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LOW REGULARITY WELL-POSEDNESS FOR SOME NONLINEAR DIRAC EQUATIONS IN ONE SPACE DIMENSION

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Abstract. We prove that the Cauchy problem for a nonlinear Dirac equation with vector self-interaction (Thirring model) and for a nonlinear system of two Dirac equations coupled through a vector-vector interaction (Federbusch model) are locally well posed, in one space dimension, for initial data in Sobolev spaces of almost critical dimension; i.e., in H^ε, the critical space being L^2, and globally well posed for initial data in $H^{1/2+\varepsilon}$, for any $\varepsilon > 0$. We also consider a nonlinear Dirac equation with quadratic nonlinearity which was studied earlier by S. Machihara and N. Bournaveas. We prove that the Cauchy problem for this equation is locally well posed for initial data in H^ε.

1. Introduction

In this paper our concern is low regularity well posedness for the following equations in one space dimension (given initial data of a certain regularity):

(I) Thirring model [15]:

$$(-i\gamma^\mu \partial_\mu + m)\,\psi = \lambda(\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi; \tag{1.1}$$

(II) Federbusch model [7]:

$$\begin{cases} (-i\gamma^\mu \partial_\mu + m_1)\,\psi = \lambda(\overline{\phi}\gamma^\mu\phi)\gamma_\mu\psi, \\ (-i\gamma^\mu \partial_\mu + m_2)\,\phi = \lambda(\overline{\psi}\gamma^\mu\psi)\gamma_\mu\phi; \end{cases} \tag{1.2}$$

(III) Dirac equation with quadratic nonlinearity:

$$(-i\gamma^\mu \partial_\mu + m)\,\psi = \lambda Q(\psi). \tag{1.3}$$

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Here, the unknowns $\psi = \binom{\psi_1}{\psi_2}$ and $\phi = \binom{\phi_1}{\phi_2}$ are $\mathbb{C}^2$-valued functions of $(t,x) \in \mathbb{R} \times \mathbb{R}$ and $m,\ m_1,\ m_2 \geq 0$ and $\lambda \in \mathbb{C}$ are constants. Repeated indices are implicitly summed over $\mu = 0,1$, we write $\partial_0 = \partial_t$ and $\partial_1 = \partial_x$, the γ^μ's are 2×2 Dirac matrices in the representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and we write $\overline{\psi} = \psi^\dagger \gamma^0$, where $\dagger$ denotes conjugate transpose. Indices are lowered using the metric $g = \operatorname{diag}(1,-1)$. The quadratic nonlinearity in (1.3) is given by

$$Q(\psi) = \begin{pmatrix} |\psi_1|^2 \\ |\psi_2|^2 \end{pmatrix}. \tag{1.4}$$

We complement (1.1) and (1.3) with initial data

$$\psi|_{t=0} = \psi_0 \in H^s, \tag{1.5}$$

and (1.2) with data

$$\psi|_{t=0} = \psi_0 \in H^s, \quad \phi|_{t=0} = \phi_0 \in H^s, \tag{1.6}$$

where $H^s = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{R})$ is the standard Sobolev space of order s.

Concerning the Cauchy problems (1.1, 1.5) and (1.2, 1.6), global existence was proved by Delgado [4] for $s = 1$. Global existence and time decay of solutions to (1.1, 1.5) with sufficiently smooth data was also studied by Dias and Figueira [5]. Recently, Machihara and Omoso [10] obtained explicit solutions for this Cauchy problem. In higher dimensions, nonlinear Dirac equations have been studied in [2, 6, 12, 11].

We prove local well posedness of (1.1, 1.5) and (1.2, 1.6) for all $s > 0$. This is almost optimal, since the problems (in 1d) are charge critical (i.e., L^2-critical), but we are then not able to use the conservation of charge to extend the local result to a global one. To get global existence, Delgado [4] first obtained an a priori bound on L_x^∞ for the Dirac spinor. Using the same idea, we show that the local well posedness of (1.1, 1.5) and (1.2, 1.6) for $s > 1/2$ extends to global well posedness, by using the Sobolev embedding in 1d, $H^{1/2+\varepsilon} \hookrightarrow L_x^\infty$.

Our results for (1.1, 1.5) and (1.2, 1.6), stated in Theorems 1 and 2 below, improve on the earlier local and global results, by exploiting the structure of the equations and by using spaces of Bourgain-Klainerman-Machedon type. We prove local well posedness for $s > 0$ and global well posedness for $s > 1/2$. W also prove unconditional uniqueness for $s > 1/4$ (the contraction argument only gives uniqueness in the iteration space).

Theorem 1. *Suppose $s > 0$. Then we have the following:*

(i) There exists a time $T > 0$ and a solution of the Cauchy problem (1.1)-(1.5) *on $(0,T) \times \mathbb{R}$,*

$$\psi \in C([0,T], H^s),$$

which depends continuously on the data. The solution is unique in a certain subspace of $C([0,T], H^s)$. Furthermore, if $s > \frac{1}{4}$, the solution is unique in $C([0,T], H^s)$.

(ii) There exists a time $T > 0$ and a solution of the Cauchy problem (1.2)-(1.6) *on $(0,T) \times \mathbb{R}$,*

$$(\psi, \phi) \in C([0,T], H^s) \times C([0,T], H^s),$$

which depends continuously on the data. The solution is unique in a certain subspace of $C([0,T], H^s) \times C([0,T], H^s)$. Furthermore, if $s > \frac{1}{4}$, the solution is unique in $C([0,T], H^s) \times C([0,T], H^s)$.

Theorem 2. *Suppose $s > 1/2$. Then we have the following.*

(i) There exists a unique global solution of the Cauchy problem (1.1)-(1.5)

$$\psi \in C([0,\infty), H^s),$$

which depends continuously on the data.

(ii) There exists a unique global solution of the Cauchy problem (1.2)-(1.6)

$$(\psi, \phi) \in C([0,\infty), H^s) \times C([0,\infty), H^s),$$

which depends continuously on the data.

Concerning the Cauchy problem (1.3, 1.5), local well posedness was proved by Machihara [8] for $s > 1/4$. The scaling properties of the system give the critical exponent $s_{cr} = -1/2$, so the result of Machihara leaves a gap. More recently, Bournaveas [1] proved local well posedness of the same problem for data $\psi_0 \in \widehat{H^{s,p}}$ with (s,p) in the region $s > \frac{1}{2p}$, $s > \frac{2}{p} - 1$, generalizing the results for $p = 2$ by Machihara. Here, $\widehat{H^{s,p}}$ is the L^p-based space with norm

$$\|f\|_{\widehat{H^{s,p}}} = \left\| \langle \xi \rangle^s \hat{f}(\xi) \right\|_{L^{p'}_\xi},$$

where $1/p + 1/p' = 1$ and

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

This result is an improvement from the scaling point of view; see [1].

In this paper we use, like Machihara, the L^2-based data space H^s, and prove local well posedness for $s > 0$, thus improving Machihara's result ($s > 1/4$). Moreover, we prove unconditional uniqueness for $s > 1/6$.

Theorem 3. *Suppose $s > 0$. Then there exists a time $T > 0$ and a solution of the Cauchy problem* (1.3)-(1.5) *on* $(0,T) \times \mathbb{R}$,

$$\psi \in C([0,T], H^s),$$

which depends continuously on the data. The solution is unique in a certain subspace of $C([0,T], H^s)$. Furthermore, if $s > \frac{1}{6}$, the solution is unique in $C([0,T], H^s)$.

This paper is organized as follows. In the next section, we fix some notation, state definitions, and recall some linear and nonlinear estimates needed in the proofs of our theorems. In Sections 3, 4 and 5 we prove Theorems 3, 1 and 2, respectively.

2. Notation, Definitions and some basic estimates

In estimates, C denotes a positive constant which can vary from line to line. We use the shorthand $X \lesssim Y$ for $X \le CY$, and if $C \ll 1$ we use the symbol $\ll$ instead of $\lesssim$. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$. Throughout, ε denotes a sufficiently small positive number. We use the notation

$$\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}.$$

The Fourier transform in space-time is defined by

$$\widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^{1+1}} e^{-i(t\tau + x\xi)} u(t,x)\, dt\, dx.$$

We use the following spaces of Bourgain-Klainerman-Machedon type: For $a, \alpha \in \mathbb{R}$, define $X_\pm^{a,\alpha}$ and $H^{a,\alpha}$ to be the completions of $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to the norms, respectively,

$$\|u\|_{X_\pm^{a,\alpha}} = \left\| \langle \xi \rangle^a \langle \tau \pm \xi \rangle^\alpha \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}},$$
$$\|u\|_{H^{a,\alpha}} = \left\| \langle \xi \rangle^a \langle |\tau| - |\xi| \rangle^\alpha \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}}.$$

Denote $S_T = (0,T) \times \mathbb{R}$. The restriction space $X_\pm^{a,\alpha}(S_T)$ is a Banach space with norm $\|u\|_{X_\pm^{a,\alpha}(S_T)} = \inf_{\tilde{u}_{|S_T} = u} \|\tilde{u}\|_{X_\pm^{a,\alpha}}$.

We need the fact that, if $\alpha > 1/2$, then

$$\|u(t)\|_{H^a} \le C \|u\|_{X_\pm^{a,\alpha}(S_T)} \qquad \text{for } 0 \le t \le T, \tag{2.1}$$

where C depends only on α. We also need the estimates

$$\|u\|_{H^{a,\alpha}} \lesssim \|u\|_{X_\pm^{a,\alpha}} \quad \text{if } \alpha \geq 0, \tag{2.2}$$

$$\|u\|_{X_\pm^{a,\alpha}} \lesssim \|u\|_{H^{a,\alpha}} \quad \text{if } \alpha \leq 0. \tag{2.3}$$

The following well-known linear estimates (for references, see [3]) will be used.

Lemma 1. *Let $1/2 < b \leq 1$, $a \in \mathbb{R}$, $0 < T \leq 1$ and $0 \leq \delta \leq 1 - b$. Then for all data $F \in X_\pm^{a,b-1+\delta}(S_T)$ and $f \in H^a$ the Cauchy problem*

$$-i(\partial_t \pm \partial_x)u = F(t,x) \quad in \quad (0,T)\times \in \mathbb{R}, \qquad u(0,x) = f(x),$$

has a unique solution $u \in X_\pm^{a,b}(S_T)$. Moreover,

$$\|u\|_{X_\pm^{a,b}(S_T)} \leq C\Big(\|f\|_{H^a} + T^\delta \|F\|_{X_\pm^{a,b-1+\delta}(S_T)}\Big),$$

where C depends only on b.

We shall need the standard product estimate for the Sobolev spaces H^s.

Lemma 2. *If $a_1, a_2, a_3 \in \mathbb{R}$ satisfy*

$$a_1 + a_2 + a_3 > 1/2, \quad a_1 + a_2 \geq 0, \quad a_1 + a_3 \geq 0, \quad a_2 + a_3 \geq 0, \tag{2.4}$$

then

$$\|fg\|_{H^{-a_3}} \lesssim \|f\|_{H^{a_1}} \|g\|_{H^{a_2}}. \tag{2.5}$$

Moreover, we can allow $a_1 + a_2 + a_3 = 1/2$ if $a_j < 1/2$ for $1 \leq j \leq 3$.

The following estimate, taken from [14], is just the analogue of Lemma 2 for the wave-Sobolev space $H^{s,b}$.

Lemma 3. *Suppose a_1, a_2, a_3 satisfy (2.4). Let $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma > \frac{1}{2}$. Then*

$$\|uv\|_{H^{-a_3,-\gamma}} \lesssim \|u\|_{H^{a_1,\alpha}} \|v\|_{H^{a_2,\beta}}. \tag{2.6}$$

Moreover, we can allow $a_1 + a_2 + a_3 = 1/2$ if $a_j \neq 1/2$ for $1 \leq j \leq 3$.

The following bilinear estimate, proved in [14], is crucial in the proof of Theorem 1.

Lemma 4. *For any $\alpha > 1/2$,*

$$\|uv\|_{L^2} \lesssim \|u\|_{X_+^{0,\alpha}} \|v\|_{X_-^{0,\alpha}}.$$

The following comparison estimate between elliptic and hyperbolic weights will also be needed in the proof of Theorem 3.

Lemma 5. *Denote*

$$\Gamma_\pm = \tau \pm \xi, \qquad \Theta_\pm = \lambda \pm \eta, \qquad \Sigma_\pm = \lambda - \tau \pm (\eta - \xi).$$

Then

$$|\xi| \le \frac{3}{2} \max(|\Gamma_\pm|, |\Theta_\mp|, |\Sigma_\mp|). \tag{2.7}$$

Proof. This follows from the identity $\Gamma_\pm - \Theta_\mp + \Sigma_\mp = \pm 2\xi$. □

3. PROOF OF THEOREM 3

3.1. **Local existence of solution when** $s > 0$**.** First, using $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and the representation of the Dirac matrices γ^0 and γ^1, we calculate

$$(\gamma^0 \partial_t + \gamma^1 \partial_x)\psi = \begin{pmatrix} \partial_t \psi_2 - \partial_x \psi_2 \\ \partial_t \psi_1 + \partial_x \psi_1 \end{pmatrix}.$$

Thus, the equation (1.3) (using (1.4)) can be rewritten as

$$\begin{cases} -i(\partial_t + \partial_x)\psi_1 = -m\psi_2 + \lambda \left|\psi_2\right|^2, \\ -i(\partial_t - \partial_x)\psi_2 = -m\psi_1 + \lambda \left|\psi_1\right|^2. \end{cases} \tag{3.1}$$

Putting $\psi_0 = \begin{pmatrix} \psi_{1,0} \\ \psi_{2,0} \end{pmatrix}$ the data (1.5) becomes

$$\psi_1|_{t=0} = \psi_{1,0} \in H^s, \quad \psi_2|_{t=0} = \psi_{2,0} \in H^s. \tag{3.2}$$

We now apply the standard Picard iteration to the Cauchy problem (3.1, 3.2). Set $\psi_1^{(-1)} = \psi_2^{(-1)} = 0$, and define inductively, for $n \ge -1$,

$$\begin{cases} -i(\partial_t + \partial_x)\psi_1^{(n+1)} = -m\psi_2^{(n)} + \lambda \big|\psi_2^{(n)}\big|^2, \\ -i(\partial_t - \partial_x)\psi_2^{(n+1)} = -m\psi_1^{(n)} + \lambda \big|\psi_1^{(n)}\big|^2, \end{cases} \tag{3.3}$$

with data

$$\psi_1^{(n+1)}(0) = \psi_{1,0} \in H^s, \qquad \psi_2^{(n+1)}(0) = \psi_{2,0} \in H^s.$$

We iterate $(\psi_1^{(n)}, \psi_2^{(n)})$ in $X_+^{s,1/2+\varepsilon}(S_T) \times X_-^{s,1/2+\varepsilon}(S_T)$. Let us write

$$A_n = \left\|\psi_1^{(n)}\right\|_{X_+^{s,1/2+\varepsilon}(S_T)} + \left\|\psi_2^{(n)}\right\|_{X_-^{s,1/2+\varepsilon}(S_T)}.$$

By Lemma 1, local existence of a solution of the Cauchy problem (3.1, 3.2) reduces to proving the bilinear estimates

$$\|v\overline{v}\|_{X_+^{s,-1/2+2\varepsilon}} \lesssim \|v\|^2_{X_-^{s,1/2+\varepsilon}}, \tag{3.4}$$

$$\|u\overline{u}\|_{X_-^{s,-1/2+2\varepsilon}} \lesssim \|u\|^2_{X_+^{s,1/2+\varepsilon}}\,, \tag{3.5}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^{1+1})$. Indeed, let us for the moment assume (3.4) and (3.5) hold true, and show local existence.

Applying Lemma 1 and estimates (3.4) and (3.5) to equation (3.3), we get

$$A_{n+1} \le C\left(\|\psi_{1,0}\|_{H^s} + \|\psi_{2,0}\|_{H^s} + T^\varepsilon A_n + T^\varepsilon A_n^2\right).$$

Let $0 < T < 1$, and set $E_0 = \|\psi_{1,0}\|_{H^s} + \|\psi_{2,0}\|_{H^s}$. Now, assume $A_n \le 2CE_0$. Then $A_{n+1} \le 2CE_0$, provided that $T \le (2C + 4C^2E_0)^{-1/\varepsilon}$. This shows that the sequence of iterates $(\psi_1^{(n)}, \psi_2^{(n)})$ is bounded in $X_+^{s,1/2+\varepsilon}(S_T) \times X_-^{s,1/2+\varepsilon}(S_T)$. One can then show, similarly, that for small enough $T > 0$,

$$\begin{aligned}\left\|\psi_1^{(n+1)} - \psi_1^{(n)}\right\|_{X_+^{s,1/2+\varepsilon}(S_T)} &+ \left\|\psi_2^{(n+1)} - \psi_2^{(n)}\right\|_{X_-^{s,1/2+\varepsilon}(S_T)} \\ &\le \frac{1}{2}\left\|\psi_1^{(n)} - \psi_1^{(n-1)}\right\|_{X_+^{s,1/2+\varepsilon}(S_T)} + \left\|\psi_2^{(n)} - \psi_2^{(n-1)}\right\|_{X_-^{s,1/2+\varepsilon}(S_T)},\end{aligned}$$

hence, the sequence of iterates $(\psi_1^{(n)}, \psi_2^{(n)})$ is Cauchy in

$$X_+^{s,1/2+\varepsilon}(S_T) \times X_-^{s,1/2+\varepsilon}(S_T),$$

and hence, converges to a solution in that space, which we recall embeds into $C([0,T], H^s) \times C([0,T], H^s)$. Stability follows from similar estimates, as does uniqueness in $X_+^{s,1/2+\varepsilon}(S_T) \times X_-^{s,1/2+\varepsilon}(S_T)$.

We now prove (3.4) (the proof of (3.5) is similar). The estimate is equivalent to

$$I \lesssim \|F\|_{L^2}\|G\|_{L^2}\,,$$

where

$$I = \left\|\int_{\mathbb{R}^{1+1}} \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)\,d\lambda\,d\eta}{\langle\xi\rangle^{-s}\langle\eta\rangle^s\langle\eta-\xi\rangle^s\langle\Gamma_+\rangle^{1/2-2\varepsilon}\langle\Theta_-\rangle^{1/2+\varepsilon}\langle\Sigma_-\rangle^{1/2+\varepsilon}}\right\|_{L^2_{\tau,\xi}}.$$

We shall reduce this to Lemma 3, using also (2.2) and (2.3).

First, if $\max(|\Gamma_+|, |\Theta_-|, |\Sigma_-|) = |\Gamma_+|$, then by (2.7) and (2.2) the estimate for I reduces to (2.6) with exponents $(a_1, a_2, a_3) = (s, s, -s + 1/2 - 2\varepsilon)$ and $\alpha, \beta, \gamma \ge 0$ such that $\alpha + \beta + \gamma > 1/2$. So Lemma 3 applies if $s \ge 3\varepsilon$.

Second, if $\max(|\Gamma_+|, |\Theta_-|, |\Sigma_-|) = |\Theta_-|$ or $|\Sigma_-|$, then by (2.7), (2.2) and (2.3), the estimate for I reduces to (2.6) with exponents $(a_1, a_2, a_3) = (s, s, -s + 1/2 + \varepsilon)$ and $\alpha, \beta, \gamma \ge 0$ such that $\alpha + \beta + \gamma > 1/2$. Lemma 3 therefore applies if $s \ge 0$.

3.2. **Unconditional uniqueness when** $s > 1/6$**.** We prove uniqueness of the solution $(\psi_1, \psi_2) \in C([0,T], H^s) \times C([0,T], H^s)$ for $s > 1/6$. Since we already have uniqueness in the iteration space $(\psi_1, \psi_2) \in X_+^{3\varepsilon,1/2+\varepsilon}(S_T) \times X_-^{3\varepsilon,1/2+\varepsilon}(S_T)$ it suffices to show that, if

$$(\psi_1, \psi_2) \in C([0,T], H^{\frac{1}{6}+3\varepsilon}) \times C([0,T], H^{\frac{1}{6}+3\varepsilon}) \tag{3.6}$$

is a weak solution of (3.1, 3.2) with data $(\psi_{1,0}, \psi_{2,0}) \in H_x^{\frac{1}{6}+3\varepsilon} \times H_x^{\frac{1}{6}+3\varepsilon}$, then

$$(\psi_1, \psi_2) \in X_+^{3\varepsilon,1/2+\varepsilon}(S_T) \times X_-^{3\varepsilon,1/2+\varepsilon}(S_T). \tag{3.7}$$

We prove (3.7) as follows: By (3.6),

$$(\psi_1, \psi_2) \in X_+^{\frac{1}{6}+3\varepsilon,0}(S_T) \times X_-^{\frac{1}{6}+3\varepsilon,0}(S_T). \tag{3.8}$$

Next, we show

$$(\psi_1, \psi_2) \in X_+^{-\frac{1}{6}+4\varepsilon,1}(S_T) \times X_-^{-\frac{1}{6}+4\varepsilon,1}(S_T). \tag{3.9}$$

Applying Lemma 1 to (3.1), we reduce (3.9) to

$$\left(|\psi_2|^2, |\psi_1|^2\right) \in X_+^{-\frac{1}{6}+4\varepsilon,0}(S_T) \times X_-^{-\frac{1}{6}+4\varepsilon,0}(S_T),$$

but this holds by Lemma 2:

$$\left\| |\psi_2|^2 \right\|_{L_t^2 H_x^{-\frac{1}{6}+4\varepsilon}(S_T)} \lesssim \|\psi_2\|^2_{L_t^\infty H_x^{\frac{1}{6}+3\varepsilon}(S_T)},$$
$$\left\| |\psi_1|^2 \right\|_{L_t^2 H_x^{-\frac{1}{6}+4\varepsilon}(S_T)} \lesssim \|\psi_1\|_{L_t^\infty H_x^{\frac{1}{6}+3\varepsilon}(S_T)}.$$

Interpolation between (3.8) and (3.9) gives

$$\psi_1 \in X_+^{(1-\theta)(\frac{1}{6}+3\varepsilon)+\theta(-\frac{1}{6}+4\varepsilon),\theta}(S_T), \qquad \psi_2 \in X_-^{((1-\theta)(\frac{1}{6}+3\varepsilon)+\theta(-\frac{1}{6}+4\varepsilon),\theta}(S_T),$$

for $0 \le \theta \le 1$. In particular, taking $\theta = 1/2 + \varepsilon$, we obtain (3.7).

4. PROOF OF THEOREM 1

4.1. **Proof of (*i*).** Using $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and the representation of the Dirac matrices γ^0 and γ^1, we rewrite (1.1) as

$$\begin{cases} -i(\partial_t + \partial_x)\psi_1 = -m\psi_2 + 2\lambda |\psi_2|^2 \psi_1, \\ -i(\partial_t - \partial_x)\psi_2 = -m\psi_1 + 2\lambda |\psi_1|^2 \psi_2. \end{cases} \tag{4.1}$$

We iterate in

$$(\psi_1, \psi_2) \in X_+^{s,1/2+\varepsilon}(S_T) \times X_-^{s,1/2+\varepsilon}(S_T).$$

Local well posedness then reduces to proving the trilinear estimates

$$\|uv\overline{v}\|_{X_+^{s,-1/2+2\varepsilon}} \lesssim \|u\|_{X_+^{s,1/2+\varepsilon}} \|v\|^2_{X_-^{s,1/2+\varepsilon}} , \tag{4.2}$$

$$\|u\overline{u}v\|_{X_-^{s,-1/2+2\varepsilon}} \lesssim \|u\|^2_{X_+^{s,1/2+\varepsilon}} \|v\|_{X_-^{s,1/2+\varepsilon}} , \tag{4.3}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^{1+1})$.

We only prove (4.2), since the proof for (4.3) is similar. By duality, (4.2) is equivalent to the estimate

$$\int_{\mathbb{R}^{1+1}} u\overline{v}v\overline{w}\,dt\,dx \lesssim \|w\|_{X_+^{-s,1/2-2\varepsilon}} \|u\|_{X_+^{s,1/2+\varepsilon}} \|v\|^2_{X_-^{s,1/2+\varepsilon}} , \tag{4.4}$$

for $w \in \mathcal{S}(\mathbb{R}^{1+1})$. But

$$\text{LHS of (4.4)} \leq \|u\overline{v}\|_{H^{s,0}} \|v\overline{w}\|_{H^{-s,0}} .$$

Thus, the trilinear estimate (4.2) reduces to the following bilinear estimates:

$$\|u\overline{v}\|_{H^{s,0}} \lesssim \|u\|_{X_+^{s,1/2+\varepsilon}} \|v\|_{X_-^{s,1/2+\varepsilon}} , \tag{4.5}$$

$$\|v\overline{w}\|_{H^{-s,0}} \lesssim \|v\|_{X_-^{s,1/2+\varepsilon}} \|w\|_{X_+^{-s,1/2-2\varepsilon}} . \tag{4.6}$$

4.1.1. *Proof of* (4.5). This reduces to showing that $J \lesssim \|F\|_{L^2} \|G\|_{L^2}$, where

$$J = \left\| \int_{\mathbb{R}^{1+1}} \frac{\langle\xi\rangle^s F(\lambda,\eta) G(\lambda-\tau,\eta-\xi)\,d\lambda\,d\eta}{\langle\eta\rangle^s \langle\eta-\xi\rangle^s \langle\Theta_+\rangle^{1/2+\varepsilon} \langle\Sigma_-\rangle^{1/2+\varepsilon}} \right\|_{L^2_{\tau,\xi}} .$$

Since $\langle\xi\rangle^s \lesssim \langle\eta\rangle^s + \langle\eta-\xi\rangle^s$ (recall $s > 0$), the estimate reduces to Lemma 4.

4.1.2. *Proof of* (4.6). This is equivalent to $K \lesssim \|F\|_{L^2} \|G\|_{L^2}$, where

$$K = \left\| \int_{\mathbb{R}^{1+1}} \frac{\langle\eta-\xi\rangle^s F(\lambda,\eta) G(\lambda-\tau,\eta-\xi)\,d\lambda\,d\eta}{\langle\xi\rangle^s \langle\eta\rangle^s \langle\Theta_-\rangle^{1/2+\varepsilon} \langle\Sigma_+\rangle^{1/2-2\varepsilon}} \right\|_{L^2_{\tau,\xi}} .$$

By the triangle inequality we reduce this to $K_i \lesssim \|F\|_{L^2} \|G\|_{L^2}$ for $i = 1, 2$, where

$$K_1 = \left\| \int_{\mathbb{R}^{1+1}} \frac{F(\lambda,\eta) G(\lambda-\tau,\eta-\xi)\,d\lambda\,d\eta}{\langle\eta\rangle^s \langle\Theta_-\rangle^{1/2+\varepsilon} \langle\Sigma_+\rangle^{1/2-2\varepsilon}} \right\|_{L^2_{\tau,\xi}} ,$$

$$K_2 = \left\| \int_{\mathbb{R}^{1+1}} \frac{F(\lambda,\eta) G(\lambda-\tau,\eta-\xi)\,d\lambda\,d\eta}{\langle\xi\rangle^s \langle\Theta_-\rangle^{1/2+\varepsilon} \langle\Sigma_+\rangle^{1/2-2\varepsilon}} \right\|_{L^2_{\tau,\xi}} .$$

Consider first K_1. In the case $|\Sigma_+| \le |\eta|$,

$$K_1 \lesssim \left\| \int_{\mathbb{R}^{1+1}} \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)\,d\lambda\,d\eta}{\langle \Theta_- \rangle^{1/2+\varepsilon} \langle \Sigma_+ \rangle^{1/2+s-2\varepsilon}} \right\|_{L^2_{\tau,\xi}},$$

and Lemma 4 gives the desired estimate if $s \ge 3\varepsilon$. In the case $|\Sigma_+| > |\eta|$,

$$K_1 \lesssim \left\| \int_{\mathbb{R}^{1+1}} \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)\,d\lambda\,d\eta}{\langle \eta \rangle^{1/2+s-2\varepsilon} \langle \Theta_- \rangle^{1/2+\varepsilon}} \right\|_{L^2_{\tau,\xi}},$$

so recalling (2.2) and (2.3) we reduce this to (2.6) with $(a_1,a_2,a_3) = (1/2 + s - 2\varepsilon, 0, 0)$ and $\alpha,\beta,\gamma \ge 0$ such that $\alpha+\beta+\gamma > 1/2$. Thus, Lemma 3 applies if $s \ge 3\varepsilon$. Alternatively one can use the Cauchy-Schwarz inequality directly here.

The same argument applies for K_2 (replace $\langle \eta \rangle$ by $\langle \xi \rangle$).

4.1.3. *Unconditional uniqueness when $s > 1/4$.* We prove uniqueness of the solution in $C([0,T],H^s)\times C([0,T],H^s)$ for $s > 1/4$. Arguing as in subsection 3.2, it suffices to show that, if

$$(\psi_1,\psi_2) \in C([0,T],H^{\frac14+3\varepsilon}) \times C([0,T],H^{\frac14+3\varepsilon}), \tag{4.7}$$

then

$$(\psi_1,\psi_2) \in X_+^{3\varepsilon,1/2+\varepsilon}(S_T) \times X_-^{3\varepsilon,1/2+\varepsilon}(S_T). \tag{4.8}$$

We prove (4.8) as follows: By (4.7),

$$(\psi_1,\psi_2) \in X_+^{\frac14+3\varepsilon,0}(S_T) \times X_-^{\frac14+3\varepsilon,0}(S_T). \tag{4.9}$$

On the other hand, we show

$$(\psi_1,\psi_2) \in X_+^{-\frac14+4\varepsilon,1}(S_T) \times X_-^{-\frac14+4\varepsilon,1}(S_T). \tag{4.10}$$

Indeed, applying Lemma 1 to (4.1), we reduce (4.10) to

$$\left(|\psi_2|^2\psi_1, |\psi_1|^2\psi_2 \right) \in X_+^{-\frac14+4\varepsilon,0}(S_T) \times X_-^{-\frac14+4\varepsilon,0}(S_T).$$

This holds by Lemma 2:

$$\left\| |\psi_2|^2\psi_1 \right\|_{L^2_t H_x^{-\frac14+4\varepsilon}(S_T)} \lesssim \|\psi_2\|^2_{L^\infty_t H_x^{\frac14+3\varepsilon}(S_T)} \|\psi_1\|_{L^\infty_t H_x^{\frac14+3\varepsilon}(S_T)},$$
$$\left\| |\psi_1|^2\psi_2 \right\|_{L^2_t H_x^{-\frac14+4\varepsilon}(S_T)} \lesssim \|\psi_1\|^2_{L^\infty_t H_x^{\frac14+3\varepsilon}(S_T)} \|\psi_2\|_{L^\infty_t H_x^{\frac14+3\varepsilon}(S_T)}.$$

Interpolation between (4.9) and (4.10) then gives

$$\psi_1 \in X_+^{(1-\theta)(\frac14+3\varepsilon)+\theta(-\frac14+4\varepsilon),\theta}(S_T),$$

$$\psi_2 \in X_-^{((1-\theta)(\frac{1}{4}+3\varepsilon)+\theta(-\frac{1}{4}+4\varepsilon),\theta}(S_T),$$

for $0 \leq \theta \leq 1$. In particular, taking $\theta = 1/2+\varepsilon$, we obtain (4.8).

4.2. **Proof of (ii).** Using $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, $\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ and the representation of the Dirac matrices γ^0 and γ^1, we rewrite (1.2) as

$$\begin{cases} -i(\partial_t+\partial_x)\psi_1 = -m_1\psi_2 + 2\lambda\left|\phi_2\right|^2\psi_1, \\ -i(\partial_t-\partial_x)\psi_2 = -m_1\psi_1 + 2\lambda\left|\phi_1\right|^2\psi_2, \\ -i(\partial_t+\partial_x)\phi_1 = -m_2\phi_2 + 2\lambda\left|\psi_2\right|^2\phi_1, \\ -i(\partial_t-\partial_x)\phi_2 = -m_2\phi_1 + 2\lambda\left|\psi_1\right|^2\phi_2. \end{cases} \tag{4.11}$$

We iterate in

$$(\psi_1,\psi_2,\phi_1,\phi_2) \in X_+^{s,1/2+\varepsilon}(S_T)\times X_-^{s,1/2+\varepsilon}(S_T)\times X_+^{s,1/2+\varepsilon}(S_T)\times X_-^{s,1/2+\varepsilon}(S_T).$$

By Lemma 1, local well posedness reduces to the trilinear estimates (4.2) and (4.3). Unconditional uniqueness when $s > 1/4$ also follows by the same argument as in subsection 4.1.3.

5. Proof of Theorem 2

5.1. **Proof of (i).** Let $s > 1/2$. It suffices to show that if $0 < T < \infty$ and

$$(\psi_1,\psi_2) \in C([0,T),H^s)\times C([0,T),H^s)$$

solves (4.1) on $S_T = (0,T)\times\mathbb{R}$, then

$$\|\psi_1\|_{L^\infty(S_T)} + \|\psi_2\|_{L^\infty(S_T)} < \infty. \tag{5.1}$$

Indeed, if (5.1) is satisfied then global existence can be shown as follows: Denote

$$A(t) = \|\psi_1(t)\|_{H^s} + \|\psi_2(t)\|_{H^s}.$$

By the energy inequality,

$$A(t) \leq A(0) + C\int_0^t \left(A(t') + \left\|\left|\psi_2\right|^2\psi_1(t')\right\|_{H^s} + \left\|\left|\psi_1\right|^2\psi_2(t')\right\|_{H^s}\right)dt'.$$

Now, we use the inequality (see [13, Lemma 1])

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}, \quad \text{for } s>0,$$

to obtain

$$\left\|\left|\psi_2\right|^2\psi_1(t')\right\|_{H^s} \lesssim \left\|\left|\psi_2\right|^2(t')\right\|_{H^s}\|\psi_1(t')\|_{L_x^\infty} + \left\|\left|\psi_2\right|^2(t')\right\|_{L_x^\infty}\|\psi_1(t')\|_{H^s}$$

$$\lesssim \left\|\psi_2(t')\right\|_{H^s} \left\|\psi_2(t')\right\|_{L_x^\infty} \left\|\psi_1(t')\right\|_{L_x^\infty} + \left\|\psi_2(t')\right\|_{L_x^\infty}^2 \left\|\psi_1(t')\right\|_{H^s}.$$

The term $\left\| |\psi_1|^2 \psi_2(t') \right\|_{H^s}$ can be estimated in a similar way, so we get

$$\left\| |\psi_2|^2 \psi_1(t') \right\|_{H^s} + \left\| |\psi_1|^2 \psi_2(t') \right\|_{H^s} \lesssim (\left\|\psi_1(t')\right\|_{L_x^\infty} + \left\|\psi_2(t')\right\|_{L_x^\infty})^2 A(t').$$

Therefore,

$$A(t) \leq A(0) + C\left(1 + (\|\psi_1\|_{L^\infty(S_T)} + \|\psi_2\|_{L^\infty(S_T)})^2\right) \int_0^t A(t')\, dt'.$$

Grönwall's lemma then implies

$$A(t) \leq A_0 e^{C\left(1+(\|\psi_1\|_{L^\infty(S_T)}+\|\psi_2\|_{L^\infty(S_T)})^2\right)t},$$

for $0 \leq t < T$, hence, $\sup_{0\leq t<T}(\|\psi_1(t)\|_{H^s} + \|\psi_2(t)\|_{H^s}) < \infty$, allowing us to extend the solution to $[0, T+\varepsilon] \times \mathbb{R}$, for some $\varepsilon > 0$, and global existence then follows.

It remains to prove the claim (5.1). Using (4.1) we derive

$$(\partial_t + \partial_x) |\psi_1|^2 = -2m\mathrm{Im}(\psi_1\overline{\psi_2}), \tag{5.2}$$

$$(\partial_t - \partial_x) |\psi_2|^2 = 2m\mathrm{Im}(\psi_1\overline{\psi_2}). \tag{5.3}$$

We remark that these are analogues of the equation

$$\Box |u|^2 = 0,$$

satisfied by solutions of the model (see [15, 5, 10])

$$\partial_t u + \alpha\partial_x u = i|u|^2 u.$$

By Duhamel's formula,

$$|\psi_1(t,x)|^2 = |\psi_1(0, x-t)|^2 - 2m\mathrm{Im}\int_0^t (\psi_1\overline{\psi_2})(t', x-t+t')\, dt',$$

$$|\psi_2(t,x)|^2 = |\psi_2(0, x+t)|^2 + 2m\mathrm{Im}\int_0^t (\psi_1\overline{\psi_2})(t', x+t-t')\, dt',$$

hence,

$$\begin{aligned}
&\left\| |\psi_1(t,\cdot)|^2 \right\|_{L_x^\infty} + \left\| |\psi_2(t,\cdot)|^2 \right\|_{L_x^\infty} \\
&\leq \left\|\psi_1(0,\cdot)^2\right\|_{L_x^\infty} + \left\|\psi_2(0,\cdot)^2\right\|_{L_x^\infty} + 2m\int_0^t \left\| |\psi_1(t',\cdot)|^2 \right\|_{L_x^\infty} + \left\| |\psi_2(t',\cdot)|^2 \right\|_{L_x^\infty} dt' \\
&\leq C\big(\|\psi_1(0,\cdot)\|_{H^s}^2 + \|\psi_2(0,\cdot)\|_{H^s}^2 \big) + 2m\int_0^t \left\| |\psi_1(t',\cdot)|^2 \right\|_{L_x^\infty} + \left\| |\psi_2(t',\cdot)|^2 \right\|_{L_x^\infty} dt'.
\end{aligned}$$

Grönwall's lemma then implies

$$\left\| |\psi_1(t,\cdot)|^2 \right\|_{L_x^\infty} + \left\| |\psi_2(t,\cdot)|^2 \right\|_{L_x^\infty} \le C \left(\|\psi_1(0,\cdot)\|_{H^s}^2 + \|\psi_2(0,\cdot)\|_{H^s}^2 \right) e^{2mt},$$

hence $\|\psi_1\|_{L^\infty(S_T)} + \|\psi_2\|_{L^\infty(S_T)} < \infty$, for $0 < T < \infty$, proving the claim.

5.2. **Proof of (ii).** Let $s > 1/2$. Using (4.11) we derive

$$\begin{aligned}
(\partial_t + \partial_x) |\psi_1|^2 &= -2m_1 \mathrm{Im}(\psi_1 \overline{\psi_2}), \\
(\partial_t - \partial_x) |\psi_2|^2 &= 2m_1 \mathrm{Im}(\psi_1 \overline{\psi_2}), \\
(\partial_t + \partial_x) |\phi_1|^2 &= -2m_2 \mathrm{Im}(\phi_1 \overline{\phi_2}), \\
(\partial_t - \partial_x) |\phi_2|^2 &= 2m_2 \mathrm{Im}(\phi_1 \overline{\phi_2}).
\end{aligned}$$

By the same argument as in the preceding subsection we obtain

$$\begin{aligned}
&\left\| |\psi_1(t,\cdot)|^2 \right\|_{L_x^\infty} + \left\| |\psi_2(t,\cdot)|^2 \right\|_{L_x^\infty} + \left\| |\phi_1(t,\cdot)|^2 \right\|_{L_x^\infty} + \left\| |\phi_2(t,\cdot)|^2 \right\|_{L_x^\infty} \\
&\le C \left(\|\psi_1(0,\cdot)\|_{H^s}^2 + \|\psi_2(0,\cdot)\|_{H^s}^2 + \|\phi_1(0,\cdot)\|_{H^s}^2 + \|\phi_2(0,\cdot)\|_{H^s}^2 \right) e^{2(m_1+m_2)t},
\end{aligned}$$

which then implies, for $0 < T < \infty$,

$$\|\psi_1\|_{L^\infty(S_T)} + \|\psi_2\|_{L^\infty(S_T)} + \|\phi_1\|_{L^\infty(S_T)} + \|\phi_2\|_{L^\infty(S_T)} < \infty.$$

Global existence of a solution therefore follows by the energy inequality and Grönwall's lemma (by the same argument as in the preceding subsection).

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