SEMI-DISCRETE WEAKLY DAMPED NONLINEAR 2-D SCHRÖDINGER EQUATION

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Abstract. We consider a semi-discrete in time Crank-Nicolson scheme to discretize a damped forced nonlinear Schrödinger equation in $2D$. This provides us with a discrete infinite-dimensional dynamical system. We prove the existence of a finite-dimensional global attractor for this dynamical system.

1. Introduction

The nonlinear Schrödinger equations (NLS) are asymptotic models for water waves. Here, we are concerned with a damped, forced nonlinear Schrödinger equation. For the dissipation term we consider a zero-order term, following here the work of Dias et al. (see [7], [8]) in which it is rigorously derived that the dissipation in Navier-Stokes equations provide a zero-order term damping for the asymptotic model in deep water. The equations read as follows:

$$\begin{cases} u_t + \alpha u + i\Delta u + i|u|^2 u = f(x), \ x \in \mathbb{T}^2 \\ u(0,x) = u_0(x), \ x \in \mathbb{T}^2, \end{cases} \tag{1.1}$$

in which $u = u(t,x)$ is a complex-valued function, $\alpha > 0$ denotes the damping parameter and f is the external force that belongs to $L^2(\mathbb{T}^2)$.

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For numerical computations, we have to introduce some suitable discretizations of these equations, both in time and in space that provide also discrete dynamical systems. In this article we consider a Crank-Nicolson approximation of the equation, keeping the space variable continuous. This scheme was introduced in the conservative case; i.e., in the case where $\alpha = 0$ and $f = 0$ in [6]. Here, we are given an infinite-dimensional dynamical system, in the framework described in [18], [14], [16], [15]. Our aim in this article is to prove that this dynamical system features a finite-dimensional global attractor. Nevertheless, since we are in the critical case and in order to avoid blow-up solutions that are likely to occur (see [4] and [19]), we have to assume some smallness condition on the $L^2(\mathbb{T}^2)$ norm of the initial data. This article extends to the two-dimensional case the work [13].

Let us now introduce the Crank-Nicolson scheme under consideration. For $\tau = \delta t$ the time step, $u^0 = u_0 \in H^1(\mathbb{T}^2)$ and $\delta = e^{-\alpha\tau}$, we solve recursively

$$\frac{1}{\tau}(u^{n+1} - \delta u^n) + \frac{i}{2}\Delta(u^{n+1} + \delta u^n) + \frac{i}{4}(|u^{n+1}|^2 + |u^n|^2)(u^{n+1} + \delta u^n) = \frac{1+\delta}{2} f. \tag{1.2}$$

This equation is nonlinear, and in order to ensure that the map $S : u^n \mapsto u^{n+1}$ is well defined, we need to perform a fixed-point argument. At this stage either we proceed to the Banach fixed-point theorem or we use some compactness argument. The drawback of the former method is that we obtain a smallness assumption on the time step τ (depending on u^n), and that therefore the map S does not satisfy the semi-group property. Since we have in mind to study the long-time behavior of the solutions by energy methods, we would rather perform some compactness argument to define the map S. The price to pay is a smallness condition on the $L^2(\mathbb{T}^2)$ norm of the data, that is consistent with the condition to avoid blow-up solutions as mentioned above. Let us then introduce the restriction on the $L^2(\mathbb{T}^2)$ norms of the initial data. Let us recall the well-known Gagliardo-Nirenberg inequality

$$\|u\|^4_{L^4(\mathbb{T}^2)} \leq c_{GN} \|u\|^2_{L^2(\mathbb{T}^2)} \|u\|^2_{H^1(\mathbb{T}^2)}. \tag{1.3}$$

Consider any fixed small $\varepsilon > 0$. We shall restrict ourselves to data that satisfy

$$c_{GN} \max(\|u_0\|^2_{L^2(\mathbb{T}^2)}, \frac{4\|f\|^2_{L^2(\mathbb{T}^2)}}{\alpha^2}) \leq 1 - \varepsilon. \tag{1.4}$$

Accordingly, let us introduce the closed subset of $H^1(\mathbb{T}^2)$ that reads

$$\mathcal{C}_\varepsilon = \{u \in H^1(\mathbb{T}^2) : c_{GN} \|u\|^2_{L^2(\mathbb{T}^2)} \leq 1 - \varepsilon\}. \tag{1.5}$$

Our main result is as follows.

Theorem 1. *Assume that* (1.4) *is valid, and that* $\alpha\tau$ *is small enough in order to ensure* $\alpha\tau \leq 2(1-\delta)$. *Then, for any* $\tau > 0$, *the mapping* $u^n \mapsto u^{n+1}$ *is well defined as a multi-valued function; i.e., for any* u^n *in* $\mathcal{C}_\varepsilon$ *there exists at least one* $u^{n+1} \in \mathcal{C}_\varepsilon$ *that solves the equation* (1.2).

Remark 1. The assumption $\alpha\tau \leq 2(1-\delta)$ is valid throughout this article. We will use this assumption without notice.

We also prove in the sequel the existence of an absorbing set $\mathcal{B}_\varepsilon$ that captures any (multi-valued) trajectory and that is positively invariant under the flow. If we consider the flow restricted to this absorbing set, we have the following.

Proposition 1. *There exists* τ_0 *that depends on the data* $\alpha, \|f\|_{L^2(\mathbb{T}^2)}$ *and on* ε *such that, for any* $\tau \leq \tau_0$, *the map* S *restricted to* $\mathcal{B}_\varepsilon$ *is one-to-one.*

Hence, we have a discrete semi-group in the classical sense and we can state our main result.

Theorem 2. *The discrete semi-group* $S : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ *defined by* $Su^n = u^{n+1}$ *possesses a compact global attractor* $\mathcal{A}_\tau$ *that is a compact subset of* $H^2(\mathbb{T}^2)$. *Moreover,* $\mathcal{A}_\tau$ *has a finite Hausdorff dimension in* $H^1(\mathbb{T}^2)$.

Moreover, if $\mathcal{A}$ denotes the global attractor for the continuous dynamical system associated to (1.1), we have the following.

Theorem 3. *The family of attractors* $\mathcal{A}_\tau$ *are upper semi continuous in* $H^2(\mathbb{T}^2)$ *when* $\tau \to 0^+$; *i.e.,*

$$\sup_{a\in\mathcal{A}} \inf_{a_\tau\in\mathcal{A}_\tau} \|a - a_\tau\|_{H^2(\mathbb{T}^2)} \to 0. \tag{1.6}$$

The remainder of the article is organized as follows. In Section 2, we discuss the well posedness of the scheme; in other words, we prove Theorem 1. In Section 3, we establish the existence of a finite-dimensional global attractor; i.e., we focus on the proof of Theorem 2. In Section 4, we prove Theorem 3.

To complete this introduction, let us introduce some notation. By the letter c we denote a numerical constant that may change from one line to another. The letter K will denote some generic constant that depends on the data $\alpha, \|f\|_{L^2(\mathbb{T}^2)}$ and ε. The scalar product of two functions in $L^2(\mathbb{T}^2)$ reads

$$\langle u, v\rangle = \mathrm{Re} \int_{\mathbb{T}^2} u\bar{v}dx. \tag{1.7}$$

The scalar product in $H^1(\mathbb{T}^2)$ is then

$$\langle u, v\rangle_{H^1(\mathbb{T}^2)} = \mathrm{Re}\int_{\mathbb{T}^2}(u\bar{v} + \nabla u\nabla\bar{v})dx. \tag{1.8}$$

2. Proof of Theorem 1: Defining a dissipative semi-group

2.1. Solving the fixed-point problem in $L^2(\mathbb{T}^2)$. In this section, we state and prove some results that ensure that the discrete semi-group defined recursively by (1.2) is well posed. We begin with a general result that provides us with the existence of $S : u^n \mapsto u^{n+1}$ by some compactness argument. The drawback of the method is that the uniqueness is not ensured.

Proposition 2. *Assume that* (1.4) *is valid. For any u^n in $\mathcal{C}_\varepsilon$, for any $\tau > 0$ there exists at least one solution u^{n+1} for the discrete scheme* (1.2).

Proof. Let us introduce the Faedo-Galerkin approximation of this nonlinear equation as follows: we consider the finite-dimensional subspace

$$V_N = \mathrm{Span}\{e_k = e^{ik.x} : k = (k_1, k_2); \max(|k_1|, |k_2|) \le N\}.$$

The orthogonal projection

$$P_N : L^2(\mathbb{T}^2) \to V_N : u \mapsto P_N(u) = \sum_{\max(|k_1|,|k_2|)\le N} \hat{u}(k)e_k$$

is the projection onto this finite-dimensional space, which commutes with the Laplace operator. We seek $w^N \in P_N H^1(\mathbb{T}^2)$ such that

$$\langle\frac{w^N - \delta u^n}{\tau} + \frac{i}{2}\Delta(w^N + \delta u^n) + \frac{i}{4}(|w^N|^2 + |u^n|^2)(w^N + \delta P_N u^n) - \frac{1+\delta}{2}f, \phi_N\rangle = 0, \tag{2.1}$$

for any ϕ_N in $P_N H^1(\mathbb{T}^2)$. The existence of w^N holds true due to the following lemma given in [2].

Lemma 1. *Consider a continuous mapping $F : \mathbb{C}^q \to \mathbb{C}^q$, equipped with a scalar product $[.,.]$, such that there exists $R_0 > 0$ such that $[F(w), w] > 0$ for all w such that $[w] = [w, w]^{\frac{1}{2}} = R_0$. Then there exists $w^*, [w^*] \le R_0$ such that $F(w^*) = 0$.*

We apply this lemma to the finite-dimensional space V_N with the L^2 scalar product, to $w = w^N + \delta P_N u^n$ and to F that reads

$$F(w) = P_N^*\Big(\frac{w^N - \delta u^n}{\tau} + \frac{i}{2}\Delta(w^N + \delta u^n) + \frac{i}{4}(|w^N|^2 + |u^n|^2)(w^N + \delta P_N u^n) - \frac{1+\delta}{2}f\Big);$$

here, $P_N^* : H^{-1}(\mathbb{T}^2) \to V_N$ is the operator defined as $\langle P_N^*\eta, v\rangle = \langle \eta, P_N v\rangle$ where $\langle ., .\rangle$ denotes either the L^2 scalar product or the duality between $H^1(\mathbb{T}^2)$ and its dual space. Using the fact that

$$[F(w), w] = \frac{1}{\tau}(\|w^N\|^2_{L^2(\mathbb{T}^2)} - \delta^2\|P_N u^n\|^2_{L^2(\mathbb{T}^2)}) - \frac{1+\delta}{2}\langle f, w^N + \delta P_N u^n\rangle \geq \tag{2.2}$$

$$(\frac{1}{\tau}(\|w^N\|_{L^2(\mathbb{T}^2)} - \delta\|P_N u^n\|_{L^2(\mathbb{T}^2)}) - \|f\|_{L^2(\mathbb{T}^2)})(\|w^N\|_{L^2(\mathbb{T}^2)} + \delta\|P_N u^n\|_{L^2(\mathbb{T}^2)})$$

is positive for $\|w^N\|_{L^2(\mathbb{T}^2)} > \delta\|P_N u^n\|_{L^2(\mathbb{T}^2)} + \tau\|f\|_{L^2(\mathbb{T}^2)}$. By taking $R_0 = 2\delta\|u^n\|_{L^2} + \tau\|f\|_{L^2}$, we get by the triangle inequality that, if $\|w\|_{L^2} = R_0$, then $\|w^N\|_{L^2} > \delta\|u^n\|_{L^2(\mathbb{T}^2)} + \tau\|f\|_{L^2(\mathbb{T}^2)}$ and hence $[F(w), w] > 0$. By Lemma 1, we obtain the existence of w^N that satisfies

$$\|w^N\|_{L^2(\mathbb{T}^2)} \leq \|w\| + \delta\|P_N u^n\|_{L^2(\mathbb{T}^2)} \leq 3\delta\|u^n\|_{L^2(\mathbb{T}^2)} + \tau\|f\|_{L^2(\mathbb{T}^2)}. \tag{2.3}$$

Going back to the equation, we observe that this bound can be improved. From $[F(w^N + \delta P_N u^n), w^N + \delta P_N u^n] = 0$, we infer that

$$\frac{1}{\tau}(\|w^N\|^2_{L^2(\mathbb{T}^2)} - \delta^2\|P_N u^n\|^2_{L^2(\mathbb{T}^2)}) \leq \|f\|_{L^2(\mathbb{T}^2)}\|w^N + \delta P_N u^n\|_{L^2(\mathbb{T}^2)},$$

and then the following estimate follows straightforwardly:

$$\|w^N\|_{L^2(\mathbb{T}^2)} \leq \delta\|u^n\|_{L^2(\mathbb{T}^2)} + \tau\|f\|_{L^2(\mathbb{T}^2)}. \tag{2.4}$$

Gathering (1.4) and (2.4), since $\alpha\tau \leq 2(1-\delta)$, it follows that w^N belongs also to $\mathcal{C}_\varepsilon$. We observe that the right-hand side of (2.4) is independent of N.

We now plan to pass to the limit as $N \to +\infty$. We have the $L^2(\mathbb{T}^2)$ estimate (2.4). To go further, we need some compactness. Taking $\phi_N = i(w^N - \delta P_N u^n)$ in (2.1) leads to

$$\frac{1}{2}\|\nabla w^N\|^2_{L^2(\mathbb{T}^2)} \leq \frac{\delta^2}{2}\|\nabla u^n\|^2_{L^2(\mathbb{T}^2)} + \frac{1}{4}\int_{\mathbb{T}^2}(|w^N|^2 + |u^n|^2)|w^N|^2 + \|f\|_{L^2(\mathbb{T}^2)}(\|w^N\|_{L^2(\mathbb{T}^2)} + \|u^n\|_{L^2(\mathbb{T}^2)}). \tag{2.5}$$

At this stage we use the fact that both u^n and w^N belong to $\mathcal{C}_\varepsilon$, to obtain, using the Gagliardo-Nirenberg inequality,

$$\|\nabla w^N\|^2_{L^2(\mathbb{T}^2)} \leq \delta^2\|\nabla u^n\|^2_{L^2(\mathbb{T}^2)} + (1-\varepsilon)(\|\nabla w^N\|^2_{L^2(\mathbb{T}^2)} + \|\nabla u^n\|^2_{L^2(\mathbb{T}^2)}) + K. \tag{2.6}$$

This provides us with a bound for w^N in $H^1(\mathbb{T}^2)$, independently of N. Hence, the sequence w^N is relatively compact in $H^s(\mathbb{T}^2)$ for $s < 1$; we are now able to pass to the limit in (2.1). This is standard and then omitted for the sake of conciseness. □

Let us observe that passing to the limit in (2.4) yields

$$\|u^{n+1}\|_{L^2(\mathbb{T}^2)} \leq \delta\|u^n\|_{L^2(\mathbb{T}^2)} + \frac{2}{\alpha}(1-\delta)\|f\|_{L^2(\mathbb{T}^2)}, \tag{2.7}$$

and then, due to (1.4), recursively on n,

$$c_{GN}\|u^{n+1}\|^2_{L^2(\mathbb{T}^2)} \leq 1-\varepsilon. \tag{2.8}$$

2.2. **An $H^1(\mathbb{T}^2)$ estimate.** Due to the result of the previous subsection, we have defined a (multi-valued) map S. We define a trajectory by choosing as u^{n+1} any function in $\{Su^n\}$. We aim to prove that any trajectory enters an absorbing set in $H^1(\mathbb{T}^2)$ after a finite transient time (that depends only on the size of the initial data). We already know that the condition $c_{GN}\|u\|^2_{L^2(\mathbb{T}^2)} \leq 1-\varepsilon$ is (positively) invariant by the flow. We introduce the energy functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{T}^2)} - \frac{1}{4}\|u\|^4_{L^4(\mathbb{T}^2)} + \frac{1+\delta}{2}\mathrm{Im}\int_{\mathbb{T}^2} f\overline{u}dx. \tag{2.9}$$

We emphasize that, due to the Gagliardo-Nirenberg inequality (1.3), the functional J is coercive for $u \in \mathcal{C}_\varepsilon$; that is,

$$J(u) \geq \frac{1+\varepsilon}{4}\|\nabla u\|^2_{L^2(\mathbb{T}^2)} - K \geq \frac{1+\varepsilon}{4}\|u\|^2_{H^1(\mathbb{T}^2)} - \tilde{K}. \tag{2.10}$$

We now state and prove the following.

Proposition 3. *There exists a constant $K > 0$ that depends on the data of the equation α, f, ε such that the set $\mathcal{B}_\varepsilon = \{u \in \mathcal{C}_\varepsilon : J(u) \leq K\}$ is a bounded absorbing set that is positively invariant under S.*

Proof. We consider the scalar product of (1.2) with $i(u^{n+1} - \delta u^n)$. This yields

$$J(u^{n+1}) = \delta^2 J(u^n) + \frac{1-\delta^2}{4}\int_{\mathbb{T}^2}|u^n|^2|u^{n+1}|^2dx + \delta\frac{1-\delta^2}{2}\mathrm{Im}\int_{\mathbb{T}^2} f\overline{u^n}dx. \tag{2.11}$$

By the Gagliardo-Nirenberg and Young inequalities and using (2.10) we obtain

$$\begin{aligned} J(u^{n+1}) \leq \delta^2 J(u^n) + \frac{1-\delta^2}{8}c_{GN}\Big(&\|u^n\|^2_{L^2(\mathbb{T}^2)}\|u^n\|^2_{H^1(\mathbb{T}^2)} \\ &+ \|u^{n+1}\|^2_{L^2(\mathbb{T}^2)}\|u^{n+1}\|^2_{H^1(\mathbb{T}^2)}\Big) + (1-\delta)K \end{aligned} \tag{2.12}$$

$$\leq \delta^2 J(u^n) + \frac{1-\delta^2}{2}\frac{1-\varepsilon}{1+\varepsilon}(J(u^n)+J(u^{n+1})) + (1-\delta)\tilde{K}.$$

We then obtain that there exists $\tilde{\delta} = \frac{2\delta^2+2\delta^2\varepsilon+(1-\delta^2)(1-\varepsilon)}{2+2\varepsilon-(1-\delta^2)(1-\varepsilon)} < 1$ that depends on δ, ε such that

$$J(u^{n+1}) \leq \tilde{\delta} J(u^n) + \frac{2(1-\delta)}{1+\delta^2}\tilde{K}. \tag{2.13}$$

Therefore,

$$J(u^n) \leq \tilde{\delta}^n J(u_0) + \frac{2(1-\delta)}{(1-\tilde{\delta})(1+\delta^2)}\tilde{K} \leq \tilde{\delta}^n J(u_0) + \frac{2\tilde{K}}{\varepsilon(1+\delta)(1+\delta^2)}. \tag{2.14}$$

The proof of the Proposition is then easily completed, by choosing $\mathcal{B}_\varepsilon = \{u \in \mathcal{C}_\varepsilon : J(u) \leq K = \frac{2\tilde{K}}{\varepsilon}\}$. □

Remark 2. Actually, due to (2.10), one can deduce that there exists $K_1 > 0$ such that

$$\mathcal{B}_\varepsilon \subset B_{H^1}(0, K_1). \tag{2.15}$$

2.3. **A uniqueness result.** Due to the results in the previous subsection, our map S is a bounded map on bounded subsets of $H^1(\mathbb{T}^2)$, and satisfies that any trajectory enters $\mathcal{B}_\varepsilon$ after a finite number of iterations (that depends on the size in $H^1(\mathbb{T}^2)$ of the initial data). We now prove a lemma that asserts that, if τ is small enough with respect to $\alpha, \varepsilon, \|f\|_{L^2(\mathbb{T}^2)}$, then $S : u^n \mapsto u^{n+1}$ is one-to-one and continuous in the $H^1(\mathbb{T}^2)$ topology. In the remainder of the paper, we consider this assumption to be valid, and this way we shall proceed to the existence of a global attractor.

Proposition 4. *There exists τ_0 which depends on $\alpha, \varepsilon, \|f\|_{L^2(\mathbb{T}^2)}$ such that for any $\tau \leq \tau_0$ the map $S : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ is one-to-one and continuous for the H^1 topology.*

Proof. For a fixed u in $H^1(\mathbb{T}^2)$ we consider the map $\mathcal{T} : H^1(\mathbb{T}^2) \to H^1(\mathbb{T}^2) : v \mapsto \mathcal{T}(v)$ such that, setting $F(v) = (|v|^2 + |u|^2)(v + \delta u)$,

$$\mathcal{T}_u(v) = (1 + \tfrac{i\tau}{2}\Delta)^{-1}(1 - \tfrac{i\tau}{2}\Delta)\delta u + \tau(1 + \tfrac{i\tau}{2}\Delta)^{-1}(-\tfrac{i}{4}F(v) + \tilde{f}), \tag{2.16}$$

where $\tilde{f} = \frac{1+\delta}{2}f$. On the one hand, if u, v_1, v_2 remain in the ball $B_{H^1}(0, K_1)$, then

$$\begin{aligned}&\|\mathcal{T}_u(v_1) - \mathcal{T}_u(v_2)\|_{H^1(\mathbb{T}^2)}\\ &\leq \tau\|(1 + \frac{i\tau}{2}\Delta)^{-1}\|_{\mathcal{L}(H^{-\frac{1}{2}}(\mathbb{T}^2), H^1(\mathbb{T}^2))}\|F(v_1) - F(v_2)\|_{H^{-\frac{1}{2}}(\mathbb{T}^2)}.\end{aligned} \tag{2.17}$$

Since $\|(1+\frac{i\tau}{2}\Delta)^{-1}\|_{\mathcal{L}(H^{-\frac{1}{2}}(\mathbb{T}^2),H^1(\mathbb{T}^2))} \leq c\tau^{-\frac{3}{4}}$ (this is straightforward using Fourier series), and since $L^{\frac{4}{3}}(\mathbb{T}^2) \subset H^{-\frac{1}{2}}(\mathbb{T}^2)$, it follows that

$$\begin{aligned} &\|\mathcal{T}_u(v_1) - \mathcal{T}_u(v_2)\|_{H^1(\mathbb{T}^2)} \\ &\leq c\tau^{\frac{1}{4}}(\|v_1\|^2_{L^4(\mathbb{T}^2)} + \|v_2\|^2_{L^4(\mathbb{T}^2)} + \|u\|^2_{L^4(\mathbb{T}^2)})\|v_1 - v_2\|_{L^4(\mathbb{T}^2)} \\ &\leq c\tau^{\frac{1}{4}} M_1^2 \|v_1 - v_2\|_{H^1(\mathbb{T}^2)}. \end{aligned} \tag{2.18}$$

Thus, for $c\tau^{\frac{1}{4}}M_1^2 \leq \frac{1}{2}$, $\mathcal{T}_u$ is a contraction map on $\mathcal{B}_\varepsilon$, and is thus one-to-one. On the other hand, since the map $u \mapsto \mathcal{T}_u(v)$ is continuous in the $H^1(\mathbb{T}^2)$ topology, the map $u \mapsto v$, in which v is the unique fixed point, is also continuous; actually, if v and $\tilde{v}$ are respectively the fixed point for $\mathcal{T}_u$ and $\mathcal{T}_{\tilde{u}}$, then

$$\begin{aligned} \|v - \tilde{v}\|_{H^1(\mathbb{T}^2)} &= \|\mathcal{T}_u(v) - \mathcal{T}_{\tilde{u}}(\tilde{v})\|_{H^1(\mathbb{T}^2)} \\ &\leq \delta\|u - \tilde{u}\|_{H^1(\mathbb{T}^2)} + c\tau^{\frac{1}{4}}K_1^2\|u - \tilde{u}\|_{H^1(\mathbb{T}^2)} + c\tau^{\frac{1}{4}}K_1^2\|v - \tilde{v}\|_{H^1(\mathbb{T}^2)}. \end{aligned} \tag{2.19}$$

The result follows promptly; if τ is small enough as above, then

$$\|v - \tilde{v}\|_{H^1(\mathbb{T}^2)} \leq (2\delta + 1)\|u - \tilde{u}\|_{H^1(\mathbb{T}^2)}. \tag{2.20}$$

3. PROOF OF THEOREM 2 : THE GLOBAL ATTRACTOR

3.1. **Existence and regularity of the global attractor.** General results concerning the existence of global attractors are given in the book of R. Temam [18, chapter 2] for both continuous and discrete dynamical systems. If $S = S_1 + S_2$ where $S_1, S_2 : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ is such that

$$S_1 \text{ is relatively compact in } \mathcal{B}_\varepsilon, \tag{3.1}$$

and for every bounded set $B \subset \mathcal{B}_\varepsilon$,

$$\sup_{u \in B} |S_2^n u|_{\mathcal{B}_\varepsilon} \to 0, \quad \text{as } n \to \infty, \tag{3.2}$$

then by Theorem I.1.1 in [18], the $\omega-$limit set of $\mathcal{B}_\varepsilon$, $\mathcal{A}_\tau = \omega(\mathcal{B}_\varepsilon)$ is a global attractor for S.

We proceed as follows. Due to the very definition of Su^n, that is a fixed point for the mapping defined in (2.16),

$$Su^n = (1 + \frac{i\tau}{2}\Delta)^{-1}(1 - \frac{i\tau}{2}\Delta)\delta u^n + \tau(1 + \frac{i\tau}{2}\Delta)^{-1}F(u^n, Su^n), \tag{3.3}$$

in which $F(u,v) = -\frac{i}{4}(|u|^2 + |v|^2)(\delta u + v) + \frac{1+\delta}{2} f$. We set

$$S_2 u = (1 + \frac{i\tau}{2}\Delta)^{-1}(1 - \frac{i\tau}{2}\Delta)\delta u,$$

and

$$S_1 u = \tau(1 + \frac{i\tau}{2}\Delta)^{-1} F(u^n, Su^n).$$

On the one hand, since $\tau(1 + \frac{i\tau}{2}\Delta)^{-1}$ is a continuous mapping from $L^2(\mathbb{T}^2)$ to $H^2(\mathbb{T}^2)$, and since $u \mapsto F(u, Su)$ is continuous from $H^1(\mathbb{T}^2)$ to $L^2(\mathbb{T}^2)$, it follows that S_1 is a compact operator. On the other hand, $\|S_2^n u\|_{H^1(\mathbb{T}^2)} = \delta^n \|u\|_{H^1(\mathbb{T}^2)}$ goes to 0, uniformly on bounded sets. Thus Theorem I.1.1 in [18] applies and the existence of the attractor in $H^1(\mathbb{T}^2)$ follows promptly.

We now proceed to the proof of the regularity of the attractor. Consider a complete trajectory $\{S^n u : n \in \mathbb{Z}\}$ that belongs to this attractor. Consider P_N the projector onto the low frequency space defined above. We then infer from (3.3) that

$$P_N S u^n = (1 + \frac{i\tau}{2}\Delta)^{-1}(1 - \frac{i\tau}{2}\Delta)\delta P_N u^n + \tau(1 + \frac{i\tau}{2}\Delta)^{-1} P_N F(u^n, Su^n). \tag{3.4}$$

Since $\tau(1 + \frac{i\tau}{2}\Delta)^{-1}$ is a bounded mapping from $H^1(\mathbb{T}^2)$ into $H^2(\mathbb{T}^2)$ (whose norm is $\mathrm{O}(\sqrt{\tau})$) and since both u^n, Su^n belong to the global attractor, we have that

$$\|\tau(1 + \frac{i\tau}{2}\Delta)^{-1} P_N F(u^n, Su^n)\|_{H^2(\mathbb{T}^2)} \leq K\sqrt{\tau}. \tag{3.5}$$

Gathering the previous inequalities, we obtain

$$\|P_N u^{n+1}\|_{H^2(\mathbb{T}^2)} \leq \delta \|P_N u^n\|_{H^2(\mathbb{T}^2)} + K\sqrt{\tau}. \tag{3.6}$$

Then, we prove recursively that

$$\|P_N u^0\|_{H^2(\mathbb{T}^2)} \leq \delta^n \|P_N u^{-n}\|_{H^2(\mathbb{T}^2)} + \frac{K\sqrt{\tau}(1 - \delta^n)}{1 - \delta}. \tag{3.7}$$

Due to a classical inverse inequality $\|P_N u^{-n}\|_{H^2(\mathbb{T}^2)} \leq cN\|u^{-n}\|_{H^1(\mathbb{T}^2)}$ and thus $P_N u^{-n}$ remains in a bounded set of $H^2(\mathbb{T}^2)$, since u^{-n} remains bounded in $H^1(\mathbb{T}^2)$. Therefore we let n go to infinity to obtain

$$\|P_N u^0\|_{H^2(\mathbb{T}^2)} \leq \frac{\tilde{K}}{\sqrt{\tau}}. \tag{3.8}$$

Letting $N \to \infty$ provides that the compact global attractor is bounded in $H^2(\mathbb{T}^2)$, but that the bound *does depend* on τ.

To prove the compactness of the attractor, we rely on the J. Ball argument. We proceed as follows: let $(u_j^{n+1})_j$ be a sequence of points in $\mathcal{A} \subset H^2(\mathbb{T}^2)$ and consider the sequence $(u_j^n)_j$ such that $u_j^{n+1} = Su_j^n$, or equivalently,

$$\frac{1}{\tau}(u_j^{n+1} - \delta u_j^n) + \frac{i}{2}\Delta(u_j^{n+1} + \delta u_j^n) + \frac{i}{4}(|u_j^{n+1}|^2 + |u_j^n|^2)(u_j^{n+1} + \delta u_j^n) = 0. \quad (3.9)$$

We are going to prove that there exists a subsequence of $(u_j^{n+1})_j$ that converges strongly in $H^2(\mathbb{T}^2)$. We know that there exists a subsequence, still denoted by $(u_j^n)_j$, such that, for all $n \in \mathbb{Z}$, $u_j^n \to u^n$ strongly in $H^1(\mathbb{T}^2)$ and weakly in $H^2(\mathbb{T}^2)$. We are going to prove that the convergence holds strongly in $H^2(\mathbb{T}^2)$.

Set

$$X_j^n = \frac{i}{4}(|u_j^{n+1}|^2 + |u_j^n|^2)(u_j^{n+1} + \delta u_j^n), \quad (3.10)$$

$$X^n = \frac{i}{4}(|u^{n+1}|^2 + |u^n|^2)(u^{n+1} + \delta u^n) \,. \quad (3.11)$$

Then X_j^n converges strongly in $H^1(\mathbb{T}^2)$ towards X^n. Set $w_j^{n+1} = u_j^{n+1} - u^{n+1}$ and $w_j^n = u_j^n - u^n$. Then w_j^n is a solution to

$$\frac{w_j^{n+1} - \delta w_j^n}{\tau} + \frac{i}{2}\Delta(w_j^{n+1} + \delta w_j^n) = X^n - X_j^n. \quad (3.12)$$

Consider the scalar product of this equation with $\Delta(w_j^{n+1} - \delta w_j^n)$. We then obtain

$$\|\Delta w_j^{n+1}\|_{L^2(\mathbb{T}^2)}^2 = \delta^2 \|\Delta w_j^n\|_{L^2(\mathbb{T}^2)}^2 - 2\langle X_j^n - X^n, \Delta(w_j^{n+1} - \delta w_j^n)\rangle. \quad (3.13)$$

Introducing

$$\beta_n = \limsup_{j \to +\infty} \|\Delta w_j^n\|_{L^2(\mathbb{T}^2)}^2, \quad (3.14)$$

we then have

$$\beta_{n+1} \le \delta^2 \beta_n, \quad (3.15)$$

and thus $\beta_n \le \delta^{2n-2p}\beta_p$. Letting $p \to -\infty$ completes the proof of the desired regularity. □

3.2. **Dimension of the attractor.** In this section, we prove that the global attractor has finite fractal dimension. For this purpose, we state a result given in [5].

Proposition 5. *Let X be a separable Hilbert space and M a bounded closed set in X. Assume that there exists a mapping $V : M \mapsto X$ such that $M \subseteq VM$ and*

(1) *V is Lipschitz on M; i.e., there exists $L_0 > 0$ such that*

$$\|Vv_1 - Vv_2\|_X \leq L_0 \|v_1 - v_2\|_X, \; v_1, v_2 \in M;$$

(2) *There exist compact seminorms $n_1(x)$ and $n_2(x)$ on X such that*

$$\|Vv_1 - Vv_2\|_X \leq \mu \|v_1 - v_2\|_X + K(n_1(v_1 - v_2) + n_2(Vv_1 - Vv_2)), \; v_1, v_2 \in M,$$

where $0 < \mu < 1$ and $K > 0$ are constants (a seminorm $n(x)$ in X is said to be compact if and only if $n(x_m) \to 0$ for any sequence $(x_m)_m \subset X$ such that $x_m \rightharpoonup 0$ weakly in X). Then M is a compact set in X with finite fractal dimension.

We apply this proposition with $S = V$, $H = H^1(\mathbb{T}^2)$, $M = \mathcal{A}_\tau$. To begin with, the first assumption is valid with $L_0 = 2(\delta + c\tau^{\frac{1}{4}} K_1^2)$ due to (2.19). On the other hand, considering the equation satisfied by the difference w of two solutions u, v we have, proceeding as in the H^1 estimate,

$$\|w^{n+1}\|_{H^1}^2 - \delta^2 \|w^n\|_{H^1}^2 \leq cM_1^2 \left(\|w^n\|_{L^4(\mathbb{T}^2)}^2 + \|w^n\|_{L^4(\mathbb{T}^2)}^2 \right). \tag{3.16}$$

Since the embedding $H^1(\mathbb{T}^2) \subset L^4(\mathbb{T}^2)$ is compact, the assumption (2) is valid. □

Remark 3. It is worth pointing out that the upper bound for the dimension of the attractor depends on τ and blows up as τ converges to 0. This is not satisfactory since the dimension of the continuous equation is finite. To our knowledge, to prove an upper bound that is independent of τ is a difficult open issue.

3.3. **Upper semicontinuity of the global attractor.** To begin with, we recall some well-known results on the dynamical system associated to the continuous equation (1.1). It is worth pointing out that the same restriction on the L^2 norm of the initial data applies, since we are in the critical focusing case. We state the following.

Proposition 6. *For all $u_0 \in \mathcal{C}_\epsilon$, the initial-value problem associated to* (1.1) *is globally well posed in $H^1(\mathbb{T}^2)$. This allows us to define a semi-group $S(t)$ on $\mathcal{C}_\varepsilon$ that has a compact global attractor $\mathcal{A}$ in $H^1(\mathbb{T}^2)$ that is, moreover, a compact subset of $H^2(\mathbb{T}^2)$.*

Proof. To prove that the initial-value problem is globally well posed is standard. To prove the regularity of the global attractor follows step by step the method in [12] for the 2D nonlinear Schrödinger equation in the subcritical case. Thus the proof is not reproduced here for the sake of conciseness. □

Actually, the time derivative of a trajectory into the global attractor is smoother than expected as stated below.

Proposition 7. *Consider a complete trajectory $u(t)$ that is included in $\mathcal{A}$. Then there exists K that depends on α and $\|f\|_{L^2(\mathbb{T}^2)}$ such that for all t in $\mathbb{R}$*

$$\|u_t(t)\|_{H^3(\mathbb{T}^2)} + \|u_{tt}(t)\|_{H^1(\mathbb{T}^2)} \le K. \tag{3.17}$$

Proof. Once again, the proof is too lengthy to be reproduced here. The idea is to derive the equation (1.1) with respect to t and prove some regularity results on $v = u_t$ that solves

$$v_t + \alpha v + i\Delta v + 2i|u|^2 v + iu^2\bar{v} = 0. \tag{3.18}$$

This uses Proposition 6 and follows line by line the method in [11]. We first prove some H^1 estimate on v (let us observe that f disappears in (3.18)), and iterate the process. □

We now proceed to the proof of Theorem 3. Let us recall that K is a constant that may vary from one line to one another, and that τ is assumed small enough. We consider a complete trajectory $u(t)$ in $\mathcal{A}$ and we introduce the consistency error as, for $t^n = n\tau$,

$$\begin{aligned}\eta_n = \frac{u(t^{n+1}) - \delta u(t^n)}{\tau} + i\Delta(\frac{u(t^{n+1}) + \delta u(t^n)}{2})\\ +\frac{i}{4}(|u(t^{n+1})|^2 + |u(t^n)|^2)(u(t^{n+1}) + \delta u(t^n)) - \frac{1+\delta}{2} f.\end{aligned} \tag{3.19}$$

We now state the following.

Lemma 2. *There exists K that depends only on the data α and $\|f\|_{L^2(\mathbb{T}^2)}$ such that*

$$\|\eta_n\|_{L^2(\mathbb{T}^2)} \le K\tau^{\frac{3}{2}}. \tag{3.20}$$

Proof. Integrating (1.1) for t in (t^n, t^{n+1}) we observe that

$$\begin{aligned}\frac{u(t^{n+1}) - \delta u(t^n)}{\tau} + \frac{i}{\tau}\Delta\int_{t^n}^{t^{n+1}} e^{\alpha(s-t^{n+1})}u(s)ds\\ +\frac{i}{\tau}\int_{t^n}^{t^{n+1}} e^{\alpha(s-t^{n+1})}|u(s)|^2 u(s)ds = \frac{1}{\tau}\int_{t^n}^{t^{n+1}} e^{\alpha(s-t^{n+1})} f ds.\end{aligned} \tag{3.21}$$

Let us set for $k \in \{-1, 0, 1\}$

$$\varepsilon_n^k = \|\frac{1}{\tau}\Delta\int_{t^n}^{t^{n+1}} e^{\alpha(s-t^{n+1})}u(s)ds - \Delta(\frac{u(t^{n+1}) + \delta u(t^n)}{2})\|_{H^k(\mathbb{T}^2)}. \tag{3.22}$$

We now apply the well-known trapezoidal formula

$$\Big|\frac{1}{\tau}\int_0^\tau f(s)ds - \frac{f(\tau)+f(0)}{2}\Big| \le c|f''|_\infty \tau^2,$$

to obtain

$$\varepsilon_n^{-1} \le K\tau^2 \sup_t \|\Delta u_{tt}\|_{H^{-1}(\mathbb{T}^2)} \le K\tau^2, \tag{3.23}$$

due to Proposition 7. On the other hand we use

$$\Big|\frac{1}{\tau}\int_0^\tau f(s)ds - \frac{f(\tau)+f(0)}{2}\Big| \le c|f'|_\infty \tau,$$

to obtain, due to the same proposition,

$$\varepsilon_n^{1} \le K\tau \sup_t \|\Delta u_t\|_{H^{1}(\mathbb{T}^2)} \le K\tau. \tag{3.24}$$

Interpolating with (3.23) provides an upper bound for $\varepsilon_n = \varepsilon_n^0$ that is bounded above by $K\tau^{\frac{3}{2}}$. To provide an upper bound on the other terms, namely the nonlinear term and the forcing term, in the consistency estimate, is simpler and thus omitted. □

The next step is to control the difference $w^n = u(t^n) - u^n$ where $u(t)$ belongs to the global attractor for the continuous equation and where $u^n = S^n u(0)$. Then w^n solves

$$\frac{1}{\tau}(w^{n+1}-\delta w^n) + \frac{i}{2}\Delta(w^{n+1}+\delta w^n) + \frac{i}{4}(|u(t^{n+1})|^2 + |u(t^n)|^2)(u(t^{n+1}) \\ + \delta u(t^n)) - \frac{i}{4}(|u^{n+1}|^2 + |u^n|^2)(u^{n+1}+\delta u^n) = \eta_n. \tag{3.25}$$

Proposition 8. *There exists K that depends on the data $\alpha, \|f\|_{L^2(\mathbb{T}^2)}$ such that*

$$\|w^n\|_{H^2(\mathbb{T}^2)} \le K(\frac{1}{\tau})^{Kn\tau}\frac{\tau^{\frac{1}{2}}}{\log(\frac{1}{\tau})}. \tag{3.26}$$

Proof. To begin with, we establish the L^2 estimate for w^n. By considering the inner product of (3.25) with $w^{n+1}+\delta w^n$ in $L^2(\mathbb{T}^2)$ we obtain

$$\frac{1}{\tau}(\|w^{n+1}\|_{L^2(\mathbb{T}^2)} - \delta\|w^n\|_{L^2(\mathbb{T}^2)}) \\ \le \|g(u(t^{n+1}),u(t^n)) - g(u^{n+1},u^n)\|_{L^2(\mathbb{T}^2)} + \|\eta_n\|_{L^2(\mathbb{T}^2)}, \tag{3.27}$$

where $g(u,v) = \frac{1}{4}(|u|^2+|v|^2)(u+\delta v)$. We now use

$$\|g(u(t^{n+1}),u(t^n)) - g(u^{n+1},u^n)\|_{L^2(\mathbb{T}^2)} \le c(\|u^{n+1}\|^2_{L^\infty(\mathbb{T}^2)} + \|u^n\|^2_{L^\infty(\mathbb{T}^2)}$$

$$+\|u(t^n)\|^2_{L^\infty(\mathbb{T}^2)} + \|u(t^{n+1})\|^2_{L^\infty(\mathbb{T}^2)})(\|w^{n+1}\|_{L^2(\mathbb{T}^2)} + \|w^n\|_{L^2(\mathbb{T}^2)}). \quad (3.28)$$

On the one hand, $\|u(t^n)\|^2_{L^\infty}$ is bounded on the global attractor $\mathcal{A}$. On the other hand, using (3.7) we can prove that if $u(0)$ belongs to a bounded subset of $H^2(\mathbb{T}^2)$ then for any n

$$\|S^n u(0)\|_{H^2(\mathbb{T}^2)} \leq \frac{K}{\sqrt{\tau}}. \quad (3.29)$$

We now invoke the so-called Brezis-Gallouet inequality [3]: for any u in $H^2(\mathbb{T}^2)$ such that $\|u\|_{H^1(\mathbb{T}^2)} \leq K$,

$$\|u\|^2_{L^\infty(\mathbb{T}^2)} \leq K(1 + \log \|u\|_{H^2(\mathbb{T}^2)}). \quad (3.30)$$

Therefore, $\|u^n\|^2_{L^\infty(\mathbb{T}^2)} \leq K \log \frac{1}{\tau}$ for all n. Hence, we are led to

$$\begin{aligned}&\|w^{n+1}\|_{L^2(\mathbb{T}^2)} \\ &\leq \|w^n\|_{L^2(\mathbb{T}^2)} + K\tau \log \frac{1}{\tau}(\|w^{n+1}\|_{L^2(\mathbb{T}^2)} + \|w^n\|_{L^2(\mathbb{T}^2)}) + \tau \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)}.\end{aligned} \quad (3.31)$$

We infer from (3.31) that for some $\beta = e^{K\tau \log \frac{1}{\tau}}$ we have

$$\|w^{n+1}\|_{L^2(\mathbb{T}^2)} \leq \beta \|w^n\|_{L^2(\mathbb{T}^2)} + 2\tau \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)}. \quad (3.32)$$

We then prove by induction, since $w^0 = 0$, that for any $n \geq 0$

$$\|w^n\|_{L^2(\mathbb{T}^2)} \leq 2\beta^n \frac{\tau}{\beta - 1} \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)} \leq \beta^n \frac{K}{\log \frac{1}{\tau}} \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)}. \quad (3.33)$$

We now prove the $H^2(\mathbb{T}^2)$ estimate. Consider the scalar product of (3.25) with $i\Delta(w^{n+1} - \delta w^n)$. We then obtain, proceeding as in (3.27)-(3.32),

$$\begin{aligned}&\|\Delta w^{n+1}\|_{L^2(\mathbb{T}^2)} - \delta \|\Delta w^n\|_{L^2(\mathbb{T}^2)} \\ &\leq K \log \frac{1}{\tau}(\|w^{n+1}\|_{L^2(\mathbb{T}^2)} + \|w^n\|_{L^2(\mathbb{T}^2)}) + \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)}.\end{aligned} \quad (3.34)$$

Therefore, we gather the two last inequalities to obtain

$$\|\Delta w^{n+1}\|_{L^2(\mathbb{T}^2)} \leq \delta \|\Delta w^n\|_{L^2(\mathbb{T}^2)} + K\beta^{n+1} \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)}. \quad (3.35)$$

We divide by δ^{n+1} and we sum from 0 to $n-1$ to obtain (using once again $w^0 = 0$)

$$\|\Delta w^n\|_{L^2(\mathbb{T}^2)} \leq \frac{K}{\frac{\beta}{\delta} - 1} \beta^{n+1} \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)} \leq \frac{K}{\tau \log \frac{1}{\tau}} \beta^{n+1} \sup_n \|\eta_n\|_{L^2(\mathbb{T}^2)}; \quad (3.36)$$

that completes the proof of the proposition due to (3.20). □

We now complete the proof of Theorem 3. Consider $a \in \mathcal{A}$. Go backward in time using the continuous semi-group and upward using the discrete semi-group to compute $v^n = S^n(S(-n\tau)a)$. On the one hand, due to the previous proposition, using $a = S(n\tau)S(-n\tau)a$,

$$\|v^n - a\|_{H^2(\mathbb{T}^2)} \leq K e^{Kn\tau \log \frac{1}{\tau}} \frac{\tau^{\frac{1}{2}}}{\log \frac{1}{\tau}}. \tag{3.37}$$

On the other hand, due to the very definition of a global attractor, for any $s > 0$ there exists n and $a_\tau \in \mathcal{A}_\tau$ such that $\|v^n - a_\tau\|_{H^2(\mathbb{T}^2)} \leq s$. Hence,

$$\|a - a_\tau\|_{H^2(\mathbb{T}^2)} \leq s + K e^{Kn\tau \log \frac{1}{\tau}} \frac{\tau^{\frac{1}{2}}}{\log \frac{1}{\tau}}. \tag{3.38}$$

For s and n fixed, we let $\tau \to 0$. This completes the proof of the theorem. □

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