T;V),$$

such that $\mathbb{P}$*-a.s. the identity*

$$\begin{aligned}(u(t), v) = (u_0, v) - \mu_1 \int_0^t (Au(s), v)ds - \int_0^t (B(u(s), u(s)), v)ds \qquad (2.15)\\ - 2\mu_0 \int_0^t (A_p(u(s)), v)ds + \int_0^t (f, v)ds + \int_0^t (G(u(s))dW(s), v)\end{aligned}$$

holds true for all $t \in [0,T]$, *and all* $v \in H^3(D)$.

Remark. If $u \in L^{\infty}(0,T;H) \cap \mathcal{C}(0,T;H^{-3}(D))$, then $u \in H$, for all $t \in [0,T]$ (see [24]).

Definition 2.4. *We say that a stationary Martingale solution consists of a stochastic basis* $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$, *a cylindrical Wiener process on the space* H *and a progressively measurable process* $u : [0,\infty) \times \Omega \to H$, *with* $\mathbb{P}$*-a.e. paths, such that*

$$u(\cdot, \omega) \in \mathcal{C}(0,T; H^{-3}(D)) \cap L^{\infty}(0,T;H) \cap L^2(0,T;V),$$

for all $T > 0$, u *is a stationary process in* H, *and* $\mathbb{P}$*-a.s. the identity*

$$\begin{aligned}(u(t), v) = (u(\tau), v) - \mu_1 \int_\tau^t (Au(s), v)ds - \int_\tau^t (B(u(s), u(s)), v)ds \qquad (2.16)\\ - 2\mu_0 \int_\tau^t (A_p(u(s)), v)ds + \int_\tau^t (f, v)ds + \int_\tau^t (G(u(s))dW(s), v)\end{aligned}$$

holds true for all $t \geq \tau \geq 0$, *and all* $v \in H^3(D)$. *Of course it is equivalent to take just* $\tau = 0$.

Obviously, for the solution, the properties of the operator A_p hold true.

Lemma 2.3. [14] *Let* $0 < \alpha < \frac{1}{2}$, $q \geq 2$; *for any progressively measurable process* $g \in L^q(\Omega; L^q(0,T;(L_2(X,Y)))$, *we have* $I(g) \in L^q(\Omega; W^{\alpha,2}(0,T;H))$.

Theorem 2.1. [14] *Let* $B_0 \subset B \subset B_1$ *be Banach spaces,* B_0 *and* B_1 *reflexive, with a compact embedding of* B_0 *in* B. *Let* $q \in (1,\infty)$ *and* $\alpha \in (0,1)$ *be given. Let* X *be the space* $X = L^q(0,T;B_0) \cap W^{\alpha,q}(0,T;B_1)$ *endowed with the natural norm; then the embedding of* X *in* $L^q(0,T;B)$ *is compact.*

Theorem 2.2. [14] *If $B_1 \subset \tilde{B}$ are two Banach spaces with compact embedding, and the real numbers $\alpha \in (0,1)$, $q > 1$ satisfy $\alpha q > 1$, then the space $W^{\alpha,q}(0,T;B_1)$ is compactly embedded into $\mathcal{C}(0,T;\tilde{B})$.*

3. Martingale solutions

In this section, we first construct approximating finite-dimensional problems, then prove the family of laws $\{\mathcal{L}(u_m(t))\}$ is tight in some chosen Hilbert spaces, by the Skorohod embedding theorem and representation theorem for martingales, the existence of martingale solutions of (2.2)-(2.3) is obtained.

Theorem 3.1. *If $u_0 \in H$, $f \in H$, and $G(u)$ satisfies the assumed conditions G_1, G_2, for $1 < p < 2$ $(n = 2,3)$, and $1 < p \le 2 + \frac{2}{n+2}$ $(n = 2,3)$, there exists a martingale solution of* (2.2)-(2.3), *in the sense of Definition* 2.3.

Proof. First step: Galerkin approximations. $B(u,u)$ is locally Lipschitz from $H^1(D)$ to V' (see [14]). From Lemma 2.1, we know that $A_p(u)$ also is locally Lipschitz continuous from V to V'.

Let P_m be the projection from V' to V defined as

$$P_m x = \sum_{i=1}^{m} (x, \phi_i)\phi_i, \qquad x \in V',$$

where ϕ_i is defined in (2.1). Its restriction to H is the orthogonal projection onto the space spanned by $\phi_1, \phi_2 ...... \phi_m$.

Noticing $V \subset H \subset V'$, denote by $(\cdot,\cdot)$ also the dual pairing between V and V'. Moreover, it satisfies

$$(P_m x, y) = (x, P_m y),$$

for all $x, y \in V'$. Let

$$u_m(t) = \sum_{i=1}^{m} (u(t), \phi_i)\phi_i.$$

Define $B_m(u,u) = \chi_m(u)B(u,u)$, $A_p^m(u) = -\nabla\cdot[\chi_m(u)(\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e_{ij}(u)]$, for $u \in P_m H$, where $\chi_m : H \to R$ is defined as $\chi_m(u) = \theta_m(\|u\|)$, with $\theta_m : R \to [0,1]$ of class C^∞, such that

$$\begin{cases} \chi_m(u) = 1 & \|u\| \le m, \\ \chi_m(u) = 0 & \|u\| > m+1. \end{cases} \tag{3.1}$$

Consider the classical Faedo-Galerkin approximation, defined by the process $u_m \in P_mH$, the solution of the following equation:

$$\begin{cases} du_m + \mu_1 Au_m dt + P_m B_m(u_m, u_m)dt + 2\mu_0 P_m A_p^m(u_m)dt \\ \qquad\qquad\qquad = P_m f dt + P_m G(u_m)dW(t), \\ u_m(0) = P_m u_0. \end{cases} \tag{3.2}$$

Noticing that all the coefficients of this system are continuous and with linear growth in P_mH, we thus know that the equation (3.2) has a martingale solution $u_m(t) \in L^2(\Omega, \mathcal{C}(0,T; P_mH))$ (see [10]).

Second step: Some energy estimates with expectation. We apply the Itô formula to $\|u_m(t)\|^{p'}$ for $p' \geq 2$, then

$$\begin{aligned} d\|u_m(t)\|^{p'} \leq & p'\|u_m(t)\|^{p'-2}(\mathcal{Q}u_m(t), u_m(t))dt + p'\|u_m(t)\|^{p'-2}(f, u_m(t))dt \\ & + p'\|u_m(t)\|^{p'-2}(P_m G(u_m(t)), u_m(t))dW(t) \\ & + \frac{1}{2}p'(p'-1)\|u_m(t)\|^{p'-2}\|P_m G(u_m(t))\|^2_{L_2^{0,0}} dt, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mathcal{Q}u_m = & 2\mu_0 P_m[\nabla \cdot \left(\chi_m(u_m)(\epsilon + |e(u_m)|^2)^{\frac{p-2}{2}} e_{ij}(u_m)\right)] - \mu_1 \Delta^2 u_m \\ & - P_m(\chi_m(u_m)(u_m \cdot \nabla u_m)) \\ = & -\mu_1 A u_m - 2\mu_0 P_m A_p^m u_m - P_m B_m(u_m, u_m), \end{aligned}$$

thus

$$\begin{aligned} & (\mathcal{Q}u_m(t), u_m(t)) \\ & = (-\mu_1 \Delta^2 u_m, u_m) - 2\mu_0(P_m A_p^m u_m, u_m) - (P_m B_m(u_m, u_m), u_m), \end{aligned}$$

owing to the divergence-free condition

$$(P_m(u_m \cdot \nabla u_m), u_m) = (u_m \cdot \nabla u_m, u_m) = 0,$$

thus

$$\begin{aligned} & (\mathcal{Q}u_m(t), u_m(t)) \\ & = -\mu_1\|\Delta u_m\|^2 - 2\mu_0 \int_D \chi_m(u_m)(\epsilon + |e(u_m)|^2)^{\frac{p-2}{2}} |e(u_m)|^2 dx, \end{aligned} \tag{3.4}$$

because the value range of $\chi_m(u_m)$ is $(0,1)$ and the second term in the right-hand side of (3.4) is less than zero.

Combining the restrictions on $G(u)$, we can deduce that

$$d\|u_m(t)\|^{p'} + \mu_1 p'\|u_m(t)\|^{p'-2}\|u_m(t)\|_2^2 dt$$

$$\leq p'\|u_m(t)\|^{p'-2}(f,u_m)dt + p'\|u_m(t)\|^{p'-2}(P_mG(u_m(t)),u_m(t))dW(t)$$
$$+\frac{1}{2}p'(p'-1)\|u_m(t)\|^{p'-2}(\lambda_0\|u_m(t)\|^2+\rho)dt. \tag{3.5}$$

Obviously, for the first term in the right-hand side of the above inequality, we have

$$p'\|u_m(t)\|^{p'-2}(f,u_m) \leq \frac{\mu_1 p'}{2}\|u_m\|^{p'-2}\|u_m\|_2^2 + \frac{p'\|f\|^2}{2\mu_1}\|u_m(t)\|^{p'-2}$$

and

$$\frac{p'\|f\|^2}{2\mu_1}\|u_m(t)\|^{p'-2} \leq \frac{(p'-2)\|f\|^2}{2\mu_1}\|u_m\|^{p'} + \frac{\|f\|^2}{\mu_1}.$$

As for the last term in (3.5),

$$\frac{\rho p'(p'-1)}{2}\|u_m(t)\|^{p'-2} \leq \frac{\rho(p'-1)(p'-2)}{2}\|u_m\|^{p'} + \rho(p'-1).$$

From the above estimates it follows that

$$d\|u_m(t)\|^{p'} + \frac{\mu_1 p'}{2}\|u_m(t)\|^{p'-2}\|u_m(t)\|_2^2 dt \tag{3.6}$$
$$\leq [\frac{\lambda_0}{2}p'(p'-1) + \frac{(p'-2)\|f\|^2}{2\mu_1} + \frac{\rho(p'-1)(p'-2)}{2}]\|u_m(t)\|^{p'}dt$$
$$+ (\frac{\|f\|^2}{\mu_1} + \rho(p'-1))dt + p'\|u_m(t)\|^{p'-2}(P_mG(u_m(t)),u_m(t))dW(t)$$

and

$$\mathbb{E}(\|u_m(t)\|^{p'}) \leq \mathbb{E}\|u_{0m}\|^{p'} \tag{3.7}$$
$$+\int_0^t [\frac{\lambda_0 p'(p'-1)}{2} + \frac{(p'-2)\|f\|^2}{2\mu_1} + \frac{\rho(p'-1)(p'-2)}{2}]\mathbb{E}(\|u_m(s)\|^{p'})ds$$
$$+\int_0^t [\frac{\|f\|^2}{\mu_1} + \rho(p'-1)]ds.$$

Applying the Gronwall inequality, we have

$$\sup_{t\in[0,T]} \mathbb{E}(\|u_m(t)\|^{p'}) \leq C, \quad m \geq 1.$$

Meanwhile, we can obtain that

$$\int_0^T \mathbb{E}(\|u_m(s)\|^{p'-2}\|u_m(s)\|_2^2)ds \leq C, \tag{3.8}$$

where the constant is independent of m. In particular, for $p' = 2$,

$$\int_0^T \mathbb{E}(\|u_m(s)\|_2^2)ds \leq C. \tag{3.9}$$

Using the martingale inequality (see [10] Theorem 3.14) and condition G_2 for $G(u)$, we have

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\int_0^s p'\|u_m(\sigma)\|^{p'-2}(P_mG(u_m(\sigma))dW(\sigma), u_m(\sigma)) \\
&\leq C\mathbb{E}((\int_0^t \|u_m(s)\|^{2p'-2}\|G(u_m(s))\|^2_{L_2^{0,0}}ds)^{\frac{1}{2}}) \\
&\leq C\mathbb{E}(\sup_{s\in[0,t]}\|u_m(s)\|^{\frac{p'}{2}}(\int_0^t \|u_m(s)\|^{p'-2}(\lambda_0\|u_m(s)\|^2+\rho)ds)^{\frac{1}{2}}) \\
&\leq \frac{1}{2}\mathbb{E}(\sup_{s\in[0,t]}\|u_m(s)\|^{p'}) + C\lambda_0\mathbb{E}\int_0^t \sup_{s\in[0,\sigma]}\|u_m(s)\|^{p'}d\sigma + C\rho\mathbb{E}\int_0^t\|u_m(s)\|^{p'-2}ds \\
&\leq \frac{1}{2}\mathbb{E}(\sup_{s\in[0,t]}\|u_m(s)\|^{p'}) + C\mathbb{E}\int_0^t \sup_{s\in[0,\sigma]}\|u_m(s)\|^{p'}d\sigma + C.
\end{aligned}$$

From the above estimates,

$$\begin{aligned}
&\mathbb{E}(\sup_{s\in[0,t]}\|u_m(s)\|^{p'}) \\
&\leq \mathbb{E}\|u_{0m}\|^{p'} + \int_0^t[\frac{\|f\|^2}{\mu_1}+\rho(p'-1)]ds \\
&+\int_0^t[\frac{\lambda_0p'(p'-1)}{2}+\frac{(p'-2)\|f\|^2}{2\mu_1}+\frac{\rho(p'-1)(p'-2)}{2}]\mathbb{E}(\sup_{r\in[0,s]}\|u_m(r)\|^{p'})ds \\
&+\frac{1}{2}\mathbb{E}(\sup_{s\in[0,t]}\|u_m(s)\|^{p'}) + C\mathbb{E}\int_0^t\sup_{s\in[0,\sigma]}\|u_m(s)\|^{p'}d\sigma + C.
\end{aligned}$$

By the Gronwall inequality, we obtain the estimate

$$\mathbb{E}(\sup_{t\in[0,T]}\|u_m(t)\|^{p'}) \leq C, \qquad p' \geq 2, \tag{3.10}$$

where the constant C is independent of m.

In order to apply Theorem 2.1 to obtain the strong convergence, we give another estimate with expectation. Recalling that

$$u_m(t) = P_mu_0 - \mu_1\int_0^t Au_m(s)ds - \int_0^t P_mB_m(u_m(s),u_m(s))ds \tag{3.11}$$

$$- 2\mu_0 \int_0^t P_m A_p^m(u_m(s))ds + \int_0^t P_m f ds + \int_0^t P_m G(u_m(s))dW(s)$$

and letting

$$u_m(t) = J_m^1 + J_m^2(t) + J_m^3(t) + J_m^4(t) + J_m^5(t) + J_m^6(t), \tag{3.12}$$

the following estimates can be obtained easily:

$$\mathbb{E}\|J_m^1\|^2 \leq C, \quad \mathbb{E}\|J_m^5(t)\|^2 \leq C, \quad \mathbb{E}\|J_m^2(t)\|^2_{H^1(0,T;V')} \leq C,$$

where $H^1(0,T;V')$ denotes the space of all $\psi \in L^2(0,T;V')$ such that $\psi_t \in L^2(0,T;V')$. Obviously, $H^1(0,T;V') \hookrightarrow W^{\alpha,2}(0,T;V')$, for all $\alpha \in (0,\frac{1}{2})$. As for the estimate of $J_m^6(t)$, applying Lemma 2.3, the uniform assumption on G, and the inequality (3.10),

$$\begin{aligned} &\mathbb{E}(\|\int_0^t P_m G(u_m)dW(s)\|^{p'}_{W^{\alpha,p'}(0,T;H)}) \leq C\mathbb{E}\int_0^T \|G(u_m)\|^{p'}_{L_2^{0,0}}ds \\ &\leq C\mathbb{E}\int_0^T (\|u_m\|^2 + \rho)^{\frac{p'}{2}} ds \leq C. \end{aligned} \tag{3.13}$$

For $p' = 2$,

$$\mathbb{E}\|J_m^6(t)\|^2_{W^{\alpha,2}(0,T;H)} \leq C.$$

For the estimate of $J_m^3(t)$, for all $v \in H^2(D)$,

$$|(B(u,u),v)| \leq \|u\|\,\|u\|_1\|v\|_{L^\infty} \leq C\|u\|\,\|u\|_1\|v\|_2,$$

where we used the Sobolev embedding $H^2(D) \hookrightarrow L^\infty(D)$, for $n = 2,3$. Thus

$$\|P_m B_m(u_m,u_m)\|^2_{L^2(0,T;V')} \leq C \sup_{0\leq t\leq T} \|u_m(t)\|^2 \int_0^T \|u_m(s)\|_1^2 ds,$$

$$\mathbb{E}\|J_m^3(t)\|^2_{H^1(0,T;V')} \leq C \sup_{0\leq t\leq T} \|u_m(t)\|^2 \int_0^T \|u_m(s)\|_1^2 ds \leq C.$$

We mainly estimate $J_m^4(t)$:

$$J_m^4(t) = \int_0^T P_m\{\nabla \cdot [\chi_m(u_m)(\epsilon + |e(u_m)|^2)^{\frac{p-2}{2}} e_{ij}(u_m)]\}ds.$$

From Lemma 2.2, we know, for $n = 2,3,\ 1 < p < 2$,

$$\|P_m\{\nabla \cdot [\chi_m(u_m)(\epsilon + |e(u_m)|^2)^{\frac{p-2}{2}} e_{ij}(u_m)]\}\|^2_{L^2(0,T;V')} \leq C\int_0^T \|u_m(s)\|_2^2 ds. \tag{3.14}$$

For $n=2$, $2<p\leq \frac{5}{2}$,

$$\|P_m\{\nabla\cdot[\chi_m(u_m)(\epsilon+|e(u_m)|^2)^{\frac{p-2}{2}}e_{ij}(u_m)]\}\|^2_{L^2(0,T;V')} \tag{3.15}$$
$$\leq C\int_0^T\|u_m(s)\|_2^2ds+CT^{\frac{5-2p}{2}}\sup_{0\leq s\leq T}\|u_m(s)\|(\int_0^T\|u_m(s)\|_2^2ds)^{\frac{2p-3}{2}}$$

and, for $n=3$, $2<p\leq\frac{12}{5}$,

$$\|P_m\{\nabla\cdot[\chi_m(u_m)(\epsilon+|e(u_m)|^2)^{\frac{p-2}{2}}e_{ij}(u_m)]\}\|^2_{L^2(0,T;V')} \tag{3.16}$$
$$\leq C\int_0^T\|u_m(s)\|_2^2ds+CT^{\frac{12-5p}{4}}\sup_{0\leq s\leq T}\|u_m(s)\|^{\frac{4-p}{2}}(\int_0^T\|u_m(s)\|_2^2ds)^{\frac{5p-8}{4}}.$$

Combining the estimates of (3.14), (3.15) and (3.16), and recalling (3.9), (3.10), we have

$$\mathbb{E}\|J_m^4(t)\|^2_{H^1(0,T;V')}\leq C.$$

From the above estimates, we can deduce

$$\mathbb{E}\|u_m(t)\|^2_{W^{\alpha,2}(0,T;V')}\leq C,\quad \text{for all}\quad \alpha\in(0,\tfrac{1}{2}), \tag{3.17}$$

where the constant is independent of m.

Third step: Taking the limit of the approximating solutions. From the above estimates (3.9) and (3.17), we can obtain that the laws $\mathcal{L}(u_m)$ are bounded in probability in $L^2(0,T;V)\cap W^{\alpha,2}(0,T;V')$; combining this with Theorem 2.1 implies that the family of laws $\mathcal{L}(u_m)$ is tight in $L^2(0,T;H^1(D))$.

Similarly, from Theorem 2.2, we know $H^1(0,T;V')$ is compactly embedded into $\mathcal{C}(0,T;H^{-3}(D))$.

On the other hand, for $W^{\alpha,p'}(0,T;H)$, here $\alpha\in(0,\frac{1}{2})$, $p'\geq 2$ and we can take $\alpha p'>1$, thus the condition in Theorem 2.2 is satisfied and $W^{\alpha,p'}(0,T;H)$ also is compactly embedded into $\mathcal{C}(0,T;H^{-3}(D))$. The family of laws $\mathcal{L}(u_m)$ also is tight in $\mathcal{C}([0,T];H^{-3}(D))$. Thus, we can find a subsequence, still denoted by u_m, such that $\mathcal{L}(u_m)$ converges weakly in $L^2(0,T;H^1(D))\cap\mathcal{C}([0,T];H^{-3}(D))$. By the Skorohod embedding theorem (see [10]), there exist a stochastic basis $(\Omega^1,\mathcal{F}^1,\{\mathcal{F}\}^1_{t\in[0,T]},\mathbb{P}^1)$, and random-valued variables u^1,u^1_m, $m\geq 1$ in $L^2(0,T;H^1(D))\cap\mathcal{C}([0,T];H^{-3}(D))$ such that u^1_m has the same law of $u_m(\mathcal{L}(u^1_m)=\mathcal{L}(u_m))$ on $L^2(0,T;H^1(D))\cap\mathcal{C}([0,T];\,H^{-3}(D))$, and

$$u^1_m\to u^1\quad\text{in}\quad L^2(0,T;H^1(D))\cap\mathcal{C}([0,T];H^{-3}(D))\quad \mathbb{P}-a.s. \tag{3.18}$$

Moreover, we also have, for each m, $\mathcal{L}(u_m^1)(\mathcal{C}([0,T];P_mH)) = 1$, and

$$\mathbb{E}(\sup_{0\le s\le T} \|u_m^1(s)\|^{p'}) \le C, \quad p' \ge 2, \qquad \mathbb{E}\int_0^T \|u_m^1(s)\|_2^2 ds \le C.$$

Hence, we have $u_m^1(\cdot,\omega) \in L^2(0,T;V) \cap L^\infty(0,T;H)$ $\mathbb{P}-a.s.$ and $u_m^1 \to u^1$ weakly in $L^2(\Omega;L^2(0,T;V))$.

For each $m \ge 1$, the process $M_m^1(t)$ defined as

$$M_m^1(t) = u_m^1(t) - P_m u^1(0) + \mu_1 \int_0^t A u_m^1(s) ds + \int_0^t P_m B_m(u_m^1(s), u_m^1(s)) ds + 2\mu_0 \int_0^t P_m A_p^m u_m^1(s) ds - \int_0^t P_m f ds$$

is a square integrable martingale for the filtration

$$(\mathcal{F}_m^1)_t = \sigma\{u_m^1(s) : 0 \le s \le t\},$$

with quadratic variation

$$<< M_m^1 >>_t = \int_0^t P_m G(u_m^1(s)) G(u_m^1(s))^* P_m ds.$$

Moreover,

$$\mathbb{E} \sup_{t\in[0,T]} (\|M_m^1(t)\|^{p'}) \le C, \quad \text{for } p' \ge 2. \tag{3.19}$$

For all $0 \le s \le t \le T$, all bounded continuous functions on $L^2(0,s;H^1(D))$ or $\mathcal{C}([0,s];H^{-3}(D))$ and $v,z \in \mathcal{V}$, since $\mathcal{L}(u_m^1) = \mathcal{L}(u_m)$, we have

$$\mathbb{E}\big((M_m^1(t) - M_m^1(s), v)\phi(u_m^1|_{[0,s]})\big) = 0, \tag{3.20}$$

and

$$\mathbb{E}\Big(\big((M_m^1(t),v)(M_m^1(t),z) - (M_m^1(s),v)(M_m^1(s),z) - \int_s^t (G(u_m^1(\sigma))^* P_m v, G(u_m^1(\sigma))^* P_m z) d\sigma\big)\phi(u_m^1|_{[0,s]})\Big) = 0. \tag{3.21}$$

If we show that we can take the limit in (3.20) and (3.21), as $m \to \infty$, then we will know that, for all $s \le t \in [0,T]$, all bounded continuous functions on $L^2(0,s;H^1(D))$ or $\mathcal{C}([0,s];H^{-3}(D))$, and all $v,z \in \mathcal{V}$, we have

$$\mathbb{E}\big((M^1(t) - M^1(s), v)\phi(u^1|_{[0,s]})\big) = 0, \tag{3.22}$$

and

$$\mathbb{E}\Big(\big((M^1(t),v)(M^1(t),z) - (M^1(s),v)(M^1(s),z)$$

$$- \int_s^t (G(u^1(\sigma))^* v, G(u^1(\sigma))^* z) d\sigma\big)\phi(u^1|_{[0,s]})\big) = 0, \tag{3.23}$$

where $M^1(t)$ is defined as

$$M^1(t) = u^1(t) - u^1(0) + \mu_1 \int_0^t Au^1(s)ds + \int_0^t B(u^1(s), u^1(s))ds + 2\mu_0 \int_0^t A_p u^1(s)ds - \int_0^t f ds.$$

This identity must be interpreted $\mathbb{P}$-a.s., as an identity in $\mathcal{C}([0,s]; H^{-3}(D))$.

In the following context, we will prove that we can obtain the limit. In other words, we need to prove that

$$(M_m^1(t), v) \to (M^1(t), v), \qquad \mathbb{P} - a.s. \quad \text{ for all } t, v. \tag{3.24}$$

Obviously, $\phi(u_m^1) \to \phi(u^1), \ \ \mathbb{P} - a.s.$ Using the continuity of G, the boundedness of ϕ, and (3.19), it can be checked that

$$\mathbb{E}\big((M_m^1(t) - M_m^1(s), v)\phi(u_m^1|_{[0,s]})\big) \to \mathbb{E}\big((M^1(t) - M^1(s), v)\phi(u^1|_{[0,s]})\big), \tag{3.25}$$

and

$$\begin{aligned}&\mathbb{E}\big(\big((M_m^1(t), v)(M_m^1(t), z) - (M_m^1(s), v)(M_m^1(s), z)\\ &- \int_s^t (G(u_m^1(\sigma))^* P_m v, G(u_m^1(\sigma))^* P_m z) d\sigma\big)\phi(u_m^1|_{[0,s]})\big) \to\\ &\mathbb{E}\big(\big((M^1(t), v)(M^1(t), z) - (M^1(s), v)(M^1(s), z)\\ &- \int_s^t (G(u^1(\sigma))^* v, G(u^1(\sigma))^* z) d\sigma\big)\phi(u^1|_{[0,s]})\big),\end{aligned} \tag{3.26}$$

where all terms in (3.25) and (3.26) are uniformly integrable in ω, and converge $\mathbb{P}$-a.s. In order to prove (3.24), we need to prove in particular that

$$(\int_0^t P_m B_m(u_m^1(s), u_m^1(s))ds, v) \to (\int_0^t B(u^1(s), u^1(s))ds, v), \tag{3.27}$$

$$(\int_0^t P_m A_p^m(u_m^1(s))ds, v) \to (\int_0^t A_p(u^1(s))ds, v), \tag{3.28}$$

where

$$\begin{aligned}P_m A_p^m(u_m^1(s)) &= -P_m\{\nabla \cdot [\chi_m(u_m^1(s))(\epsilon + |e(u_m^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u_m^1(s))]\},\\ A_p(u^1(s)) &= -\nabla \cdot [(\epsilon + |e(u^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u^1(s))].\end{aligned}$$

The convergence in (3.27) is similar to [14]; we omit the details. For (3.28), we only need to prove that

$$(\int_0^t A_p^m(u_m^1(s))ds, v) \to (\int_0^t A_p(u^1(s))ds, v). \tag{3.29}$$

By the property of the projector P_m, and the triangle inequality, we easily have the required convergence (3.28).

Now, we pay attention to the proof of (3.29), namely,

$$\int_0^t \int_D \chi_m(u_m^1(s))(\epsilon+|e(u_m^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u_m^1(s))e_{ij}(v)dxds \to \int_0^t \int_D (\epsilon + |e(u^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u^1(s))e_{ij}(v)dxds.$$

This holds on a set of $\mathbb{P}$-measure 1.

First, for all $\epsilon > 0$, take any subsequence $\{m_k\}$; there exists a subsequence $\{\nu_h\}$ of $\{m_k\}$ such that $u_{\nu_h}^1 \to u^1$ almost surely on $[0, T]$ with values in $L^2(D)$, therefore $\chi_{\nu_h}(u_{\nu_h}^1)$ converges to 1 almost surely on $[0, T]$, and the convergence is uniformly bounded by 1, thus

$$|\int_0^t \int_D \chi_m(u_m^1(s))(\epsilon+|e(u_m^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u_m^1(s))e_{ij}(v)dxds - \int_0^t \int_D (\epsilon + |e(u_m^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u_m^1(s))e_{ij}(v)dxds| \le \frac{\delta}{2};$$

to obtain the result of (3.29), we discuss p according to two cases.

Case 1: $n = 2, 3,\ 1 < p < 2$; by the mean value theorem, we can easily obtain

$$\int_0^t \int_D [(\epsilon + |e(u_m^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u_m^1(s)) - (\epsilon + |e(u^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u^1(s))]e_{ij}(v)dxds$$
$$\le C \int_0^t \|u_m^1(s) - u^1(s)\|_1 \|v\|_1 ds \le C(\int_0^t \|u_m^1(s) - u^1(s)\|_1^2 ds)^{\frac{1}{2}} (\int_0^t \|v\|_1^2 ds)^{\frac{1}{2}},$$

where the constant is independent of m.

Case 2: $n = 2, 3,\ 2 < p \le 2 + \frac{2}{n+2}$;

$$\int_0^t \int_D [(\epsilon + |e(u_m^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u_m^1(s)) - (\epsilon + |e(u^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u^1(s))]e_{ij}(v)dxds$$
$$= \int_0^t \int_D (\epsilon + |e(u_m^1(s))|^2)^{\frac{p-2}{2}} e_{ij}(u_m^1(s) - u^1(s))e_{ij}(v)dxds$$
$$+ \int_0^t \int_D [(\epsilon + |e(u_m^1(s))|^2)^{\frac{p-2}{2}} - (\epsilon + |e(u^1(s))|^2)^{\frac{p-2}{2}}]e_{ij}(u^1(s))e_{ij}(v)dxds$$

$=J_1+J_2$;

we estimate J_1 and J_2 respectively.

$$J_1 \le C\int_0^t (\int_D (\epsilon+|e(u_m^1(s))|^2)^{p-2}dx)^{\frac{1}{2}}\|u_m^1(s)-u^1(s)\|_1\|\nabla v\|_{L^\infty}ds;$$

notice the Sobolev embedding $H^2(D)\hookrightarrow L^\infty(D)$, thus

$$\|\nabla v\|_{L^\infty}\le C\|v\|_3\le C.$$

From the assumed condition $2<p\le\frac{5}{2}$ (respectively $2<p\le\frac{12}{5}$), $0<p-2\le\frac{1}{2}$ (respectively $0<p-2\le\frac{2}{5}$), and $(a+b)^m\le a^m+b^m$, for $0<m<1$, we have

$$J_1 \le C(\int_0^t\int_D(\epsilon^{p-2}+|e(u_m^1(s))|^{2(p-2)})dxds)^{\frac{1}{2}}(\int_0^t\|u_m^1(s)-u^1(s)\|_1^2ds)^{\frac{1}{2}}. \tag{3.30}$$

We mainly estimate the first term in the right-hand side of the above inequality:

$$\int_0^t\int_D|e(u_m^1(s))|^{2(p-2)}dxds\le CT^{3-p}|D|^{3-p}(\int_0^t\|u_m^1(s)\|_2^2ds)^{p-2}, \tag{3.31}$$

where we have used the Hölder inequality. Then we can deduce that

$$J_1\le C(\int_0^t\int_D\epsilon^{p-2}dxds+(\int_0^t\|u_m^1(s)\|_2^2ds)^{p-2})^{\frac{1}{2}}(\int_0^t\|u_m^1(s)-u^1(s)\|_1^2ds)^{\frac{1}{2}}, \tag{3.32}$$

where the constant is independent of m.

For the estimate of J_2, first, we consider the case $|e(u_m)|<|e(u)|$; by the mean value theorem, there exists η such that $|e(u_m)|<\eta<|e(u)|$, thus

$$\begin{aligned}
&(\epsilon+|e(u_m^1)|^2)^{\frac{p-2}{2}}-(\epsilon+|e(u^1)|^2)^{\frac{p-2}{2}}\\
&=\frac{p-2}{2}(\epsilon+\eta^2)^{\frac{p-4}{2}}(|e(u_m^1)|^2-|e(u^1)|^2)\\
&\le\frac{p-2}{2}(\epsilon+|e(u^1)|^2)^{\frac{p-4}{2}}(2|e(u^1)|)(|e(u_m^1)|-|e(u^1)|)\\
&\le(p-2)(\epsilon+|e(u^1)|^2)^{\frac{p-4}{2}}|e(u^1)|(|e(u_m^1)-e(u^1)|),
\end{aligned} \tag{3.33}$$

thus we can obtain

$$\begin{aligned}
J_2&\le C\int_0^t\int_D(\epsilon+|e(u^1)|^2)^{\frac{p-4}{2}}|e(u^1)|(|e(u_m^1)-e(u^1)|)|e(v)|dxds\\
&\le C\int_0^t\int_D(\epsilon+|e(u^1)|^2)^{\frac{p-2}{2}}(|e(u_m^1)-e(u^1)|)|e(v)|dxds
\end{aligned} \tag{3.34}$$

and we can obtain the following estimate similar to J_1 :

$$J_2 \leq C(\int_0^t \int_D \epsilon^{p-2} dxds + (\int_0^t \|u^1(s)\|_2^2 ds)^{p-2})^{\frac{1}{2}} (\int_0^t \|u_m^1(s) - u^1(s)\|_1^2 ds)^{\frac{1}{2}}, \tag{3.35}$$

where the constant is independent of m.

In the case of $|e(u_m)| < |e(u)|$, a similar estimate can be obtained.

In both of the two cases, because $u_m^1(s) \to u^1(s)$ converges strongly in $L^2(0,T;H^1(D))$, it is easy by the triangle inequality to see that (3.28) is true. We deduce that $M^1(t)$ is a square integrable martingale on $[0,T]$, for the filtration $(\mathcal{F}^1)_t = \sigma\{u^1(s) : s \leq t\}$ with quadratic variation $<< M^1 >>_t = \int_0^t G(u^1(s))G(u^1(s))^* ds$. To prove Theorem 3.1, it suffices to apply Theorem 8.2 P_{220} in [10] on the representation of martingale solutions. □

4. STATIONARY SOLUTION

Theorem 4.1. *In addition to the hypothesis of G and the restriction of p, assume that $\lambda_1\mu_1 > \lambda_0$, where λ_1 is the first eigenvalue of A. Then there exists a stationary martingale solution.*

Proof. First, we want to construct a stationary process for the approximating equation.

Let G_n be a Lipschitz continuous function such that $G_n \to P_m G$ uniformly on bounded sets in $P_m H$, and u_m^n be a solution of the equation

$$du_m^n + \mu_1 A u_m^n dt + P_m B_m(u_m^n, u_m^n)dt + 2\mu_0 P_m A_p^m(u_m^n)dt$$
$$= P_m f dt + G_n(u_m^n) dW(t), \tag{4.1}$$
$$u_m^n(0) = 0. \tag{4.2}$$

Then from (3.5), and the ϵ-Young inequality

$$ab \leq \frac{\epsilon}{q} a^q + \frac{1}{q' \epsilon^{\frac{q'}{q}}} b^{q'},$$

we can deduce

$$\frac{p'\|f\|^2}{2\mu_1} \|u_m^n(t)\|^{p'-2} \leq \frac{\epsilon}{2} \|u_m^n(t)\|^{p'} + \mathbb{C}_1,$$
$$\frac{\rho p'(p'-1)}{2} \|u_m^n(t)\|^{p'-2} \leq \frac{\epsilon}{2} \|u_m^n(t)\|^{p'} + \mathbb{C}_2,$$

where

$$\mathbb{C}_1 = \frac{1}{p'(\frac{\epsilon p'}{p'-2})^{\frac{p'-2}{2}}} (\frac{p'\|f\|^2}{\mu_1})^{\frac{p'}{2}}, \quad \mathbb{C}_2 = \frac{1}{p'(\frac{\epsilon p'}{p'-2})^{\frac{p'-2}{2}}} (\rho p'(p'-1))^{\frac{p'}{2}}.$$

Let $\mathbb{C} = \mathbb{C}_1 + \mathbb{C}_2$; $\mathbb{C}$ depends on $\epsilon, p', \|f\|, \rho$, thus

$$\frac{d}{dt}\mathbb{E}\|u_m^n(t)\|^{p'} + \frac{\mu_1 p'}{2}\mathbb{E}(\|u_m^n\|^{p'-2}\|u_m^n\|_2^2) \leq [\frac{1}{2}\lambda_0 p'(p'-1)+\epsilon]\mathbb{E}\|u_m^n(t)\|^{p'} + \mathbb{C},$$

$$\frac{d}{dt}\mathbb{E}\|u_m^n(t)\|^{p'} + \frac{\mu_1 p'}{2}\lambda_1\mathbb{E}(\|u_m^n\|^2) \leq [\frac{1}{2}\lambda_0 p'(p'-1)+\epsilon]\mathbb{E}\|u_m^n(t)\|^{p'} + \mathbb{C}, \quad (4.3)$$

where λ_1 is the first eigenvalue of A.

If $\lambda_1\mu_1 > \lambda_0$, then there exists $\epsilon > 0$ and $p' > 2$ such that

$$p'\lambda_1\mu_1 > p'(p'-1)\lambda_0 + 2\epsilon.$$

By the Gronwall inequality, we can obtain

$$\mathbb{E}(\|u_m^n(t)\|^{p'}) \leq C, \quad \forall\, t \geq 0, \quad m \geq 1,$$

where the constant is independent of m and n. This implies that the process $u_m^n(t)$ is bounded in probability, since the semigroup generated by A is compact; by Theorem 11.29 P_{338} in [10], there exists an invariant measure μ_m^n for equation (4.1)-(4.2).

At the same time, the law $\mathcal{L}(u_m^n(t))$ is tight on $\mathcal{C}(0,T;P_mH)$. Letting $n \to \infty$, there exists a stationary solution with laws μ_m for equation (3.2).

Thus, we can construct a new stochastic basis $(\Omega', \mathcal{F}', \mathcal{F}'_{t\in[0,T]}, \mathbb{P}')$, a cylindrical Wiener process $W'(t)$ with values in H, and $\mathcal{F}'$-measurable P_mH-valued random variables u'_{0m} with laws μ_m satisfying

$$\mathbb{E}'(\|u'_{0m}\|^{p'}) \leq C, \quad \forall\, m \geq 1.$$

The corresponding solution $u'_m(t)$ of (3.2) is a stationary process in P_mH.

Noting that we only prove the existence of stationary solutions over a finite time interval, we shall use a localization method to prove tightness.

Endow $L^2_{loc}(0,\infty;H)$ with the distance

$$d_2(u,v) = \sum_{k=1}^{\infty}\frac{1}{2^k}(\|u-v\|_{L^2(0,k;H)} \wedge 1),$$

and $\mathcal{C}(0,\infty;H^{-3}(D))$ with the distance

$$d_\infty(u,v) = \sum_{k=1}^{\infty}\frac{1}{2^k}(\|u-v\|_{C([0,k];H^{-3}(D))} \wedge 1).$$

Recalling that we can deduce the following estimates similarly,

$$\mathbb{E}'(\sup_{t\in[0,T]}\|u'_m(t)\|^2) \leq C, \qquad \int_0^T \mathbb{E}'(\|u'_m(s)\|_2^2)ds \leq C,$$

$$\mathbb{E}'(\|u'_m(t)\|_{W^{\alpha,2}(0,T;V')}) \leq C, \qquad \mathbb{E}'(\|u'_m(t)\|_{W^{\alpha,p'}(0,T;H)}) \leq C.$$

Then the laws of $u'_m(t)$ are tight in $L^2(0,T;H)\bigcap \mathcal{C}(0,T;H^{-3}(D))$.

Noting that the convergence with respect to $d_2 + d_\infty$ is equivalent to the convergence on every finite time interval, it follows that if $Y \subset L^2_{loc}(0,\infty;H)$ $\cap\mathcal{C}(0,\infty;H^{-3}(D))$ has the property that for all k the set $Y_k = \{u|_{[0,k]} : u \in Y\}$ is compact, then Y is compact in $L^2_{loc}(0,\infty;H)\cap\mathcal{C}(0,\infty;H^{-3}(D))$. Thus, we obtain that the laws of $u'_m(t)$ are tight in $L^2_{loc}(0,\infty;H)\cap\mathcal{C}(0,\infty;H^{-3}(D))$. Let $u''_m(t)$, $u''(t)$ be given by the Skorohod embedding theorem, as in the proof above. $\{\mathcal{L}(u''_m(t)) : t \geq 0; m \geq 1\}$ is tight on $\mathcal{C}(0,\infty;H^{-3}(D))$, thus there exists an invariant measure and the corresponding solution $u''_m(t)$ is stationary in $H^{-3}(D)$. Recalling that $\{\mathcal{L}(u''_m(t)) : t \geq 0; m \geq 1\}$ is tight on $\mathcal{C}(0,\infty;H^{-3}(D))$, we can let m tend to infinity and deduce that $u''(t)$ is stationary in $H^{-3}(D)$; at the same time, for all $t \geq 0$, $u''(t)$ is an H-value random variable. This fact implies that $u''(t)$ also is stationary in H.

On the other hand, the process $u''(t)$ also is a martingale solution, by the same proof as in the previous section 3; we omit the details. □

Remark. If the uniqueness of solutions to the stochastic non-Newtonian fluid can be obtained under some restricted condition, it is possible to define the transition semigroup, and by the existence of stationary martingale solution, the existence of the invariant measures can be proved.

Acknowledgements. The authors express their deep gratitude to the referees for their careful reading and helpful comments which have improved the original version of the manuscript.

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