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A VARIATIONAL PRINCIPLE ASSOCIATED WITH A CERTAIN CLASS OF BOUNDARY-VALUE PROBLEMS

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Abstract. A variational principle is introduced to provide a new formulation and resolution for several boundary-value problems. Indeed, we consider systems of the form

$$\begin{cases} \Lambda u = \nabla\Phi(u), \\ \beta_2 u = \nabla\Psi(\beta_1 u), \end{cases}$$

where Φ and Ψ are two convex functions and Λ is a possibly unbounded self-adjoint operator modulo the boundary operator $\mathcal{B} = (\beta_1, \beta_2)$. We shall show that solutions of the above system coincide with critical points of the functional

$$I(u) = \Phi^*(\Lambda u) - \Phi(u) + \Psi^*(\beta_2 u) - \Psi(\beta_1 u),$$

where Φ^* and Ψ^* are the Fenchel-Legendre dual of Φ and Ψ respectively. Note that the standard Euler-Lagrange functional corresponding to the system above is of the form,

$$F(u) = \tfrac{1}{2}\langle \Lambda u, u\rangle - \Phi(u) - \Psi(\beta_1 u).$$

An immediate advantage of using the functional I instead of F is to obtain more regular solutions and also the flexibility to handle boundary-value problems with nonlinear boundary conditions. Applications to Hamiltonian systems and semi-linear Elliptic equations with various linear and nonlinear boundary conditions are also provided.

1. Introduction

Let X be a reflexive Banach space and X^* its topological dual. Assume that $\Phi : X \to \mathbb{R} \cup \{\infty\}$ is convex and lower semi-continuous and $\Lambda : Dom(\Lambda) \subset X \to X^*$ is a self-adjoint operator. It follows that the Euler-Lagrange functional corresponding to the equation

$$\Lambda u = \nabla\Phi(u), \qquad u \in X,$$

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is of the form $F(u) = \frac{1}{2}\langle \Lambda u, u\rangle - \Phi(u)$ and critical points of F are weak solutions of these equations. In this work, we propose a new functional for which critical points and classical solutions of the above equation coincide. To be more precise we first recall the notion of weak and classical solutions for the equation

$$\Lambda u = \nabla\Phi(u), \quad u \in X,$$

as follows:

- Say that $u \in X$ is a *weak solution* of this equation if
$$\langle u, \Lambda\eta\rangle - \langle \nabla\Phi(u), \eta\rangle = 0, \quad \text{for all } \eta \in Dom(\Lambda).$$
- Say that $u \in X$ is a *classical solution* of the above equation, if $\Lambda u \in X^*$ and
$$\Lambda u - \nabla\Phi(u) = 0, \quad \text{in } X^*.$$

We shall state our first theorem.

Theorem 1.1. *Let $\Phi : X \to \mathbb{R}$ be a Gâteaux differentiable convex and lower semi-continuous and the linear operator $\Lambda : Dom(\Lambda) \subset X \to X^*$ be self-adjoint and onto. If Λ is a non-negative operator then critical points of the functional*

$$I(u) := \Phi^*(\Lambda u) - \Phi(u) \tag{1.1}$$

are classical solutions of the equation

$$\Lambda u = \nabla\Phi(u), \qquad u \in X, \tag{1.2}$$

and vise-versa.

Note that the Fenchel-Legendre dual has been usually used to construct a dual problem from the original problem. However, in the above theorem, they are actually used to provide a formulation for the original problem.

Remark 1.2. In our forthcoming paper, we shall show that, by some assumptions on the convex function Φ and the operator Λ, this result can be extended to the case where Λ is a -possibly unbounded- self-adjoint operator that may have an infinite sequence of eigenvalues going from $-\infty$ to $+\infty$. In this case, the standard Euler-Lagrange function

$$F(u) = \tfrac{1}{2}\langle \Lambda u, u\rangle - \Phi(u)$$

is strongly indefinite and in general one can not make use of classical techniques in order to find critical points of F except in very specific cases where the linking theorem might be useful. To overcome this difficulty, Clarke and Ekeland [2, 3] proposed an interesting dual variational formulation for Hamiltonian systems. Indeed, they showed that the function

$$\tilde{F}(u) = \Phi^*(\Lambda u) - \tfrac{1}{2}\langle \Lambda u, u\rangle$$

is the dual of F in a sense that F and $\tilde{F}$ have the same critical points modulo $Ker(\Lambda)$. This dual formulation turns out to be extremely useful in finite-dimensional Hamiltonian systems. However, for the infinite-dimensional case, even though this functional might be bounded from below, because of the quadratic term $\langle \Lambda u, u\rangle$ it is not in general lower semi-continuous. That is why critical points of $\tilde{F}$ can never be obtained by a simple minimization. Taking into account the new functional

$$I(u) := \Phi^*(\Lambda u) - \Phi(u)$$

we have indeed replaced the quadratic term by a more manageable term which is in most cases weakly continuous.

Theorem 1.1 applies readily to many equations giving a new formulation and resolution. In the following example, we shall show how the new functional I given by (1.1) will be useful in the calculus of variations to obtain solutions with more regularity. Suppose Ω is a bounded domain in $\mathbb{R}^N$ and consider

$$\begin{cases} -\Delta u + u = |u|^{p-2}u + f(x), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Let $X = L^p(\Omega)$. It follows that $X^* = L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. We have

$$Dom(\Lambda) = \Big\{ u \in L^p(\Omega) : -\Delta u + u \in L^{p'}(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \Big\}.$$

As a consequence of Theorem 1.1 we can establish the following existence result.

Corollary 1.3. *Let* $2 < p < \frac{2N}{N-2}$ *and* $f \in L^{p'}(\Omega)$. *Then, the functional* $I : W^{2,p'}(\Omega) \to \mathbb{R}$ *defined by*

$$I(u) = \frac{1}{p'}\int_\Omega |-\Delta u(x) + u(x) - f(x)|^{p'}\,dx - \frac{1}{p}\int_\Omega |u(x)|^p\,dx - \int_\Omega u(x)f(x)\,dx$$

is continuously differentiable and, if $\widetilde{u}$ *is a critical point of* I, *then* $\widetilde{u} \in W^{2,p'}(\Omega)$ *and it is indeed a solution of* (1.3).

We shall also deal with situations where the operator Λ is not purely self-adjoint provided one takes into account certain boundary terms. In fact, the operator Λ modulo the boundary operator $\mathcal{B} := (\beta_1, \beta_2) : X \to Y \times Y^*$ for some reflexive Banach space Y, corresponds to the "Green formula"

$$\langle \Lambda u, v\rangle_{X\times X^*} = \langle u, \Lambda v\rangle_{X\times X^*} + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} - \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*}.$$

We introduce the following notion and refer the interested reader to [6] where a similar notion was first introduced for skew-symmetric operators.

Definition 1.4. *We say that an operator Λ is self-adjoint modulo the boundary operator $\mathcal{B} = (\beta_1, \beta_2)$ if the following properties are satisfied:*

(1) *The space $X_0 = Dom(\Lambda) \cap ker(\beta_1, \beta_2)$ is dense in X.*
(2) *The operator $(\Lambda, \beta_2) : Dom(\Lambda) \subset X \to X^* \times Y^*$ is onto.*
(3) *For every $u, v \in Dom(\Lambda)$ we have*

$$\langle \Lambda u, v\rangle_{X\times X^*} = \langle u, \Lambda v\rangle_{X\times X^*} + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} - \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*}.$$

Our definition of non-negative self-adjoint operators modulo the boundary operator $\mathcal{B} := (\beta_1, \beta_2) : X \to Y \times Y^*$ will change accordingly. Indeed, we have the following.

Definition 1.5. *We say that an operator Λ is non-negative modulo the boundary operator $\mathcal{B} = (\beta_1, \beta_2)$ if the following property is satisfied:*

- *For every $u \in Dom(\Lambda)$ we have $\langle \Lambda u, u\rangle_{X\times X^*} + \langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*} \geq 0$.*

Here is our variational formulation when the operator Λ is not purely self-adjoint.

Theorem 1.6. *Let $\Phi : X \to \mathbb{R}$ and $\Psi : Y \to \mathbb{R}$ be Gâteaux differentiable convex and lower semi-continuous functions and $\Lambda : Dom(\Lambda) \subset X \to X^*$ be a linear operator. If Λ is a non-negative and self-adjoint operator modulo the boundary operator $\mathcal{B} = (\beta_1, \beta_2)$, then critical points of the functional*

$$I(u) := \Phi^*(\Lambda u) - \Phi(u) + \Psi^*(\beta_2 u) - \Psi(\beta_1 u),$$

are classical solutions of the boundary-value problem

$$\begin{cases} \Lambda u = \nabla \Phi(u), \\ \beta_2 u = \nabla \Psi(\beta_1 u), \end{cases}$$

and vise-versa.

As an application of Theorem 1.6, we consider a second-order Hamiltonian system with nonlinear boundary conditions as follows:

$$\begin{cases} -\ddot{u} + A(t)u = \nabla H(t, u), & t \in [0, T], \\ -\dot{u}(T) = \nabla \Psi_1(u(0)), \quad \dot{u}(0) = \nabla \Psi_2(u(T)), \end{cases} \tag{1.4}$$

where

$$\Phi(u) = \int_0^T H(t,u)\,dt : L^p([0,T]) \to \mathbb{R},$$

and $\Psi_1, \Psi_1 : \mathbb{R}^N \to \mathbb{R}$ are convex and lower semi-continuous and $A \in C^0([0,T], \mathbb{R}^N \times \mathbb{R}^N)$ is a strictly positive symmetric matrix. Here is one useful corollary of the variational principle proposed in Theorem 1.6.

Corollary 1.7. *Suppose $\Phi : L^p([0,T]) \to \mathbb{R}$ and $\Psi_1, \Psi_2 : \mathbb{R}^N \to \mathbb{R}$ are continuously differentiable and strictly convex. If the following conditions hold:*

(A) *Φ is coercive,*

$$\frac{\Phi(u)}{\|u\|_{L^p}} \to \infty \text{ as } \|u\|_{L^p} \to \infty;$$

(B) *Ψ_i is coercive, that is, $\frac{\Psi_i(u)}{|u|} \to \infty$ as $|u| \to \infty$ for $i = 1, 2$;*
then the functional

$$\begin{aligned} I(u) = \int_0^T &[H^*(t, -\ddot{u} + A(t)u) - H(t,u)]\,dt + \Psi_1^*(\dot{u}(T)) \\ &+ \Psi_2^*(\dot{u}(0)) - \Psi_2(u(T)) - \Psi_1(u(0)) \end{aligned}$$

has a critical point $\widetilde{u} \in W^{2,p'}([0,T])$ which is a solution of (1.4).

2. Preliminaries

In this section, we recall some standard results in convex analysis [4, 5]. Let X be a reflexive Banach space and X^* its topological dual. Let $\Phi : X \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. Define the sub-differential $\partial\Phi$ of Φ to be the following set-valued operator: if $u \in Dom(\Phi)$, set

$$\partial\Phi(u) = \{p \in X^* : \langle p, v-u\rangle + \Phi(u) \leq \Phi(v) \text{ for all } v \in X\},$$

and if $u \notin Dom(\Phi)$, set $\partial\Phi(u) = \varnothing$. If Φ is Gâteaux differentiable at u then $\partial\Phi(u) = \{\nabla\Phi(u)\}$.

The Fenchel-Legendre dual of Φ is denoted by Φ^*; that is, a function on X^* defined by $\Phi^*(p) = \sup\{\langle p, x\rangle - \Phi(x) : x \in X\}$. It follows from this definition that, for every $(u,p) \in X \times X^*$, we have $\Phi(u) + \Phi^*(p) \geq \langle u, p\rangle$. The following result is standard (see [5] Proposition 5.1).

Proposition 2.1. *If Φ is convex and lower-semi continuous then $\Phi^{**} = \Phi$ and the following are equivalent:*

(i) $\Phi(u) + \Phi^*(p) = \langle u, p\rangle$.

(ii) $p \in \partial\Phi(u)$.
(iii) $u \in \partial\Phi^*(p)$.

Since the functional proposed in Theorems 1.1 and 1.6 may not be Gâteaux differentiable, we are required to give a meaning to a critical point of such a functional.

Definition 2.1. *Let $\Phi : X \to \mathbb{R}$ and $\Psi : Y \to \mathbb{R}$ be Gâteaux differentiable convex and lower semi-continuous functions. Assume Λ, β_1 and β_2 are as in Theorem 1.6. Say that $u \in X$ is a critical point of*

$$I(u) := \Phi^*(\Lambda u) - \Phi(u) + \Psi^*(\beta_2 u) - \Psi(\beta_1 u),$$

if there exist $v \in \partial\Phi^*(\Lambda u)$ and $w \in \partial\Psi^*(\beta_2 u)$ such that

$$\langle v, \Lambda\eta\rangle - \langle \nabla\Phi(u), \eta\rangle + \langle w, \beta_2\eta\rangle - \langle \nabla\Psi(\beta_1 u), \beta_1\eta\rangle = 0,$$

for all $\eta \in Dom(\Lambda)$.

3. Proof of Theorems 1.1 and 1.6.

Proof of Theorem 1.1. Suppose u is a critical point of I. Then, there exists $v \in \partial\Phi^*(\Lambda u)$ such that

$$\langle v, \Lambda\eta\rangle - \langle \nabla\Phi(u), \eta\rangle = 0 \qquad \text{for all } \eta \in Dom(\Lambda). \tag{3.1}$$

Since Λ is onto, there exists $w \in Dom(\Lambda)$ such that $\Lambda w = \nabla\Phi(u)$. This together with (3.1) implies that

$$\langle v, \Lambda\eta\rangle = \langle \Lambda w, \eta\rangle \qquad \text{for all } \eta \in Dom(\Lambda). \tag{3.2}$$

It follows from the fact that $\Lambda w = \nabla\Phi(u)$ and $v \in \partial\Phi^*(\Lambda u)$ that

$$\Phi^*(\Lambda w) + \Phi(u) = \langle \Lambda w, u\rangle \tag{3.3}$$

$$\Phi^*(\Lambda u) + \Phi(v) = \langle \Lambda u, v\rangle. \tag{3.4}$$

By adding up (3.3) and (3.4) we obtain

$$\begin{aligned}\langle \Lambda w, u\rangle + \langle \Lambda u, v\rangle &= \Phi^*(\Lambda w) + \Phi(u) + \Phi^*(\Lambda u) + \Phi(v)\\ &= \Phi^*(\Lambda w) + \Phi(v) + \Phi^*(\Lambda u) + \Phi(u)\\ &\geq \langle \Lambda w, v\rangle + \langle \Lambda u, u\rangle. \end{aligned}\tag{3.5}$$

This implies that

$$\langle \Lambda w, u\rangle + \langle \Lambda u, v\rangle \geq \langle \Lambda w, v\rangle + \langle \Lambda u, u\rangle. \tag{3.6}$$

On the other hand, we obtain from (3.2) that $\langle v, \Lambda u\rangle = \langle \Lambda w, u\rangle$ and $\langle v, \Lambda w\rangle = \langle \Lambda w, w\rangle$, which together with the above inequality yield that

$$\langle \Lambda w, u\rangle + \langle \Lambda w, u\rangle \geq \langle \Lambda w, w\rangle + \langle \Lambda u, u\rangle.$$

It follows that
$$\langle \Lambda w - \Lambda u, w - u\rangle \le 0.$$
On the other hand, since Λ is non-negative we have that the latter is indeed zero,
$$\langle \Lambda w - \Lambda u, w - u\rangle = 0.$$
This implies that the inequality in (3.6) is in fact an equality which together with (3.5) implies that
$$\langle \Lambda w, v\rangle + \langle \Lambda u, u\rangle = \Phi^*(\Lambda w) + \Phi(v) + \Phi^*(\Lambda u) + \Phi(u),$$
from which we obtain
$$\big[\Phi^*(\Lambda w) + \Phi(v) - \langle \Lambda w, v\rangle\big] + \big[\Phi^*(\Lambda u) + \Phi(u) - \langle \Lambda u, u\rangle\big] = 0.$$
This together with the fact that
$$\Phi^*(\Lambda w) + \Phi(v) - \langle \Lambda w, v\rangle \ge 0, \qquad \Phi^*(\Lambda u) + \Phi(u) - \langle \Lambda u, u\rangle \ge 0,$$
implies that $\Phi^*(\Lambda u) + \Phi(u) - \langle \Lambda u, u\rangle = 0$. Taking into account Proposition 2.1, it follows that $\Lambda u = \nabla\Phi(u)$ and u is a solution of (1.2).

Conversely, suppose u is a solution of problem (1.2). It follows that $\Lambda u = \nabla\Phi(u)$ which implies $u \in \nabla\Phi^*(\Lambda u)$ and therefore for $\eta \in Dom(\Lambda)$ we obtain
$$\langle u, \Lambda\eta\rangle - \langle \nabla\Phi(u), \eta\rangle = \langle \Lambda u - \nabla\Phi(u), \eta\rangle = 0,$$
thereby giving that u is a critical point of I. □

Proof of Theorem 1.6. Suppose u is a critical point of I. Then, there exist $v \in \partial\Phi^*(\Lambda u)$ and $w \in \partial\Psi^*(\beta_2 u)$ such that
$$\langle v, \Lambda\eta\rangle - \langle \nabla\Phi(u), \eta\rangle + \langle w, \beta_2\eta\rangle - \langle \nabla\Psi(\beta_1 u), \beta_1\eta\rangle = 0, \tag{3.7}$$
for all $\eta \in Dom(\Lambda)$. Since $(\Lambda, \beta_2) : Dom(\Lambda) \subset X \to X^* \times Y^*$ is onto, there exists $x \in Dom(\Lambda)$ such that
$$\begin{cases} \Lambda x = \nabla\Phi(u), \\ \beta_2 x = \nabla\Psi(\beta_1 u). \end{cases}$$
This together with (3.7) implies that
$$\langle v, \Lambda\eta\rangle - \langle \Lambda x, \eta\rangle + \langle w, \beta_2\eta\rangle - \langle \beta_2 x, \beta_1\eta\rangle = 0, \qquad \text{for all } \eta \in Dom(\Lambda).$$
Now, we show that $x = v$ and $w = \beta_1(v)$. Indeed, it follows from the above and part (3) of Definition 1.4 that
$$\langle v, \Lambda\eta\rangle - \langle \Lambda\eta, x\rangle + \langle w, \beta_2\eta\rangle - \langle \beta_2\eta, \beta_1 x\rangle = 0, \qquad \text{for all } \eta \in Dom(\Lambda),$$
thereby giving
$$\langle v - x, \Lambda\eta\rangle + \langle \beta_2\eta, \beta_1 x - w\rangle = 0, \qquad \text{for all } \eta \in Dom(\Lambda).$$

This, together with the fact that $(\Lambda, \beta_2) : Dom(\Lambda) \subset X \to X^* \times Y^*$ is onto, implies that $x = v$ and $w = \beta_1(x) = \beta_1(v)$.

In this step, we prove the inequality

$$\langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*} + \langle \beta_1 v, \beta_2 v\rangle_{Y\times Y^*} \le \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} + \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*}.$$

Indeed, it follows from the system

$$\begin{cases} \beta_1 v \in \partial\Psi^*(\beta_2 u), \\ \beta_2 v = \nabla\Psi(\beta_1 u), \end{cases}$$

that

$$\begin{aligned} \langle \beta_1 u, \beta_2 v\rangle + \langle \beta_1 v, \beta_2 u\rangle &= \Psi^*(\beta_2 v) + \Psi(\beta_1 u) + \Psi^*(\beta_2 u) + \Psi(\beta_1 v) \\ &= \Psi^*(\beta_2 v) + \Psi(\beta_1 v) + \Psi^*(\beta_2 u) + \Psi(\beta_1 u) \\ &\ge \langle \beta_1 v, \beta_2 v\rangle_{Y\times Y^*} + \langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*}, \end{aligned}$$

from which we obtain

$$\langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*} + \langle \beta_1 v, \beta_2 v\rangle_{Y\times Y^*} \le \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} + \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*}. \quad (3.8)$$

By the same argument, it follows from the fact that $v \in \partial\Phi^*(\Lambda u), \Lambda v = \nabla\Phi(u)$ that

$$\langle \Lambda v, v\rangle + \langle \Lambda u, u\rangle \le \langle \Lambda v, u\rangle + \langle \Lambda v, u\rangle. \quad (3.9)$$

Taking the sum of inequalities (3.8) and (3.9), we have

$$\begin{aligned} &\langle \Lambda u, u\rangle + \langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*} + \langle \Lambda v, v\rangle + \langle \beta_1 v, \beta_2 v\rangle_{Y\times Y^*} \\ &\le \langle \Lambda v, u\rangle + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} + \langle \Lambda v, u\rangle + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*}. \end{aligned} \quad (3.10)$$

On the other hand, it follows from part (3) of Definition 1.4 that

$$\langle \Lambda v, u\rangle + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*} = \langle \Lambda u, v\rangle + \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*},$$

from which, together with (3.10), we obtain

$$\begin{aligned} &\langle \Lambda u, u\rangle + \langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*} + \langle \Lambda v, v\rangle + \langle \beta_1 v, \beta_2 v\rangle_{Y\times Y^*} \\ &\le \langle \Lambda u, v\rangle + \langle \beta_1 v, \beta_2 u\rangle_{Y\times Y^*} + \langle \Lambda v, u\rangle + \langle \beta_1 u, \beta_2 v\rangle_{Y\times Y^*}. \end{aligned}$$

This inequality is equivalent to

$$\langle \Lambda v - \Lambda u, v - u\rangle + \langle \beta_1(v - u), \beta_2(v - u)\rangle_{Y\times Y^*} \le 0. \quad (3.11)$$

On the other hand, the operator Λ is non-negative modulo the boundary operator $\mathcal{B} = (\beta_1, \beta_2)$, from which together with (3.11) we have that the latter is indeed zero and we have equality in (3.8) and (3.9). As argued in the proof of Theorem 1.1, it then follows that

$$\Phi^*(\Lambda u) + \Phi(u) = \langle \Lambda u, u\rangle$$

$$\Psi^*(\beta_2 u) + \Psi(\beta_1 u) \quad = \quad \langle \beta_1 u, \beta_2 u\rangle_{Y\times Y^*},$$

and therefore, by Proposition 2.1, u is a solution of

$$\begin{cases} \Lambda u = \nabla\Phi(u), \\ \beta_2 u = \nabla\Psi(\beta_1 u). \end{cases}$$

Conversely, suppose u is a solution of the above problem. It follows from $\Lambda u = \nabla\Phi(u)$ and $\beta_2 u = \nabla\Psi(\beta_1 u)$ that $u \in \partial\Phi^*(\Lambda u)$ and $\beta_1 u \in \partial\Psi^*(\beta_2 u)$ respectively. Fix $\eta \in Dom(\Lambda)$. We need to show that

$$\langle u, \Lambda\eta\rangle - \langle \nabla\Phi(u), \eta\rangle + \langle \beta_1 u, \beta_2\eta\rangle - \langle \nabla\Psi(\beta_1 u), \beta_1\eta\rangle = 0.$$

In fact, since u is a solution of the above system we have

$$\begin{cases} \langle \Lambda u, \eta\rangle = \langle \nabla\Phi(u), \eta\rangle, \\ \langle \beta_2 u, \beta_1\eta\rangle = \langle \nabla\Psi(\beta_1 u), \beta_1\eta\rangle, \end{cases}$$

from which we obtain

$$\begin{aligned} 0 &= \langle \Lambda u, \eta\rangle - \langle \nabla\Phi(u), \eta\rangle + \langle \beta_2 u, \beta_1\eta\rangle - \langle \nabla\Psi(\beta_1 u), \beta_1\eta\rangle \\ &= \langle u, \Lambda\eta\rangle + \langle \beta_1 u, \beta_2\eta\rangle - \langle \beta_2 u, \beta_1\eta\rangle - \langle \nabla\Phi(u), \eta\rangle \\ &\quad + \langle \beta_2 u, \beta_1\eta\rangle - \langle \nabla\Psi(\beta_1 u), \beta_1\eta\rangle \\ &= \langle u, \Lambda\eta\rangle - \langle \nabla\Phi(u), \eta\rangle + \langle \beta_1 u, \beta_2\eta\rangle - \langle \nabla\Psi(\beta_1 u), \beta_1\eta\rangle, \end{aligned}$$

thereby giving that u is a critical point of I. □

4. Applications in boundary-value problems

In this section we first proceed with a sketch of the proof of Corollaries 1.3 and 1.7 and then make some further remarks about a generalization of this theory; regularity of solutions obtained by the new functional will follow.

4.1. **Example 1- An Elliptic equation with Neumann boundary conditions.** Suppose Ω is a bounded domain in $\mathbb{R}^N$ and consider

$$\begin{cases} -\Delta u + u = |u|^{p-2}u + f(x), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Let $X = L^p(\Omega)$. It follows that $X^* = L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\Lambda : Dom(\Lambda) \subset X \to X^*$ be defined as $\Lambda = -\Delta + id$ with

$$Dom(\Lambda) = \{u \in L^p(\Omega) : -\Delta u + u \in L^{p'}(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}.$$

It is standard that $\Lambda : Dom(\Lambda) \subset L^p(\Omega) \to L^{p'}(\Omega)$ is self-adjoint and non-negative. Consider the Banach space X_Λ that is the completion of $Dom(\Lambda)$

for the norm $\|u\|_{X_\Lambda} = \|\Lambda u\|_{X^*}$. It follows from elliptic regularity theory that there exists a positive constant c such that that

$$\|u\|_{W^{2,p'}(\Omega)} \le c\|u\|_{X_\Lambda}.$$

This implies that $X_\Lambda = \{u \in W^{2,p'}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. On the other hand, by Sobolev embedding we have that $W^{2,p'}(\Omega) \hookrightarrow L^q(\Omega)$ is compact as soon as $1 < q < \frac{Np'}{N-2p'}$. It follows that the functional

$$I(u) = \frac{1}{p'}\int_\Omega |-\Delta u(x) + u(x) - f(x)|^{p'}\,dx - \frac{1}{p}\int_\Omega |u(x)|^p\,dx - \int_\Omega u(x)f(x)\,dx$$

is lower semi-continuous and continuously differentiable in $W^{2,p'}(\Omega)$ as soon as $1 < p < \frac{Np'}{N-2p'}$. The last inequality indeed implies that $1 < p < \frac{2N}{N-2}$. It can be easily deduced that I satisfies the mountain pass geometry for $f(x) = 0$ when $2 < p < \frac{2N}{N-2}$ and therefore there exists a critical point $\widetilde{u} \in W^{2,p'}(\Omega)$ which is a solution of (4.1). We refer the interested reader to [10] for more discussion.

Remark 4.1. Assuming $f(x) = 0$ in the above example, one can think of the functional I as a function of two parameters $p, q' > 1$ as follows:

$$I_{p,q'}(u) = \frac{1}{q'}\int_\Omega |-\Delta u(x) + u(x)|^{q'}\,dx - \frac{1}{p}\int_\Omega |u(x)|^p\,dx.$$

It follows that $I_{p,p'} = I$. As mentioned above, if we set $X_{\Lambda,q'}$ to be the completion of $Dom(\Lambda)$ for the norm $\|u\|_{X_{\Lambda,q'}} = \|\Lambda u\|_{L^{q'}(\Omega)}$, then $X_{\Lambda,q'} = \{u \in W^{2,q'}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. Therefore, because of the compact embedding $W^{2,q'}(\Omega) \hookrightarrow L^p(\Omega)$, for $1 < p < \frac{Nq'}{N-2q'}$, the functional $I_{p,q'}$ is lower semi-continuous. It is also straightforward that $I_{p,q'}$ satisfies the mountain pass geometry and condition for $q' < p$. Note that this functional is bounded from below and coercive for $q' > p$. It then follows that the functional $I_{p,q'}$ has a critical point $u \in W^{2,q'}(\Omega)$. If we set $v(x) = |-\Delta u(x) + u(x)|^{q'-2}(-\Delta u(x) + u(x))$ then (u, v) is a solution of the elliptic Hamiltonian system

$$\begin{cases} -\Delta v + v = |u|^{q-2}u, & x \in \Omega, \\ -\Delta u + u = |v|^{p-2}v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Also the restriction $1 < p < \frac{Nq'}{N-2q'}$ is nothing but the well-known hyperbola $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$.

In [10], we have used this observation to treat elliptic Hamiltonian systems when p and q live on the critical hyperbola $\frac{1}{p}+\frac{1}{q}=\frac{N-2}{N}$.

4.2. **Example 2- Second-order Hamiltonian systems with nonlinear boundary conditions.** Let $H(t,.), \Psi_1$ and Ψ_2 be convex and lower semi-continuous on $\mathbb{R}^N$ and $A \in C^0([0,T], \mathbb{R}^N \times \mathbb{R}^N)$ be a symmetric strictly positive definite matrix. Consider the problem

$$\begin{cases} -\ddot{u}+A(t)u=\nabla H(t,u), & t\in[0,T], \\ -\dot{u}(0)=\nabla\Psi_1(u(T)), \quad \dot{u}(T)=\nabla\Psi_2(u(0)). \end{cases} \tag{4.2}$$

It was shown by J. P. Aubin and I. Ekeland [1] that u is a solution of (4.2) if and only if

$$0=K(u)\leq K(v), \qquad \text{for all } v\in H^1[0,T],$$

where by some modification

$$\begin{aligned} K(u) = \int_0^T &\left[H^*(t,-\ddot{u}+A(t)u)+H(t,u)-\langle -\ddot{u}+A(t)u,u\rangle\right]dt \\ &+\Psi_1^*(\dot{u}(T))+\Psi_2^*(-\dot{u}(0))+\Psi_2(u(T))+\Psi_1(u(0))-\langle\dot{u}(T),u(0)\rangle \\ &+\langle\dot{u}(0),u(T)\rangle, \end{aligned}$$

and using the Ky-Fan lemma some existence results were established. Taking into account the same functional some existence results were proved by N. Ghoussoub and the author [7, 8] for the sub-quadratic and also the nonlinear case. As stated in Corollary 1.7 to obtain solutions for this system one just needs to find critical points of the functional

$$\begin{aligned} I(u) = \int_0^T &[H^*(t,-\ddot{u}+A(t)u)-H(t,u)]\,dt+\Psi_1^*(\dot{u}(T)) \\ &+\Psi_2^*(\dot{u}(0))-\Psi_2(u(T))-\Psi_1(u(0)), \end{aligned}$$

even for the super-quadratic case. A more general version of Corollary 1.7 is proved in [9] to which we refer the interested reader.

4.3. **Regularity.** Suppose $\Lambda : Dom(\Lambda)\subset X\to X^*$ is a closed self-adjoint operator that satisfies $\langle\Lambda u,u\rangle\geq\lambda\|u\|_X^2$ for some $\lambda>0$ and all $u\in Dom(\Lambda)$. We consider the Banach space X_Λ that is the completion of $Dom(\Lambda)$ for the norm $\|u\|_{X_\Lambda}=\|\Lambda u\|_{X^*}$. One can also consider the Hilbert space H_Λ that is the completion of $Dom(\Lambda)$ for the norm $\|u\|_{H_\Lambda}=\langle\Lambda u,u\rangle$ induced by the scalar product $\langle u,v\rangle_{H_\Lambda}=\langle\Lambda u,v\rangle$ for all $u,v\in Dom(\Lambda)$. Note that we have $\|u\|_X^2 \leq \frac{1}{\lambda}\|u\|_{H_\Lambda} \leq \frac{1}{\lambda^2}\|u\|_{X_\Lambda}$ and the injections $X_\Lambda \hookrightarrow H_\Lambda \hookrightarrow X$ are therefore continuous. Now consider the equation

$$\Lambda u = \nabla\Phi(u), \qquad u \in X,$$

where $\Phi : X \to \mathbb{R}$ is convex and lower semi-continuous. As shown critical points of the standard Euler-Lagrange functional $F(u) = \frac{1}{2}\langle \Lambda u, u\rangle - \Phi(u)$ and also critical points of the new functional $I(u) := \Phi^*(\Lambda u) - \Phi(u)$ are solutions of the above equation. However, the right spaces in which to look for critical points of the functional F and I are H_Λ and X_Λ respectively and because of the embedding $X_\Lambda \hookrightarrow H_\Lambda$ one can expect solutions with more regularity taking into account the functional I instead of F.

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