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EXTRAPOLATION METHOD AND SOME NONDENSELY DEFINED IMPULSIVE SEMILINEAR NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we use the extrapolation method combined with a fixed-point theorem for the sum of completely continuous and contraction operators, to establish sufficient conditions for the existence of mild solutions and extremal mild solutions for some classes of nondensely defined impulsive semilinear neutral functional differential inclusions in separable Banach spaces with infinite delay.

1. Introduction

In this paper, we shall be concerned with the existence of mild and extremal mild solutions defined on a compact real interval for first-order nondensely defined impulsive semilinear neutral functional inclusions in a separable Banach space. More precisely, we will consider the following first-order impulsive semilinear neutral functional differential inclusions of the form

$$\frac{d}{dt}[y(t) - g(t, y_t)] - A[y(t) - g(t, y_t)] \in F(t, y_t), \tag{1.1}$$

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for almost every $t \in J = [0, b],\ t \neq t_k,\ k = 1, \ldots, m,$

$$\Delta y|_{t=t_k} \in I_k(y(t_k^-)), \quad k = 1, \ldots, m \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{1.3}$$

where $F : J \times D \to \mathcal{P}(E)$ is a compact and convex-valued multivalued map, $g : J \times D \to E$ is a given function, $\phi \in D$, where D is the phase space that will be specified later, $I_k : E \to \mathcal{P}(E),\ k = 1, 2, \ldots, m$ are bounded multivalued maps, $\mathcal{P}(E)$ is the collection of all E-subsets, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, and E is a real separable Banach space with norm $|.|$. For any function y defined on $(-\infty, b] \backslash \{t_1, t_2, ..., t_m\}$ and any $t \in J$, we denote by y_t the element of D defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0].$$

The theory and applications of impulsive functional differential equations are emerging as an important area of investigation, since it is far richer than the corresponding theory of nonimpulsive functional differential equations. We refer to the monographs of Bainov and Simeonov [11], Benchohra *et al* [15], Lakshmikantham *et al* [46], and Samoilenko and Perestyuk [56] where numerous properties of their solutions are studied, and a detailed bibliography is given.

Functional and neutral functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books of Hale [31], Hale and Verduyn Lunel [33], Kolmanovskii and Myshkis [43], Kuang [44] and Wu [60], and the references therein.

Semilinear functional differential equations and inclusions with or without impulses have been extensively studied, see, for example, the books of Ahmed [6, 7], Benchohra *et al* [14], Heikkila and Lakshmikantam [36] and Kamenskii *et al* [40] and the papers by Liu [48] and Rogovchenko [54, 55] and the references therein. In [4], Abada *et al* have studied the controllability of a class of impulsive semilinear functional differential inclusions in Fréchet spaces using the extrapolation method [22, 28], and in [2] the existence of mild and extremal mild solutions for first-order semilinear densely defined impulsive functional inclusions in separable Banach spaces with local and nonlocal conditions has been considered. In [3], they have studied the existence and controllability of nondensely defined impulsive semilinear functional differential equations. To the best of our knowledge, there are few results for impulsive evolution inclusions with multivalued jump operators; see [1, 2, 20, 50].

The notion of the phase space D plays an important role in the study of both qualitative and quantitative theory. A common choice is a semi-normed space satisfying suitable axioms, which were introduced by Hale and Kato [32] (see also Kappel and Schappacher [41] and Schumacher [57]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [38]. For the case where the impulses are absent (i.e $I_k = 0,\ k = 1, \ldots, m$), an extensive theory has been developed for the problem (1.1)-(1.3); we refer to Adimy *et al* [5], Belmekki *et al* [12], Corduneanu and Lakshmikantham [21], Hale and Kato [32], Hino *et al* [38], Lakshmikantham *et al* [47] and Shin [58].

First-order abstract neutral functional differential equations with unbounded delay were initiated by Hernandez and Henriquez [34, 35]. The goal of this paper is to give existence results for first-order impulsive neutral nondensely defined functional differential inclusions with multivalued jump and infinite delay. The main results extend to the infinite delay problems considered by Benchohra *et al* [12, 13, 16, 17]. In particular, we extend to nondensely defined operators the problems considered in [1].

This paper will be organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout this work. In Section 3, we give some examples of operators with nondense domain. In Section 4, we shall establish sufficient conditions for the existence of mild solutions for the problem (1.1)-(1.3) by relying on a fixed-point theorem due to Dhage. In Section 5, sufficient conditions for the existence of extremal mild solutions for the problem (1.1)-(1.3) are established. The last section is devoted to an example illustrating the abstract theory.

2. Preliminaries

In this section, we state some facts about semigroups, notation and definitions that are used throughout this paper. $C(J, E)$ is the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\};$$

also $B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(J,E)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

For properties of the Bochner integral, see, for instance, Yosida [61].

In order to define the phase space and the solutions of (1.1)–(1.3) we shall consider the space

$$PC = \Big\{ y : (-\infty, b] \to E,\ y(t_k^-), y(t_k^+),\ \text{exist with}\ \ y(t_k) = y(t_k^-),$$
$$y(t) = \phi(t), t \le 0,\ y_k \in C(J_k, E) \Big\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \ldots, m$. Let $\|\cdot\|_{PC}$ be the norm in PC defined by

$$\|y\|_{PC} = \sup\{|y(s)| : 0 \le s \le b\},\ y \in PC.$$

We will assume that D satisfies the following axioms:

(A) If $y : (-\infty, b] \to E, b > 0$ and $y(t_k^-),\ y(t_k^+)$, exist with $y(t_k) = y(t_k^-),\ k = 1, \ldots, m$ and $y_0 \in D$, then for every t in $[0, b)\backslash\{t_1 \ldots, t_m\}$ the following conditions hold:
  (i) $\|y_t\|_D \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_D$,
  (ii) $|y(t)| \le H\|y_t\|_D$,
  where $H \ge 0$ is a constant, $K : [0, \infty) \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.

(A-1) The space D is complete.

Denote $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$. Let A_0 be the part of A in $X_0 = \overline{D(A)}$ which is defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},\ \text{and}\ A_0 x = Ax,\ \text{for}\ x \in D(A_0).$$

Definition 2.1. *We say that a linear operator A satisfies the "Hille-Yosida condition" if there exist $M \ge 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ (the resolvent set of A) and* $\sup\{(\lambda - \omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N}, \lambda > \omega\} \le M$.

Here and hereafter, we assume:

(HY) A satisfies the *Hille-Yosida condition.*

Lemma 2.1. [28] *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\ge 0}$ on X_0 and $|T_0(t)| \le N_0 e^{\omega t}$, for $t \ge 0$. Moreover, $\rho(A) \subset \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)/X_0$, for $\lambda \in \rho(A)$.*

For a fixed $\lambda_0 \in \rho(A)$, we introduce on X_0 a new norm defined by $\|x\|_1 = |R(\lambda_0, A_0)x|$ for $x \in \overline{D(A_0)}$. The completion X_1 of $(X_0, \|\cdot\|_1)$ is called the *extrapolation space of* X *associated with* A. Note that $\|\cdot\|_1$ and the norm on X_0 given by $|R(\lambda, A_0)x|$, for $\lambda \in \rho(A)$, are extensions $T_1(t)$ to the Banach space X_1, and $(T_1(t))_{t\geq 0}$ is a strongly continuous semigroup on X_1. $(T_1(t))_{t\geq 0}$ is called *the extrapolated semigroup of* $(T_0(t))_{t\geq 0}$, and we denote its generator by $(A_1, D(A_1))$.

Lemma 2.2. [30] *The following properties hold:*

(i) $|T(t)|_{B(X_1)} = |T_0(t)|_{L(X_0)}$.
(ii) $D(A_1) = X_0$.
(iii) $A_1 : X_0 \to X_1$ *is the unique continuous extension of* $A_0 : D(A_0) \subset (X_0, |\cdot|) \to (X_0, \|\cdot\|_1)$, *and* $(\lambda I - A_1)^{-1}$ *is an isometry from* $(X_0, |\cdot|) \to (X_0, \|\cdot\|_1)$.
(iv) *If* $\lambda \in \rho(A_0)$, *then* $(\lambda I - A_1)$ *is invertible and* $(\lambda I - A_1)^{-1} \in B(X_1)$. *In particular* $\lambda \in \rho(A_1)$ *and* $R(\lambda, A_1)/X_0 = R(\lambda, A_0)$.
(v) *The space* $X_0 = \overline{D(A)}$ *is dense in* $(X_1, \|\cdot\|_1)$. *Hence the extrapolation space* X_1 *is also the completion of* $(X, \|.\|_1)$ *and* $X \hookrightarrow X_1$.
(vi) *The operator* A_1 *is an extension of* A. *In particular if* $\lambda \in \rho(A)$, *then* $R(\lambda, A_1)/X = R(\lambda, A)$ *and* $(\lambda I - A_1)X = D(A)$.

Abstract extrapolation spaces have been introduced by Da Prato and Grisvard [22]. (See also Engel and Nagel [28]) and used for various purposes ([8, 9, 10, 37, 49, 51, 52]). For the properties of semigroup theory we refer the interested reader to the books of Ahmed [6] and Pazy [53]. We finish this section, with notation, definitions, and some results from multivalued analysis.

Let (X, d) be a metric space. We use the following notations:

$$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\},\ P_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$$

$$P_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\},\ P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \bigcup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{bd,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [42]). A multivalued map $N : J \to P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = d(x, N(t)) = \inf\{d(x, z) : z \in N(t)\}$$

is measurable.

Definition 2.2. *A measurable multivalued function $F : J \to P_{bd,cl}(X)$ is said to be integrably bounded if there exists a function $w \in L^1(J, \mathbb{R}^+)$ such that $\|v\| \leq w(t)$ for almost every $t \in J$ for each $v \in F(t)$.*

A multivalued map $F : X \to \mathcal{P}(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for each $x \in X$. F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in P_b(X)$; i.e.,

$$\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty.$$

F is called upper semi-continuous (u.s.c for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subseteq \mathcal{U}$. G is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_{bd}(X)$. If the multivalued map F is completely continuous with nonempty compact-valued range, then G is u.s.c if and only if F has closed graph; i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_*)$ imply $y_* \in G(x_*)$.

Definition 2.3. *A multivalued map $F : J \times D \to \mathcal{P}(E)$ is said to be L^1 Carathéodory if*

(i) *$t \longmapsto F(t, u)$ is measurable for each $u \in D$,*
(ii) *$u \longmapsto F(t, u)$ is u.s.c. for almost each $t \in J$,*
(iii) *for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}^+)$ such that*

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{|v| : v \in F(t, u)\} \leq \varphi_q(t),$$

for each $\|u\|_D \leq q$ and for a.e. $t \in J$.

Definition 2.4. *Suppose $N : J \to P_{cl}(X)$ is a multivalued operator.*

(a) *N is a contraction if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad for \quad each \quad x, y \in X,$$

with $\gamma < 1$.

(b) *N has a fixed point if there exists $x \in X$ such that $x \in N(x)$.*

For more details on multivalued maps and the proof of the known results cited in this section we refer interested readers to the books of Deimling [24], Gorniewicz [29] and Hu and Papageorgiou [39]. The key tool in our approach is the following form of the fixed-point theorem of Dhage ([25, 27]).

Theorem 2.1. *Let X be a Banach space and $\mathcal{A}$: $X \to P_{cl,cv,bd}(X)$ and $\mathcal{B}$: $X \to P_{cp,cv}(X)$ be two multivalued operators satisfying the following.*

(a) *$\mathcal{A}$ is a contraction, and*
(b) *$\mathcal{B}$ is completely continuous.*

Then either

(i) *the operator inclusion $\lambda x \in \mathcal{A}x + \mathcal{B}x$ has a solution for $\lambda = 1$, or*
(ii) *the set $\mathcal{E} = \{u \in X : u \in \lambda\mathcal{A}u + \lambda\mathcal{B}u, \quad 0 \le \lambda \le 1\}$ is unbounded.*

3. Examples of Operators with Nondense Domain

In this section, we shall present examples of linear operators with nondense domain satisfying the Hille-Yosida estimate. More details can be found in the paper by Da Prato and Sinestrari [23].

Example 3.1. Let $E = C([0,1],\mathbb{R})$ and the operator $A : D(A) \to E$ be defined by $Ay = y'$, where $D(A) = \{y \in C^1((0,1),\mathbb{R}) : y(0) = 0\}$. Then

$$\overline{D(A)} = \{y \in C((0,1),\mathbb{R}) : y(0) = 0\} \neq E.$$

Example 3.2. Let $E = C([0,1],\mathbb{R})$ and the operator $A : D(A) \to E$ defined by $Ay = y''$, where $D(A) = \{y \in C^2((0,1),\mathbb{R}) : y(0) = y(1) = 0\}$. Then

$$\overline{D(A)} = \{y \in C((0,1),\mathbb{R}) : y(0) = y(1) = 0\} \neq E.$$

Example 3.3. Let us set for some $\alpha \in (0,1)$ $E = C_0^\alpha([0,1],\mathbb{R}) = \{y : [0,1] \to \mathbb{R} : y(0) = 0$ and $\sup_{0\le t<s\le 1} \frac{|y(t)-y(s)|}{|t-s|^\alpha} < \infty\}$ and the operator $A : D(A) \to E$ defined by $Ay = -y'$, where

$$D(A) = \{y \in C^{1+\alpha}((0,1),\mathbb{R}) : y(0) = y'(0) = 0\}.$$

Then

$$\overline{D(A)} = h_0^\alpha(0,1),\mathbb{R}) = \{y : [0,1] \to \mathbb{R} : \lim_{\delta\to 0} \sup_{0<|t-s|\le\delta} \frac{|y(t)-y(s)|}{|t-s|^\alpha} = 0\} \neq E.$$

Here, $C^{1+\alpha}([0,1],\mathbb{R}) = \{y : [0,1] \to \mathbb{R} : y' \in C^\alpha([0,1],\mathbb{R})\}$. The elements of $h^\alpha((0,1),\mathbb{R})$ are called little Holder functions and it can be proved that the closure of $C^1((0,1),\mathbb{R})$ in $C^\alpha((0,1),\mathbb{R})$ is $h^\alpha((0,1),\mathbb{R})$ (see [59] Theorem 5.3).

Example 3.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary Γ and define $E = C(\overline{\Omega},\mathbb{R})$ and the operator $A : D(A) \to E$ defined by $Ay = \Delta y$, where $D(A) = \{y \in C(\overline{\Omega},\mathbb{R}) : y = 0 \text{ on } \Gamma;\ \Delta y \in C(\overline{\Omega},\mathbb{R})\}$. Here, Δ is the Laplacian in the sense of distributions on Ω. In this case we have $\overline{D(A)} = \{y \in C(\overline{\Omega},\mathbb{R}) : y = 0 \text{ on } \Gamma\} \neq E$.

4. Existence of Mild Solutions

We shall consider the space $D_b = \{y : (-\infty, b] \to E : \ y \in PC \cap D\}$, and let $\|\cdot\|_b$ be the seminorm in D_b defined by

$$\|y\|_b := \|y_0\|_D + \sup\{|y(t)| : 0 \le s \le b\}, \ y \in D_b.$$

Assume that F is a compact and convex valued multivalued map. Let us start by defining what we mean by a mild solution of problem (1.1)-(1.3).

Definition 4.1. *A function $y \in D_b$, is said to be a mild solution of system (1.1)-(1.3) if $y(t) = \phi(t)$ for each $t \in (-\infty, 0]$, the restriction of $y(\cdot)$ to the interval $[0, b]$ is continuous, and there exist $v(\cdot) \in L^1(J_k, E)$ and $\mathcal{I}_k \in I_k(y(t_k^-))$, such that $v(t) \in F(t, y_t)$ for almost every $t \in [0, b]$, and y satisfies the integral equation*

$$y(t) = T_0(t)\left(\phi(0) - g(0, \phi(0))\right) + g(t, y_t) + \int_0^t T_1(t-s)v(s)ds \qquad (4.1)$$
$$+ \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k, \quad t \in J.$$

In our proof we use the following result due to Lazota and Optial [45].

Lemma 4.1. *Let E be a Banach space, and F be an L^1-Carathéodory multivalued map with compact convex values, and let $\Gamma : L^1(J, E) \to C(J, E)$ be a linear continuous mapping. Then the operator*

$$\Gamma \circ S_F : C(J, E) \to P_{cp,cv}(C(J, E))$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Here, $S_{F,y} = \{v \in L^1(J, E) : \quad v(t) \in F(t, y_t) \quad a.e. \quad t \in J\}$ is the set of selection of F. Let us introduce the following hypotheses:

(H1) There exists a constant $M > 0$ such that

$$\|T_1(t)\|_{B(E)} \le M; \quad t \in J.$$

(H2) There exist constants $c_k > 0$, $k = 1, \dots, m$ such that

$$H_d(I_k(y), I_k(x))| \le c_k|y - x| \ \text{ for each } \quad x, y \in E.$$

(H3) F is L^1-Carathéodory with compact convex values.

(H4) There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \le p(t)\psi(\|x\|_D) \ \text{ for a.e. } t \in J \text{ and each } x \in D,$$

with

$$\limsup_{u\to\infty}\frac{\Big(1-\alpha_1K_b-M\sum_{k=1}^{m}c_k\Big)u}{C_0+M\|p\|_{L^1}\psi(K_bu+(MK_b+M_b)\|\phi\|_D)}>1, \qquad (4.2)$$

where

$$\begin{aligned} C_0 &= \alpha_1(MK_b+M_b)\|\phi\|_D+\alpha_2+M(\alpha_1\|\phi\|_D+\alpha_2)\\ &\quad +M^2\sum_{k=1}^{m}c_k\|\phi\|_D+M\sum_{k=1}^{m}|I_k(0)|. \end{aligned}$$

(H5) The function $g(t,.)$ is continuous on J and there exists a constant $l_g>0$ such that

$$|g(t,u)-g(t,v)|\le l_g\|u-v\| \quad \text{for each} \quad u,v\in D.$$

(H6) There exist constants $\alpha_1,\ \alpha_2>0$ such that

$$\|g(t,u)\|\le\alpha_1\|u\|_D+\alpha_2 \ \text{ for each } \quad (t,u)\in[0,b]\times D.$$

Theorem 4.1. *Assume that* (H1)-(H6), $\phi\in D,\ \phi(0)\in X_0,\ g(0,\phi(0))\in X_0$ *hold. If* $l_g+M\sum_{k=1}^{m}c_k<1$, *then the IVP* (1.1)–(1.3) *has at least one mild solution on* $(-\infty,b]$.

Proof. Transform the problem (1.1)-(1.3) into a fixed-point problem. Consider the multivalued operator $N:D\to\mathcal{P}(D)$ defined by $N(y)=\{h\}$ such that

$$h(t)=\begin{cases}\phi(t), & \text{if } t\le 0,\\ T_0(t)\left(\phi(0)-g(0,\phi(0))\right)+g(t,y_t)+\displaystyle\int_0^t T_1(t-s)v(s)ds & \\ \qquad+\displaystyle\sum_{0<t_k<t}T_1(t-t_k)\mathcal{I}_k,v\in S_{F,y},\mathcal{I}_k\in I_k(y(t_k^-)) & \text{if } t\in J.\end{cases}$$

Now, we shall show that the operator N has a fixed point. This fixed point is then the mild solution of the IVP (1.1)–(1.3). For $\phi\in D$ define the function $x(\cdot):(-\infty,b]\to E$ such that

$$x(t)=\begin{cases}\phi(t), & \text{if } t\le 0\\ T_0(t)\phi(0), & \text{if } t\in J.\end{cases}$$

Then $x(\cdot)$ is an element of D_b and $x_0 = \phi(0)$. Set $y(t) = z(t)+x(t)$. Obviously if y satisfies the integral equation

$$y(t) = T_0(t)\left(\phi(0) - g(0,\phi(0))\right) + g(t,y_t) + \int_0^t T_1(t-s)v(s)ds + \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k, \quad t \in J,$$

then z satisfies $z_0 = 0$ and

$$z(t) = g(t, z_t + x_t) - T_0(t)g(0,\phi(0)) + \int_0^t T_1(t-s)v(s)ds + \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k, \quad t \in J,$$

where $v(t) \in F(t, z_t+x_t)$ for almost every $t \in [0,b]$ and $\mathcal{I}_k \in I_k(z(t_k^-+x(t_k^-)$. Let $D_b^0 = t\{z \in D_b : z_0 = 0\}$. For any $z \in D_b^0$, we have

$$\|z\|_b = \|z_0\|_D + \sup\left\{|z(s)| : 0 \le s \le b\right\} = \sup\left\{|z(s)| : 0 \le s \le b\right\}.$$

Thus, $(D_b^0, \|\cdot\|_b)$ is a Banach space. Let the operator $P : D_b^0 \to \mathcal{P}(D_b^0)$ be defined by $P(z) = \{h \in D_b^0\}$ such that

$$h(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0]; \\ g(t, z_t + x_t) - T_0(t)g(0,\phi(0)) + \displaystyle\int_0^t T_1(t-s)v(s)ds & \\ + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k, & \text{if } \ t \in J. \end{cases}$$

The operator N having a fixed point is equivalent to P having one, so it remains to prove that P has a fixed point. Consider the multivalued operators $\mathcal{A}, \mathcal{B} : D_b^0 \to \mathcal{P}(D_b^0)$ defined by $\mathcal{A}(z) := \{h \in D_b^0\}$ such that

$$h(t) = \begin{cases} 0, & \text{if } t \le 0; \\ g(t, z_t + x_t) - T_0(t)g(0,\phi(0)) + \displaystyle\sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k, & \\ \mathcal{I}_k \in I_k(z(t_k^-) + x(t_k^-)) & \text{if } t \in J, \end{cases}$$

and $\mathcal{B}(z) := \{h \in D_b^0\}$ such that

$$h(t) = \begin{cases} 0, & \text{if } t \le 0; \\ \displaystyle\int_0^t T_1(t-s)v(s)ds & \text{if } t \in J, \end{cases}$$

where $v \in S_{F,z} = \{v \in L^1([0,b],E) : v(t) \in F(t, z_t + x_t) \text{ for a.e. } t \in [0,b]\}$. It is clear that $P = \mathcal{A} + \mathcal{B}$. Then the problem of finding mild solutions of (1.1)-(1.3) is then reduced to finding mild solutions of the operator inclusion $z \in \mathcal{A}(z) + \mathcal{B}(z)$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Theorem 2.1. The proof will be given in several steps.

Step 1: $\mathcal{A}$ *is a contraction.* Let $z_1, z_2 \in D_b^0$, then from $(H2)$ and $(H5)$ we have

$$H_d\big(\mathcal{A}(z_1), \mathcal{A}(z_2)\big) \leq \|g(t, z_{1_t} + x_t) - g(t, z_{2_t} + x_t)\|$$
$$+H_d\Big(\sum_{0<t_k<t} T_1(t-t_k) I_k(z_1(t_k^-) + x(t_k^-)), \sum_{0<t_k<t} T_1(t-t_k) I_k(z_2(t_k^-) + x(t_k^-))\Big)$$
$$\leq l_g\|z_1 - z_2\| + M\sum_{k=1}^{m} c_k |z_1(t_k^-) - z_2(t_k^-)| \leq \big(l_g + M\sum_{k=1}^{m} c_k\big)\|z_1 - z_2\|.$$

Then $\mathcal{A}$ is a contraction.

Step 2. $\mathcal{B}$ *has compact, convex values, and it is completely continuous.* This will be given in several claims.

Claim 1: $\mathcal{B}$ *has compact values.* The operator $\mathcal{B}$ is equivalent to the composition $\mathcal{L} \circ S_F$ on $L^1(J,E)$, where $\mathcal{L} : L^1(J,E) \to D_b^0$ is the continuous operator defined by

$$\mathcal{L}v)(t) = \int_0^t T_1(t-s)v(s)ds, \quad t \in J.$$

Thus, it suffices to show that $\mathcal{L} \circ S_F$ has compact values on D_b^0. Let $z \in D_b^0$ be arbitrary, v_n a sequence in $S_{F,z}$; then, by definition of S_F, $v_n(t)$ belongs to $F(t, z_t)$, for almost every $t \in J$. Since $F(t, z_t)$ is compact, we may pass to a subsequence. Suppose that $v_n \to v$ in $L^1(J,E)$, where $v(t) \in F(t, z_t)$, for almost every $t \in J$. From the continuity of $\mathcal{L}$, it follows that $(\mathcal{L}v_n)(t) \to (\mathcal{L}v)(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equicontinuous sequence. Let $\tau_1, \tau_2 \in J$; then we have

$$\big|\mathcal{L}(v_n(\tau_1)) - \mathcal{L}(v_n(\tau_2))\big| = \Big|\int_0^{\tau_1} T_1(\tau_1 - s)v_n(s)ds - \int_0^{\tau_2} T_1(\tau_2 - s)v_n(s)ds\Big|$$
$$\leq \int_0^{\tau_1} \big|\big(T_1(\tau_1 - s) - T_1(\tau_2 - s)\big)\big||v_n(s)|ds + \int_{\tau_1}^{\tau_2} |T_1(\tau_2 - s)||v_n(s)|ds.$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. Since $T_1(t)$ is a strongly continuous operator, the compactness of $T_1(t),\ t >$

0, implies the continuity in the uniform topology. Hence, $\{\mathcal{L}v_n\}$ is equicontinuous, and an application of the Arzéla-Ascoli theorem implies that there exists a subsequence which is uniformly convergent. Then we have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S_F)(z)$ as $j \mapsto \infty$, and so $(\mathcal{L} \circ S_F)(z)$ is compact. Therefore, $\mathcal{B}$ is a compact valued multivalued operator on D_b^0.

Claim 2: *$\mathcal{B}(z)$ is convex for each $z \in D_b^0$.* Let $h_1, h_2 \in \mathcal{B}(z)$; then there exists $v_1, v_2 \in S_{F,z}$ such that, for each $t \in J$, we have

$$h_i(t) = \left\{ \begin{array}{ll} 0, & \text{if } t \in (-\infty, 0], \\ \int_0^t T_1(t-s)v_i(s)ds & \text{if } t \in J, \end{array} \right\}, \ i = 1, 2.$$

Let $0 \le \delta \le 1$. Then, for each $t \in J$, we have

$$(\delta h_1 + (1-\delta)h_2)(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ \int_0^t T_1(t-s)[\delta v_1(s) + (1-\delta)v_2(s)]ds & \text{if } t \in J. \end{cases}$$

Since $F(t, z_t)$ has convex values, one has $\delta h_1 + (1-\delta)h_2 \in \mathcal{B}(z)$.

Claim 3: *$\mathcal{B}$ maps bounded sets into bounded sets in D_b^0.* Let $B = \{z \in D_b^0 : \|z\|_\infty \le q\}, q >, 0$ a bounded set in D_b^0. For each $h \in \mathcal{B}(z)$, and each $z \in B$, there exists $v \in S_{F,z}$ such that

$$h(t) = \int_0^t T_1(t-s)v(s)ds.$$

From (H4) we have

$$\|z_s + x_s\|_D \le \|z_s\|_D + \|x_s\|_D \le K_b q + K_b M|\phi(0)| + M_b\|\phi\|_D = q_*.$$

Thus

$$|h(t)| \le M\psi(q_*)\int_0^t p(s)ds \le M\psi(q_*)\|p\|_{L^1} = l.$$

This further implies that $\|h\|_\infty \le l$, for all $h \in \mathcal{B}(z) \subset \mathcal{B}(B) = \bigcup_{z\in B} \mathcal{B}(z)$. Hence, $\mathcal{B}(B)$ is bounded.

Claim 4: *$\mathcal{B}$ maps bounded sets into equicontinuous sets.* Let B be as above, a bounded set, and $h \in \mathcal{B}(z)$ for some $z \in B$. Then, there exists $v \in S_{F,z}$ such that

$$h(t) = \int_0^t T_1(t-s)v(s)ds, \quad t \in J.$$

Let $\tau_1, \tau_2 \in J \backslash \{t_1, t_2, ...t_m\}, \tau_1 < \tau_2$. Thus, if $\epsilon > 0$, we have

$$|h(\tau_2) - h(\tau_1)| \le \int_0^{\tau_1-\epsilon} \|T_1(\tau_2 - s) - T_1(\tau_1 - s)\||v(s)|ds$$

$$+\int_{\tau_1-\epsilon}^{\tau_1}\|T_1(\tau_2-s)-T_1(\tau_1-s)\||v(s)|ds+\int_{\tau_1}^{\tau_2}\|T_1(\tau_2-s)\||v(s)|ds$$
$$\le\psi(q_*)\int_0^{\tau_1-\epsilon}\|T_1(\tau_2-s)-T_1(\tau_1-s)\|_{B(E)}p(s)ds$$
$$+\psi(q_*)\int_{\tau_1-\epsilon}^{\tau_1}\|T_1(\tau_2-s)-T_1(\tau_1-s)\|_{B(E)}p(s)ds$$
$$+\psi(q_*)\int_{\tau_1}^{\tau_2}\|T_1(\tau_2-s)\|_{B(E)}p(s)ds.$$

As $\tau_1\to\tau_2$ and ϵ becomes sufficiently small, the right-hand side of the above inequality tends to zero, since $T_1(t)$ is a strongly continuous operator and the compactness of $T_1(t)$ for $t>0$ implies the continuity in the uniform operator topology. This proves the equicontinuity for the case where $t\neq t_i, i=1,...,m+1$. It remains to examine the equicontinuity at $t=t_i$. First, we prove the equicontinuity at $t=t_i^-$; we know that, for some $z\in B$, there exists $v\in S_{F,z}$ such that

$$h(t)=\int_0^t T_1(t-s)v(s)ds,\quad t\in J.$$

Fix $\delta_1>0$ such that $\{t_k,k\neq i\}\cap[t_i-\delta_1,t_i+\delta_1]=\emptyset$. For $0<\rho<\delta_1$, we have

$$|h(t_i-\rho)-h(t_i)|$$
$$\le\int_0^{t_i-\rho}\|T_1(t_i-\rho-s)-T_1(t_i-s)\||v(s)|ds+\psi(q_*)M\int_{t_i-\rho}^{t_i}p(s)ds,$$

which tends to zero as $\rho\to 0$. Define

$$\hat{h}_0(t)=h(t),\ t\in[0,t_1]\ \text{ and }\ \hat{h}_i(t)=\begin{cases}h(t), & \text{if } t\in(t_i,t_{i+1}]\\ h(t_i^+), & \text{if } \ t=t_i.\end{cases}$$

Next, we prove equicontinuity at $t=t_i^+$. Fix $\delta_2>0$ such that $\{t_k,k\neq i\}\cap[t_i-\delta_2,t_i+\delta_2]=\emptyset$. Then

$$\hat{h}(t_i)=\int_0^{t_i}T_1(t_i-s)v(s)ds.$$

For $0<\rho<\delta_2$, we have

$$|\hat{h}(t_i+\rho)-\hat{h}(t_i)|\le\int_0^{t_i}\|T_1(t_i+\rho-s)-T_1(t_i-s)\||v(s)|ds+\psi(q_*)M\int_{t_i}^{t_i+\rho}p(s)ds.$$

The right-hand side tends to zero as $\rho \to 0$. The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follows from the uniform continuity of ϕ on the interval $(-\infty, 0]$. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that $\mathcal{B}$ maps B into a precompact set in E. Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B$, we define

$$h_\epsilon(t) = T_1(\epsilon) \int_0^{t-\epsilon} T_1(t-s-\epsilon)v(s)ds,$$

where $v \in S_{F,z}$. Since $T_1(t)$ is a compact operator, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in \mathcal{B}(z)\}$ is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover, for every $h \in \mathcal{B}(z)$ we have

$$\begin{aligned} |h(t) - h_\epsilon(t)| &= \Big| \int_0^t T_1(t-s)v(s)ds - T_1(\epsilon) \int_0^{t-\epsilon} T_1(t-s-\epsilon)v(s)ds \Big| \\ &= \Big| \int_{t-\epsilon}^t T_1(t-s)v(s)ds \Big| \leq M\psi(q_*) \int_{t-\epsilon}^t p(s)ds. \end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h(t) : h \in \mathcal{B}(z)\}$. Hence the set $H(t) = \{h(t) : h \in \mathcal{B}(B)\}$ is precompact in E. Hence the operator $\mathcal{B}$ is totally bounded.

Claim 5 : *$\mathcal{B}$ has closed graph.* Let $z_n \to z_*$, $h_n \in \mathcal{B}(z_n)$, and $h_n \to h_*$. We shall show that $h_* \in \mathcal{B}(z_*)$. $h_n \in \mathcal{B}(z_n)$ means that there exists $v_n \in S_{F,z_n}$ such that

$$h_n(t) = \int_0^t T_1(t-s)v_n(s)ds, \quad t \in J.$$

We must prove that there exists $v_* \in S_{F,z_*}$ such that

$$h_*(t) = \int_0^t T_1(t-s)v_*(s)ds.$$

Consider the linear and continuous operator $\mathcal{K} : L^1(J, E) \to D_b^0$ defined by

$$(\mathcal{K}v)(t) = \int_0^t T_1(t-s)v(s)ds.$$

We have

$$|(h_n(t) - (h_*(t))| \leq \|h_n - h_*\|_\infty \to 0, \quad \text{as } n \mapsto \infty.$$

From Lemma 4.1 it follows that $\mathcal{K} \circ S_F$ is a closed graph operator and from the definition of $\mathcal{K}$ one has $h_n(t) \in \mathcal{K} \circ S_{F,z_n}$. As $z_n \to z_*$ and $h_n \to h_*$, there is a $v_* \in S_{F,z_*}$ such that

$$h_*(t) = \int_0^t T_1(t-s)v_*(s)ds.$$

Hence, the multivalued operator $\mathcal{B}$ is upper semi-continuous.

Step 3: A priori bounds on solutions. Now, it remains to show that the set $\mathcal{E} = \{z \in D_b^0 : z \in \lambda\mathcal{A}z + \lambda\mathcal{B}z, \quad 0 \le \lambda \le 1\}$ is unbounded. Let $z \in \mathcal{E}$ be any element; then there exist $v \in S_{F,z}$ and $\mathcal{I}_k \in I_k(z(t_k^-))$ such that

$$\begin{aligned} z(t) &= \lambda g(t, z_t + x_t) - \lambda T_0(t) g(0, \phi(0)) \\ &\quad + \lambda \int_0^t T_1(t-s)v(s)ds + \lambda \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k. \end{aligned}$$

Thus

$$\begin{aligned} |z(t)| &\le \alpha_1 \|z_t + x_t\| + \alpha_2 + M\big(\alpha_1\|\phi\|_D + \alpha_2\big) + M\int_0^t p(s)\psi(\|z_s + x_s\|)ds \\ &\quad + M\sum_{k=1}^m c_k|z(t_k^-) + x(t_k^-)| + M\sum_{k=1}^m \|I_k(0)\| \\ &\le \alpha_1\big(K_b|z(t)| + (MK_b + M_b)\|\phi\|_D\big) + \alpha_2 + M\big(\alpha_1\|\phi\|_D + \alpha_2\big) \\ &\quad + M\int_0^t p(s)\psi\big(K_b|z(s)| + (MK_b + M_b)\|\phi\|_D\big)\big)ds \\ &\quad + M\sum_{k=1}^m c_k|z(t)| + M\sum_{k=1}^m c_k|x(t)| + M\sum_{k=1}^m \|I_k(0)\| \\ &\le \alpha_1\big(K_b|z(t)| + (MK_b + M_b)\|\phi\|_D\big) + \alpha_2 + M\big(\alpha_1\|\phi\|_D + \alpha_2\big) \\ &\quad + M\int_0^t p(s)\psi\big(K_b|z(s)| + (MK_b + M_b)\|\phi\|_D\big)\big)ds \\ &\quad + M\sum_{k=1}^m c_k|z(t)| + M\sum_{k=1}^m c_k|x(t)| + M\sum_{k=1}^m \|I_k(0)\|. \end{aligned}$$

Then we have

$$\begin{aligned} |z(t)| &\le \alpha_1\big((K_b|z(t)| + (MK_b + M_b)\|\phi\|_D\big)\big) \\ &\quad + M\int_0^t p(s)\psi\big(K_b|z(s)| + (MK_b + M_b)\|\phi\|_D\big)\big)ds \end{aligned}$$

$$+ M\sum_{k=1}^{m} c_k|z(t)| + M^2\sum_{k=1}^{m} c_k\|\phi\|_D + M\sum_{k=1}^{m}\|I_k(0)\|.$$

Thus

$$\Big(1-\alpha_1 K_b - M\sum_{k=0}^{m} c_k\Big)\|z\|_{D_b^0} \le \alpha_1 + (MK_b+M_b)\|\phi\|_D$$
$$+\alpha_2 + M(\alpha_1\|\phi\|_D+\alpha_2) + M\|p\|_{L^1}\psi\Big(K_b\|z\|_{D_b^0} + (MK_b+M_b)\|\phi\|_D\Big)$$
$$+ M^2\sum_{k=1}^{m} c_k\|\phi\|_D + M\sum_{k=1}^{m}\|I_k(0)\|$$
$$= C_0 + M\|p\|_{L^1}\psi\big(K_b\|z\|_{D_b^0} + (MK_b+M_b)\|\phi\|_D\big).$$

Then, by the previous inequality, we have

$$\frac{\big(1-\alpha_1 K_b - M\sum_{k=1}^{m} c_k\big)\|z\|_{D_b^0}}{C_0 + M\|p\|_{L^1}\psi\big(K_b\|z\|_{D_b^0} + (MK_b+M_b)\|\phi\|_D\big)} \le 1. \tag{4.3}$$

From (4.2) it follows that there exists a constant $R > 0$ such that for each $z \in \mathcal{E}$ with $\|z\|_{D_b^0} > R$ the property is not satisfied. Hence, $\|z\|_{D_b^0} \le R$ for each $z \in \mathcal{E}$ which means that $\mathcal{E}$ is bounded. As a consequence of Theorem 2.1, $\mathcal{A}+\mathcal{B}$ has a fixed point z^* on the interval $(-\infty, b]$, so $y^* = z^* + x$ is a fixed point of the operator N which is the mild solution of problem (1.1)-(1.3).

5. Existence of Extremal Mild Solutions

In this section, we shall prove the existence of maximal and minimal solutions of problem (1.1)-(1.3) under suitable monotonicity conditions on the multivalued functions involved in it.

Definition 5.1. *A nonempty closed subset C of a Banach space $(X, \|\cdot\|)$ is said to be a cone if (i) $C + C \subset C$, (ii) $\lambda C \subset C$, (iii) $\{-C\}\bigcap\{C\} = \{0\}$.*

A cone C is called normal if the norm $\|\cdot\|$ is semi-monotone on C; i.e., there exists a constant $N > 0$ such that $\|x\| \le N\|y\|$, whenever $x \le y$. We equip the space $X = C(J, E)$ with the order relation $\le$ induced by a regular cone $\mathcal{C}$ in E; that is, for all $y, \overline{y} \in X$, $y \le \overline{y}$ if and only if $\overline{y}(t) - y(t) \in \mathcal{C}$,for all $t \in J$. In what follows we will assume that the cone C is normal. Cones and their properties are detailed in [36]. Let $a, b \in X$ be such that $a \le b$. Then, by an order interval $[a, b]$, we mean a set of points in X given by

$$[a, b] = \{x \in X : a \le x \le b\}.$$

Let $D, Q \in P_{cl}(X)$. Then by $D \leq Q$ we mean $a \leq b$ for all $a \in D$ and $b \in Q$. Thus $a \leq D$ implies that $a \leq b$ for all $b \in Q$; in particular, if $D \leq D$, then it follows that D is a singleton set.

Definition 5.2. *Let X be an ordered Banach space. A multivalued mapping $T : X \to P(X)$ is called isotone increasing if $T(x) \leq T(y)$ for any $x, y \in X$ with $x < y$. Similarly, T is called isotone decreasing if $T(x) \geq T(y)$ whenever $x < y$.*

Definition 5.3. *We say that $x \in X$ is the least fixed point of G in X if $x \in Gx$ and $x \leq y$ whenever $y \in X$ and $y \in Gy$. The greatest fixed point of G in X is defined similarly by reversing the inequality. If both least and greatest fixed points of G in X exist, we call them extremal fixed points of G in X.*

Very recently Dhage has proved the following.

Theorem 5.1. [26]. *Let $[a, b]$ be an order interval in a Banach space and let $B_1, B_2 : [a, b] \to P(X)$ be two functions satisfying the following:*

(a) *B_1 is a contraction,*
(b) *B_2 is completely continuous,*
(c) *B_1 and B_2 are strictly monotone increasing, and*
(d) *$B_1(x) + B_2(x) \in [a, b]$, $\forall\, x \in [a, b]$.*

Further, if the cone C in X is normal, then the inclusion $x \in B_1(x) + B_2(x)$ has a least fixed point x_* and a greatest fixed point $x^* \in [a, b]$. Moreover, $x_* = \lim_{n\to\infty} x_n$ and $x^* = \lim_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by

$$x_{n+1} \in B_1(x_n) + B_2(x_n),\ x_0 = a \ \text{ and } \ y_{n+1} \in B_1(y_n) + B_2(y_n),\ y_0 = b.$$

We need the following definitions in the sequel.

Definition 5.4. *We say that a continuous function $\tilde{v} \in D_b$ is a lower mild solution of problem* (1.1)-(1.3) *if $\tilde{v}(t) = \phi(t)$, $t \in (-\infty, 0]$, and there exist $v(\cdot) \in L^1(J_k, E)$ and $\mathcal{I}_k \in I_k(\tilde{v}(t_k^-))$, such that $v(t) \in F(t, \tilde{v}_t)$ for almost every $t \in [0, b]$, and $\tilde{v}$ satisfies*

$$\begin{aligned}\tilde{v}(t) &\leq T_0(t)\left(\phi(0) - g(0, \phi(0))\right) + g(t, \tilde{v}_t) + \int_0^t T_1(t-s)v(s)ds \\ &\quad + \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k, \quad t \in J,\ t \neq t_k,\end{aligned}$$

and $\tilde{v}(t_k^+)-\tilde{v}(t_k^-) \leq \mathcal{I}_k$ *where* $\mathcal{I}_k \in I_k(\tilde{v}(t_k)),\ t = t_k,\ k = 1, \dots, m$. *Similarly an upper mild solution* $\tilde{w}$ *of IVP* (1.1)-(1.3) *is defined by reversing the order.*

Definition 5.5. *A solution* x_M *of IVP* (1.1)-(1.3) *is said to be maximal if, for any other solution* x *of IVP* (1.1)-(1.3) *on* J*, we have that* $x(t) \leq x_M(t)$ *for each* $t \in J$*. Similarly, a minimal solution of IVP (1.1)-(1.3) is defined by reversing the order of the inequalities.*

Definition 5.6. *A multivalued function* $F(t, x)$ *is called strictly monotone increasing in* x *almost everywhere for* $t \in J$*, if* $F(t, x) \leq F(t, y)$ *for almost every* $t \in J$ *for all* $x, y \in X$ *with* $x < y$*. Similarly,* $F(t, x)$ *is called strictly monotone decreasing in* x *almost everywhere for* $t \in J$*, if* $F(t, x) \geq F(t, y)$ *for almost every* $t \in J$ *for all* $x, y \in X$ *with* $x < y$*.*

We consider the following assumptions in the sequel.

- (H7) The multivalued function $F(t, y)$ is strictly monotone increasing in y for almost every $t \in J$.
- (H8) The IVP (1.1)-(1.3) has a lower mild solution $\tilde{v}$ and an upper mild solution $\tilde{w}$ with $\tilde{v} \leq \tilde{w}$.
- (H9) $T_1(t)$ is preserving the order, that is $T_1(t)v \geq 0$ whenever $v \geq 0$.
- (H10) The functions $I_k,\ k = 1, \dots m$ are continuous and nondecreasing.

Theorem 5.2. *Assume that assumptions* (H1)-(H10) *hold. If* $\phi \in D,\ \phi(0) \in X_0$ *and* $g(0, \phi(0)) \in X_0$*, then IVP* (1.1)-(1.3) *has minimal and maximal solutions on* D_b*.*

Proof. We can write $\tilde{v}$ and $\tilde{w}$ as $\tilde{v}(t) = v^*(t) + x(t)$, $\tilde{w}(t) = w^*(t) + x(t)$, where $v^* \in D_b^0$ and $w^* \in D_b^0$ and $x(t)$ is defined in the above section. Then $\tilde{v}$ is a lower solution to IVP(1.1)-(1.3) if v^* satisfies

$$\begin{aligned} v^*(t) \leq & -T_0(t)g(0, \phi(0)) + g(t, v_t^* + x_t) + \int_0^t T_1(t-s)v(s)ds \\ & + \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k, \quad t \in J,\ t \neq t_k, \end{aligned}$$

and $v^*(t_k^+) - v^*(t_k^-) \leq \mathcal{I}_k$ such that $\mathcal{I}_k \in I_k(v^*(t_k)),\ t = t_k,\ k = 1, \dots, m$. Analogously, $\tilde{w}$ is an upper solution to IVP(1.1)-(1.3) if w^* satisfies the reversed inequality.

It can be shown, as in the proof of Theorem 4.1, that $\mathcal{A}$ is completely continuous and $\mathcal{B}$ is a contraction on $[v^*, w^*]$. We shall show that $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v^*, w^*]$. Let $z, \overline{z} \in [v^*, w^*]$ be such that $z \leq \overline{z}$, $z \neq \overline{z}$.

Then by (H10), we have for each $t \in J$

$$\begin{aligned}\mathcal{A}(z) &= \{h \in D_b^0 : h(t) = -T_0(t)g(0,\phi(0)) + g(t, z_t + x_t) \\ &\quad + \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k,\ \mathcal{I}_k \in I_k(z(t_k^-))\} \\ &\leq \{h \in D_b^0 : h(t) = -T_0(t)g(0,\phi(0)) + g(t, \overline{z}_t + x_t) \\ &\quad + \sum_{0<t_k<t} T_1(t-t_k)\mathcal{I}_k,\ \mathcal{I}_k \in I_k(\overline{z}(t_k^-))\} = \mathcal{A}(\overline{z}).\end{aligned}$$

Similarly, by $(H7)$ and $(H9)$,

$$\begin{aligned}\mathcal{B}(z) &= \{h \in D_b^0 : h(t) = \int_0^t T_1(t-s)v(s)ds, \quad v \in S_{F,z}\} \\ &\leq \{h \in D_b^0 : h(t) = \int_0^t T_1(t-s)v(s)ds, \quad f \in S_{F,\overline{z}}\} = \mathcal{B}(\overline{z}).\end{aligned}$$

Therefore, $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v^*, w^*]$.

Finally, let $y \in [v^*, w^*]$ be any element. By (H8) and (H9) we deduce that

$$v^* \leq \mathcal{A}(v^*) + \mathcal{B}(v^*) \leq \mathcal{A}(y) + \mathcal{B}(y) \leq \mathcal{A}(w^*) + \mathcal{B}(w^*) \leq w^*,$$

which shows that $\mathcal{A}(y) + \mathcal{B}(y) \in [v^*, w^*]$ for all $y \in [v^*, w^*]$. Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Theorem 5.1, hence IVP (1.1)-(1.3) has maximal and minimal solutions on J. This completes the proof. □

6. EXAMPLE

To apply our previous results, we consider the following impulsive partial neutral functional differential equation:

$$\begin{aligned}&\frac{\partial}{\partial t}\big[v(t,\xi) - \int_{-\infty}^0 K_1(\theta, v(t+\theta), \xi)d\theta\big] \\ &= \frac{\partial^2}{\partial t^2}\big[v(t,\xi) - \int_{-\infty}^0 K_1(\theta, v(t+\theta), \xi)d\theta\big] \\ &+ \int_{-\infty}^0 K_2(\theta)[Q_1(t,\phi(\theta,\xi)), Q_2(t,\phi(\theta,\xi))]d\theta, \\ &t \in J = [0,b],\ t \neq t_k,\ k = 1,\dots,m,\ 0 \leq \xi \leq 1\end{aligned} \tag{6.1}$$

$$v(t_k^+,\xi) - v(t_k^-,\xi) \in [-b_k|v(t_k^-,\xi)|, b_k|v(t_k^-,\xi)|], \quad \xi \in [0,1], \quad k = 1,\dots,m \tag{6.2}$$

$$v(t,0) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta), 0)d\theta = 0, \ t \in J, \tag{6.3}$$

$$v(t,1) - \int_{-\infty}^{0} K_1(\theta, v(t+\theta), 1)d\theta = 0, \ t \in J, \tag{6.4}$$

$$v(\theta,\xi) = v_0(\theta,\xi) - \infty < \theta \leq 0, \ 0 \leq \xi \leq 1, \tag{6.5}$$

where $b_k > 0, \ k = 1, \ldots, m, \ K_1 : (-\infty, 0] \times \mathbb{R} \to \mathbb{R}, \ K_2 : (-\infty, 0] \to \mathbb{R}, \ Q_1, \ Q_2 : J \times \mathbb{R} \to \mathbb{R}$ and $v_0 : (-\infty, 0] \times [0,1] \to \mathbb{R}$ are continuous functions, $v(t_k^-) = \lim_{(h,\xi)\to(0^-,\xi)} v(t_k + h, \xi), \ v(t_k^+) = \lim_{(h,\xi)\to(0^+,\xi)} v(t_k + h, \xi)$. We assume that, for each $t \in J, \ Q_1(t,\cdot)$ is lower semi-continuous (i.e, the set $\{y \in \mathbb{R} : Q_1(t,y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that, for each $t \in J, \ Q_2(t,\cdot)$ is upper semi-continuous (i.e., the set $\{y \in \mathbb{R} : Q_2(t,y) < \mu\}$ is open for each $\mu \in \mathbb{R}$).

We choose $E = C([0,1]; \mathbb{R})$ endowed with the uniform topology and consider the operator $A : D(A) \subset E \to E$ defined by

$$D(A) = \{y \in C^2([0,1], \mathbb{R}) : y(0) = y(1) = 0\} \quad Ay = y''.$$

It is well known (see [23]) that the operator A satisfies the Hille-Yosida condition with $(0, +\infty) \subset \rho(A), \ \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ for $\lambda > 0$, and

$$X_0 = \overline{D(A)} = \{y \in E : y(0) = y(1) = 0\} \neq E.$$

Thus the extrapolation method can be applied. We define

$$I_k(y(t_k^-))(\xi) = [-b_k|v(t_k^-, \xi)|, b_k|v(t_k^-, \xi)|], \ \xi \in [0,1], \ k = 1, \ldots, m$$

$$F(t,\phi)(\xi) = \int_{-\infty}^{0} K_2(\theta)[Q_1(t, \phi(\theta,\xi)), Q_2(t, \phi(\theta,\xi))]d\theta, \ t \in J, \ \xi \in [0,1],$$

$$g(t,\phi)(\xi) = \int_{-\infty}^{0} K_1(\theta, \phi(\theta)(\xi))d\theta, \ t \in J, \ \xi \in [0,1],$$

$$y(t)(\xi) = v(t,\xi), \ t \in J, \ \xi \in [0,1],$$

$$\phi(\theta)(\xi) = v_0(\theta,\xi), \ \theta \leq 0, \ \xi \in [0,1].$$

Then Problem (1.1)-(1.3) is an abstract formulation of the problem (6.1)-(6.5) with F assuming compact and convex values, and it is upper semi-continuous (see [24]). Assume that there are $p \in C(J, \mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\max(|Q_1(t,y)|, \ |Q_2(t,y)|) \leq p(t)\psi(|y|), \quad t \in J, \text{ and } \ y \in \mathbb{R}.$$

Under suitable conditions, the problem (6.1)-(6.5) has by Theorem 4.1 a solution on $(-\infty, b] \times [0,1]$.

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