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A SIMPLE PROOF OF GLOBAL SOLVABILITY OF 2-D NAVIER-STOKES EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. In this paper we provide an elementary proof of the classical result of J.L. Lions and G. Prodi on the global unique solvability of two-dimensional Navier-Stokes equations that avoids compact embedding and strong convergence. The method applies to unbounded domains without special treatment. The essential idea is to utilize the local monotonicity of the sum of the Stokes operator and the inertia term. This method was first discovered in the context of stochastic Navier-Stokes equations by J.L. Menaldi and S.S. Sritharan.

1. Introduction

In 1959, J. L. Lions and G. Prodi [8] proved the uniqueness of global weak solutions to two-dimensional Navier-Stokes equations in bounded domains with finite energy (L^2) initial data. A key feature of the proof is the strong convergence of approximate solutions to facilitate the limit of the approximations of the inertia term which is the only nonlinearity in the equation. For unbounded domains one needs to cut the domain into a sequence of bounded domains and construct Lions-Prodi solutions in each of these domains and take a limit. In the paper [10], which deals with stochastic two-dimensional

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Navier-Stokes equations, J. L. Menaldi and S. S. Sritharan devised a direct method that exploits the local monotonicity of the sum of the Stokes operator and the inertia term and used a generalization of the Minty-Browder technique thus completely eliminating the need for the compactness theorem. In this paper, we present this method in the deterministic setting to make the technique transparent so that it will be accessible to a wider audience and also be applied to other nonlinear partial differential equations in bounded and unbounded domains without searching for a strong convergence.

The local monotonicity of the sum of the Stokes operator and the inertia term has been exploited in [1] and [2], where a modification in the $W^{1,2}$-ball was made in order to achieve m-accretivity and to obtain solvability by nonlinear semigroup methods when the initial data also belongs to this space. In [5],[7] and [9], local m-accretivity and the Lyapunov property of the three dimensional Navier-Stokes equations have been studied in various function spaces.

The structure of this paper is as follows: in Section 2, we set up the function spaces and recall some classic results of the nonlinear operator B. The main L^p local monotonicity result in two dimensions is proved in Section 3; some inequalities that are needed in the proof of the existence result will also be given in this section. Section 4 is devoted to the proof of the existence of weak solutions of 2-D Navier-Stokes equations using the Minty-Browder technique. We give the three-dimensional generalized local monotonicity result as an appendix.

2. PRELIMINARIES

In this section, we define function spaces and recall some results on the Stokes operator and nonlinear operators. Let us denote by $\mathbf{u}$ and p the velocity and the pressure fields. The Navier-Stokes problem in $\mathbb{R}^n (n = 2, 3)$ is to find $(\mathbf{u}, p)$ such that

$$\partial_t \mathbf{u} - \nu \triangle \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \mathbb{R}^n \times (0, T), \tag{2.1}$$

with the conditions

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \mathbb{R}^n \times (0, T) \\ \mathbf{u} = \mathbf{u}_0 & \text{in} \quad \mathbb{R}^n \times \{0\} \end{cases}. \tag{2.2}$$

Here, $\mathbf{f}$ is a given vector function in $\mathbb{R}^n$, ν is the coefficient of kinematic viscosity. Let $\mathcal{V} = \{\mathbf{u} \in C_0^\infty(\mathbb{R}^n) : \nabla \cdot \mathbf{u} = 0\}$. Define the spaces H and V as the completion of $\mathcal{V}$ in the $L^2(\mathbb{R}^n)$ norm and the semi-norm $\|\nabla \mathbf{u}\|_{L^2}$

respectively:

$$\text{For } \mathrm{v} \in \mathrm{V}, \|\mathrm{v}\|_{\mathrm{V}} := \Big(\int_{\mathbb{R}^n} |\nabla \mathrm{v}|^2 dx\Big)^{1/2}. \tag{2.3}$$

$$\text{For } \mathrm{v} \in \mathrm{H}, \|\mathrm{v}\|_{\mathrm{H}} := \Big(\int_{\mathbb{R}^n} |\mathrm{v}|^2 dx\Big)^{1/2}. \tag{2.4}$$

Define $P_{\mathrm{H}} : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{H}$ as the Hodge projection. Let

$$\mathbf{A} : \ \mathrm{H}^2(\mathbb{R}^n) \cap \mathrm{V} \to \mathrm{H}, \qquad \mathbf{Au} = -\nu P_{\mathrm{H}} \triangle \mathbf{u}, \tag{2.5}$$

$$\mathbf{B} : \ \mathrm{D}_{\mathbf{B}} \subset \mathrm{H} \times \mathrm{V} \to \mathrm{H}, \qquad \mathbf{B}(\mathbf{u}, \mathbf{v}) = P_{\mathrm{H}}(\mathbf{u} \cdot \nabla \mathbf{v}). \tag{2.6}$$

Under the Hodge projection, the Navier-Stokes equation can be formulated in an abstract form as follows:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{Au} + \mathbf{B}(\mathbf{u}) = \mathbf{f} \qquad & \text{in} \quad \mathrm{L}^2(0, T; \mathrm{V}'), \\ \mathbf{u}(0) = \mathbf{u}_0 \qquad & \text{in} \quad \mathrm{H}, \end{aligned} \tag{2.7}$$

where $\mathbf{f} \in \mathrm{L}^2(0, T; \mathrm{V}')$ and $\mathbf{B}(\mathbf{u}) := \mathbf{B}(\mathbf{u},\mathbf{u})$. We can write

$$< \mathbf{Au}, \mathbf{w} >= \nu \sum_{i,j} \int_{\mathbb{R}^2} \partial_i \mathbf{u}_j \partial_i \mathbf{w}_j dx, \tag{2.8}$$

$$< \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} >= \sum_{i,j} \int_{\mathbb{R}^2} \mathbf{u}_i (\partial_i \mathbf{v}_j) \mathbf{w}_j dx. \tag{2.9}$$

The following properties are well known (R. Temam [11]).

For any $\mathbf{u} \in \mathrm{H}, \mathbf{v}, \mathbf{w} \in \mathrm{V}$

$$< \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} >= - < \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v} >, \tag{2.10}$$

$$< \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{v} >= 0. \tag{2.11}$$

The next lemma is known as the Gagliardo-Nirenberg inequality (Section 1.2, Theorem 2.1 in E. DiBenedetto [3]).

Lemma 2.1. *Let $\boldsymbol{v} \in \mathrm{W}^{1,p}(\mathbb{R}^n)$, where n is the dimension of the space. For every fixed number $p, s \geq 1$, there exists a constant C depending only upon n, p and s such that*

$$\|\boldsymbol{v}\|_{\mathrm{L}^q} \leq C \|\nabla \boldsymbol{v}\|_{\mathrm{L}^p}^{\alpha} \|\boldsymbol{v}\|_{\mathrm{L}^s}^{1-\alpha}, \tag{2.12}$$

where $\alpha \in [0, 1], p, q \geq 1$ and s satisfies the following relation:

$$\alpha = \Big(\frac{1}{s} - \frac{1}{q}\Big)\Big(\frac{1}{n} - \frac{1}{p} + \frac{1}{s}\Big)^{-1}. \tag{2.13}$$

The following two special cases of Lemma 2.1 were proved by O.A. Ladyzhenskaya[6].

Lemma 2.2. *For $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^2)$, we have the following estimate:*

$$\|\varphi\|^4_{\mathrm{L}^4(\mathbb{R}^2)} \leq 2\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^2)}\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^2)}. \tag{2.14}$$

Lemma 2.2 tells us that $\mathrm{V}\cap\mathrm{H}\subset\mathrm{L}^4(\mathbb{R}^2)$. Moreover,

$$\mathrm{L}^2(0,T;\mathrm{V})\cap\mathrm{L}^\infty(0,T;\mathrm{H})\subset\mathrm{L}^4((0,T)\times\mathbb{R}^2). \tag{2.15}$$

Lemma 2.3. *Let $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^3)$; then*

$$\|\varphi\|^4_{\mathrm{L}^4(\mathbb{R}^3)} \leq 4\|\varphi\|_{\mathrm{L}^2(\mathbb{R}^3)}\|\nabla\varphi\|^3_{\mathrm{L}^2(\mathbb{R}^3)}. \tag{2.16}$$

This lemma tells us that $\mathrm{V}\cap\mathrm{H}\subset\mathrm{L}^4(\mathbb{R}^3)$, but, in contrast to the two-dimensional result, we have $\mathrm{L}^2(0,T;\mathrm{V})\cap\mathrm{L}^\infty(0,T;\mathrm{H})\subset\mathrm{L}^{\frac{8}{3}}((0,T);\mathrm{L}^4(\mathbb{R}^3))$.

3. Local Monotonicity of the operator $\mathbf{A}+\mathbf{B}$ in two dimensions

Theorem 3.1. (Local Monotonicity of A + B) *For a given $r>0$, $(p>2)$ we consider the following (closed) L^p − ball $\mathbb{B}_r$ in the space* V*: $\mathbb{B}_r := \{\boldsymbol{v}\in \mathrm{V} : \|\boldsymbol{v}\|_{\mathrm{L}^p(\mathbb{R}^2)}\leq r\}$; then for any $\boldsymbol{u}\in\mathrm{V}$, $\boldsymbol{v}\in\mathbb{B}_r$ and $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{v}$ we have*

$$<\boldsymbol{A}\boldsymbol{w},\boldsymbol{w}> + <\boldsymbol{B}(\boldsymbol{u})-\boldsymbol{B}(\boldsymbol{v}),\boldsymbol{w}> + C_3\|\boldsymbol{w}\|^2_{\mathrm{H}}r^{\frac{2p}{p-2}} \geq \frac{\nu}{2}\|\boldsymbol{w}\|^2_{\mathrm{V}}. \tag{3.1}$$

Similarly, if $\boldsymbol{r}(t)$ is a positive and measurable real function and $\mathbb{B}_{\boldsymbol{r}}(t)$ is the following (closed) time-variable L^p − ball of $\mathrm{L}^2(0,T;\mathrm{V})$:

$$\mathbb{B}_{\boldsymbol{r}}(t) := \{\boldsymbol{v}(.)\in\mathrm{L}^2(0,T;\mathrm{V}) : \|\boldsymbol{v}(t)\|_{\mathrm{L}^p(\mathbb{R}^2)}\leq\boldsymbol{r}(t)\}, \tag{3.2}$$

then, for any $\boldsymbol{u}(.)$ in $\mathrm{L}^2(0,T;\mathrm{V})$, $\boldsymbol{v}(.)$ in $\mathbb{B}_{\boldsymbol{r}}(t)$, $\boldsymbol{w}(.)=\boldsymbol{u}(.)-\boldsymbol{v}(.)$ and any measurable function $\rho(t)$, we have

$$\int_0^T[<\boldsymbol{A}\boldsymbol{w},\boldsymbol{w}> + <\boldsymbol{B}(\boldsymbol{u})-\boldsymbol{B}(\boldsymbol{v}),\boldsymbol{w}>]e^{\rho(t)}dt + C_3\int_0^T\|\boldsymbol{w}\|^2_{\mathrm{H}}\boldsymbol{r}^{\frac{2p}{p-2}}(t)e^{\rho(t)}dt$$
$$\geq\frac{\nu}{2}\int_0^T\|\boldsymbol{w}\|^2_{\mathrm{V}}e^{\rho(t)}dt, \tag{3.3}$$

where C_3 is a constant which depends on ν and p.

We call the operator $\mathbf{u}\longmapsto\mathbf{Au}+\mathbf{B}(\mathbf{u})$ locally monotone if it satisfies inequality (3.2). For example, when p=4, $C_3=\frac{16}{\nu^3}$.

Proof. The following lemma is needed to prove this theorem. □

Lemma 3.1. *Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be in the spaces* $\mathrm{L}^p(\mathbb{R}^2)$ *and* V *respectively; then the following estimates hold:*

(i) For $p \geq 2$,

$$| < \boldsymbol{B}(\boldsymbol{w}), \boldsymbol{v} > | \leq C_1 \|\boldsymbol{w}\|_{\mathrm{V}}^{\frac{p+2}{p}} \|\boldsymbol{w}\|_{\mathrm{H}}^{\frac{p-2}{p}} \|\boldsymbol{v}\|_{\mathrm{L}^p(\mathbb{R}^2)}, \tag{3.4}$$

where C_1 is a constant which depends on p.

(ii) For $1 \leq p < 2$, $q \geq 2$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v})\|_{\mathrm{L}^q} \leq C_2 \|\boldsymbol{u}\|_{\mathrm{W}^{1,q}} \|\boldsymbol{v}\|_{\mathrm{W}^{1,q}}, \tag{3.5}$$

for $\boldsymbol{u},\boldsymbol{v} \in \mathrm{W}^{1,q}(\mathbb{R}^2)$ and

$$| < \boldsymbol{B}(\boldsymbol{w}), \boldsymbol{v} > | \leq C_2 \|\boldsymbol{w}\|_{\mathrm{W}^{1,q}}^2 \|\boldsymbol{v}\|_{\mathrm{L}^p(\mathbb{R}^2)}, \tag{3.6}$$

for $\boldsymbol{v} \in \mathrm{L}^p(\mathbb{R}^2)$, $\boldsymbol{w} \in \mathrm{W}^{1,q}(\mathbb{R}^2)$. Here C_2 is a constant which depends on p.

Proof. (i) It is clear that

$$| < \mathbf{B}(\mathbf{w}), \mathbf{v} > | \leq \sum_{i,j} \|\mathbf{w}_i \mathbf{v}_j\|_{\mathrm{L}^2(\mathbb{R}^2)} \|\partial_i \mathbf{w}_j\|_{\mathrm{L}^2(\mathbb{R}^2)}. \tag{3.7}$$

Applying Hölder's inequality, the above inequality can be reduced to the following form:

$$\begin{aligned} | < \mathbf{B}(\mathbf{w}), \mathbf{v} > | &\leq \sum_{i,j} \|\mathbf{w}_i\|_{\mathrm{L}^{\frac{2p}{p-2}}(\mathbb{R}^2)} \|\mathbf{v}_j\|_{\mathrm{L}^p(\mathbb{R}^2)} \|\partial_i \mathbf{w}_j\|_{\mathrm{L}^2(\mathbb{R}^2)} \\ &\leq \|\mathbf{w}\|_{\mathrm{L}^{\frac{2p}{p-2}}(\mathbb{R}^2)} \|\mathbf{v}\|_{\mathrm{L}^p(\mathbb{R}^2)} \|\mathbf{w}\|_{\mathrm{V}}. \end{aligned} \tag{3.8}$$

The Gagliardo-Nirenberg inequality gives us

$$\|\mathbf{w}\|_{\mathrm{L}^{\frac{2p}{p-2}}(\mathbb{R}^2)} \leq C_1 \|\mathbf{w}\|_{\mathrm{V}}^{\frac{2}{p}} \|\mathbf{w}\|_{\mathrm{H}}^{\frac{p-2}{p}}. \tag{3.9}$$

Substituting (3.9) into (3.8), we get

$$| < \mathbf{B}(\mathbf{w}), \mathbf{v} > | \leq C_1 \|\mathbf{w}\|_{\mathrm{V}}^{\frac{p+2}{p}} \|\mathbf{w}\|_{\mathrm{H}}^{\frac{p-2}{p}} \|\mathbf{v}\|_{\mathrm{L}^p(\mathbb{R}^2)}. \tag{3.10}$$

(ii) The proof of inequality (3.5) is given by Y. Giga and T. Miyakawa[4].

Now, suppose $\mathbf{v} \in \mathrm{L}^p(\mathbb{R}^2)$ and $\mathbf{w} \in \mathrm{W}^{1,q}(\mathbb{R}^2)(q \geq 2)$; using (3.5) and Hölder's inequality, we have

$$| < \mathbf{B}(\mathbf{w}), \mathbf{v} > | \leq \|\mathbf{B}(\mathbf{w})\|_{\mathrm{L}^q} \|\mathbf{v}\|_{\mathrm{L}^p} \leq C_2 \|\mathbf{w}\|_{\mathrm{W}^{1,q}}^2 \|\mathbf{v}\|_{\mathrm{L}^p}. \tag{3.11}$$

Inequality (3.6) is proved. □

Proof of Theorem 3.1. First we will show that $< \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w} >= - < \mathbf{B}(\mathbf{w}), \mathbf{v} >$ for $\mathbf{u}, \mathbf{v} \in \mathrm{V}$ and $\mathbf{w} = \mathbf{u} - \mathbf{v}$. We first derive two equalities based on the trilinearity of the nonlinear operator $\mathbf{B}(\cdot)$:

$$
\begin{aligned}
< \mathbf{B}(\mathbf{u}), \mathbf{w} > =< \mathbf{B}(\mathbf{u}, \mathbf{u}), \mathbf{w} >= - < \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{u} > \\
= - < \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{w} + \mathbf{v} >= - < \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v} > .
\end{aligned}
$$

Thus,

$$
< \mathbf{B}(\mathbf{v}), \mathbf{w} >= - < \mathbf{B}(\mathbf{v}, \mathbf{w}), \mathbf{v} > . \tag{3.12}
$$

The above two equalities give us

$$
\begin{aligned}
&< \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w} >=< \mathbf{B}(\mathbf{u}), \mathbf{w} > - < \mathbf{B}(\mathbf{v}), \mathbf{w} > \\
&= - < \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v} > + < \mathbf{B}(\mathbf{v}, \mathbf{w}), \mathbf{v} >= - < \mathbf{B}(\mathbf{u}, \mathbf{w}) - \mathbf{B}(\mathbf{v}, \mathbf{w}), \mathbf{v} > \\
&= - < \mathbf{B}(\mathbf{u} - \mathbf{v}, \mathbf{w}), \mathbf{v} >= - < \mathbf{B}(\mathbf{w}), \mathbf{v} > .
\end{aligned}
$$

Next, using (3.4) and Young's inequality,

$$
\begin{aligned}
&| < \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w} > | = | < \mathbf{B}(\mathbf{w}), \mathbf{v} > | \leq C_1 \|\mathbf{w}\|_{\mathrm{V}}^{\frac{p+2}{p}} \|\mathbf{w}\|_{\mathrm{H}}^{\frac{p-2}{p}} \|\mathbf{v}\|_{\mathrm{L}^p(\mathbb{R}^2)} \\
&= \left(\frac{p\nu}{p+2}\right)^{\frac{p+2}{2p}} \|\mathbf{w}\|_{\mathrm{V}}^{\frac{p+2}{p}} \frac{\left[C_1 \|\mathbf{w}\|_{\mathrm{H}}^{\frac{p-2}{p}} \|\mathbf{v}\|_{\mathrm{L}^p(\mathbb{R}^2)}\right]}{\left(\frac{p\nu}{p+2}\right)^{\frac{p+2}{2p}}} \leq \frac{\nu}{2} \|\mathbf{w}\|_{\mathrm{V}}^2 + C_3 \|\mathbf{w}\|_{\mathrm{H}}^2 \|\mathbf{v}\|_{\mathrm{L}^p(\mathbb{R}^2)}^{\frac{2p}{p-2}}.
\end{aligned} \tag{3.13}
$$

Hence,

$$
< \mathbf{A}\mathbf{w}, \mathbf{w} > + < \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w} >\geq \nu \|\mathbf{w}\|_{\mathrm{V}}^2 - \frac{\nu}{2} \|\mathbf{w}\|_{\mathrm{V}}^2 - C_3 \|\mathbf{w}\|_{\mathrm{H}}^2 \|\mathbf{v}\|_{\mathrm{L}^p(\mathbb{R}^2)}^{\frac{2p}{p-2}}.
$$

This gives (3.1). Inequality (3.3) can be obtained easily by (3.1). □

4. Existence result for Weak Solutions of the 2-D Navier-Stokes Equation

Definition 4.1. *We call* $\boldsymbol{u} \in \mathrm{L}^\infty(0, T; \mathrm{H}) \cap \mathrm{L}^2(0, T; \mathrm{V})$ *a Leray weak solution of the Navier Stokes equation if, for* $\boldsymbol{f} \in \mathrm{L}^2(0, T; \mathrm{V}')$, $\boldsymbol{u}_0 \in \mathrm{H}$ *and* $\boldsymbol{v} \in \mathrm{L}^2(0, T; \mathrm{V})$, *it satisfies*

$$
\frac{d}{dt}(\boldsymbol{u}(t), \boldsymbol{v}) + < \boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{B}(\boldsymbol{u}(t)), \boldsymbol{v} >=< \boldsymbol{f}(t), \boldsymbol{v} >, \tag{4.1}
$$

$$
(\boldsymbol{u}(0), \boldsymbol{v}) = (\boldsymbol{u}_0, \boldsymbol{v}). \tag{4.2}
$$

The following energy identity and inequality are well known.

Proposition 4.1. *Let $\boldsymbol{u}$ be a solution of the Navier-Stokes equation, where $\boldsymbol{u} \in \mathrm{L}^2(0,T;\mathrm{V}), \partial_t \boldsymbol{u} \in \mathrm{L}^2(0,T;\mathrm{V}')$ and $\boldsymbol{f} \in \mathrm{L}^2(0,T;\mathrm{V}')$. Then we have the following energy equality and a priori estimate:*

$$\|\boldsymbol{u}(T)\|_{\mathrm{H}}^2 + 2\nu \int_0^T \|\boldsymbol{u}\|_{\mathrm{V}}^2 dt = \|\boldsymbol{u}(0)\|_{\mathrm{H}}^2 + 2\int_0^T < \boldsymbol{f}(t), \boldsymbol{u}(t) > dt, \tag{4.3}$$

$$\sup_{0\le t\le T} \|\boldsymbol{u}(t)\|_{\mathrm{H}}^2 + \nu \int_0^T \|\boldsymbol{u}(t)\|_{\mathrm{V}}^2 dt \le \|\boldsymbol{u}(0)\|_{\mathrm{H}}^2 + \frac{1}{\nu}\int_0^T \|\boldsymbol{f}(t)\|_{\mathrm{V}'}^2 dt. \tag{4.4}$$

Remark. Proposition 4.1 remains true in the three-dimensional case.

Theorem 4.2. (2-D Existence) *Let $\boldsymbol{f} \in \mathrm{L}^2(0,T;\mathrm{H})$, $\boldsymbol{u}_0 \in \mathrm{H}$; then there exists a weak solution of the Navier-Stokes equation $\boldsymbol{u}$ with $\boldsymbol{u} \in \mathrm{C}([0,T];\mathrm{H}) \cap \mathrm{L}^2(0,T;\mathrm{V}) \cap \mathrm{L}^4((0,T)\times\mathbb{R}^2)$.*

Proof. We show only the existence result since uniqueness is straightforward in bounded and unbounded domains.

(1) *Finite-dimensional* (*Galerkin*) *approximation of the Navier-Stokes equation*: Let $\{e_1, e_2, \ldots\}$ be a complete orthogonal system in H belonging to V. Denote by H_n the n-dimensional subspace of H. Consider the following ODE in H_n :

$$\frac{d}{dt}(\mathbf{u}^n(t), \mathbf{v}) + < \mathbf{A}\mathbf{u}^n(t) + \mathbf{B}(\mathbf{u}^n(t)), \mathbf{v} > = < \mathbf{f}(t), \mathbf{v} > \tag{4.5}$$

in $(0,T)$ for any $\mathbf{v} \in \mathrm{H}_n$ where $\mathbf{u}_0^n$ is the orthogonal projection of $\mathbf{u}_0$ into *span*$\{e_1, e_2, \ldots, e_n\}$. Denoting by $\mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}) - \mathbf{f}$, we have

$$\frac{d}{dt}\mathbf{u}^n(t) + \mathbf{F}(\mathbf{u}^n(t)) = 0 \ \ in \ \ \mathrm{H}_n, \tag{4.6}$$

and the corresponding energy equality

$$\|\mathbf{u}^n(t)\|_{\mathrm{H}}^2 + 2\int_0^t < \mathbf{F}(\mathbf{u}^n(s)), \mathbf{u}^n(s) > ds = \|\mathbf{u}^n(0)\|_{\mathrm{H}}^2. \tag{4.7}$$

(2) *Weak convergent sequences*: By Proposition 4.1, we can extract two subsequences $\{\mathbf{u}^n\}$ and $\{\mathbf{F}(\mathbf{u}^n)\}$, such that $\{\mathbf{u}^n\} \longrightarrow \mathbf{u}$ weak star in $\mathrm{L}^\infty(0,T;\mathrm{H})$ and weakly in $\mathrm{L}^2(0,T;\mathrm{V})$, and $\{\mathbf{F}(\mathbf{u}^n)\} \longrightarrow \mathbf{F}_0$ weakly in $\mathrm{L}^2(0,T;\mathrm{V}')$. We define $\bar{\mathbf{u}}^n(t) = e^{-\mathbf{r}(t)}\mathbf{u}^n(t)$;

$$\frac{d}{dt}\bar{\mathbf{u}}^n(t) = e^{-\mathbf{r}(t)}\frac{d}{dt}\mathbf{u}^n(t) - \mathbf{r}'(t)\bar{\mathbf{u}}^n(t). \tag{4.8}$$

Now, consider

$$< \frac{d}{dt}\bar{\mathbf{u}}^n(t), \mathbf{u}^n(t) > = < e^{-\mathbf{r}(t)}\frac{d}{dt}\mathbf{u}^n(t) - \mathbf{r}'(t)\bar{\mathbf{u}}^n(t), \mathbf{u}^n(t) > \tag{4.9}$$

$$= e^{-\mathbf{r}(t)} < \frac{d}{dt}\mathbf{u}^n(t), \mathbf{u}^n(t) > -\mathbf{r}'(t) < \bar{\mathbf{u}}^n(t), \mathbf{u}^n(t) >,$$

which is

$$\frac{d}{dt} < \bar{\mathbf{u}}^n(t), \mathbf{u}^n(t) >= e^{-\mathbf{r}(t)}\frac{d}{dt}\|\mathbf{u}^n(t)\|_{\mathrm{H}}^2 - \mathbf{r}'(t)e^{-\mathbf{r}(t)} < \mathbf{u}^n(t), \mathbf{u}^n(t) > .$$

By (4.7), this reduces to the energy equality

$$e^{-\mathbf{r}(t)}\|\mathbf{u}^n(t)\|_{\mathrm{H}}^2 + \int_0^t e^{-\mathbf{r}(s)} < 2\mathrm{F}(\mathbf{u}^n(s)) + \mathbf{r}'(s)\mathbf{u}^n(s), \mathbf{u}^n(s) > ds = \|\mathbf{u}^n(0)\|_{\mathrm{H}}^2. \tag{4.10}$$

Here, $\mathbf{r}'(t)$ denotes the time derivative of $\mathbf{r}(t)$. Note that $\mathbf{u}$ satisfies

$$\frac{d}{dt}\mathbf{u}(t) + \mathbf{F}_0(t) = 0 \;\; in \;\; \mathrm{L}^2(0, T; \mathrm{V}'), \tag{4.11}$$

and the energy equality

$$\|\mathbf{u}(t)\|_{\mathrm{H}}^2 + 2\int_0^t < \mathbf{F}_0(s), \mathbf{u}(s) > ds = \|\mathbf{u}(0)\|_{\mathrm{H}}^2. \tag{4.12}$$

We can show that $\mathbf{u}(t)e^{-\mathbf{r}(t)}$ satisfies the following energy equality:

$$e^{-\mathbf{r}(t)}\|\mathbf{u}(t)\|_{\mathrm{H}}^2 + \int_0^t e^{-\mathbf{r}(s)} < 2\mathrm{F}_0(s) + \mathbf{r}'(s)\mathbf{u}(s), \mathbf{u}(s) > ds = \|\mathbf{u}(0)\|_{\mathrm{H}}^2. \tag{4.13}$$

Also note that, at the initial time, $\{\mathbf{u}^n(0)\}$ converges to $\mathbf{u}(0)$ strongly; i.e.,

$$\lim_{n\to\infty} \|\mathbf{u}^n(0) - \mathbf{u}(0)\|_{\mathrm{L}^2(\mathbb{R}^2)} = 0. \tag{4.14}$$

(3) *Local Minty-Browder technique*: For $\mathbf{v} \in \mathrm{L}^\infty(0, T; \mathrm{H}_m)$ with $m < n$, we define

$$\mathbf{r}(t) = \frac{16}{\nu^3}\int_0^t \|\mathbf{v}(s,\cdot)\|_{\mathrm{L}^4(\mathbb{R}^2)}^4 ds. \tag{4.15}$$

Here, $\|\mathbf{v}(s,\cdot)\|_{\mathrm{L}^4(\mathbb{R}^2)}^4$ can be controlled using Lemma 2.2(see Remark 2).

From Theorem 3.1 (with p=4), we have

$$\int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{v}(t)) + \mathbf{r}'(t)\mathbf{v}(t), \mathbf{v}(t) - \mathbf{u}^n(t) > dt \tag{4.16}$$

$$\geq \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{u}^n(t)) + \mathbf{r}'(t)\mathbf{u}^n(t), \mathbf{v}(t) - \mathbf{u}^n(t) > dt.$$

By the energy equality (4.10), the right-hand side of (4.16) can be written as

$$\int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{u}^n(t)) + \mathbf{r}'(t)\mathbf{u}^n(t), \mathbf{v}(t) - \mathbf{u}^n(t) > dt \tag{4.17}$$

$$= \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{u}^n(t)) + \mathbf{r}'(t)\mathbf{u}^n(t), \mathbf{v}(t) > dt$$

$$- \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{u}^n(t)) + \mathbf{r}'(t)\mathbf{u}^n(t), \mathbf{u}^n(t) > dt$$

$$= \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{u}^n(t)) + \mathbf{r}'(t)\mathbf{u}^n(t), \mathbf{v}(t) > dt + e^{-\mathbf{r}(T)}\|\mathbf{u}^n(T)\|_{\mathrm{H}}^2 - \|\mathbf{u}^n(0)\|_{\mathrm{H}}^2;$$

we thus have

$$\int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{v}(t)) + \mathbf{r}'(t)\mathbf{v}(t), \mathbf{v}(t) - \mathbf{u}^n(t) > dt \tag{4.18}$$

$$\geq \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{u}^n(t)) + \mathbf{r}'(t)\mathbf{u}^n(t), \mathbf{v}(t) > dt + e^{\mathbf{r}(T)}\|\mathbf{u}^n(T)\|_{\mathrm{H}}^2 - \|\mathbf{u}^n(0)\|_{\mathrm{H}}^2.$$

Taking lim inf on both sides,

$$\int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{v}(t)) + \mathbf{r}'(t)\mathbf{v}(t), \mathbf{v}(t) - \mathbf{u}(t) > dt \tag{4.19}$$

$$\geq \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}_0(t) + \mathbf{r}'(t)\mathbf{u}(t), \mathbf{v}(t) > dt$$

$$+ \liminf_{n\to\infty}(e^{-\mathbf{r}(T)}\|\mathbf{u}^n(T)\|_{\mathrm{H}}^2 - \|\mathbf{u}^n(0)\|_{\mathrm{H}}^2).$$

By lower semi-continuity of the L^2-norm and strong convergence of the initial data $\mathbf{u}^n(0)$, the second term on the right-hand side of (4.19) satisfies the following inequality:

$$\liminf_{n\to\infty}(e^{-\mathbf{r}(T)}\|\mathbf{u}^n(T)\|_{\mathrm{H}}^2 - \|\mathbf{u}^n(0)\|_{\mathrm{H}}^2) \geq e^{-\mathbf{r}(T)}\|\mathbf{u}(T)\|_{\mathrm{H}}^2 - \|\mathbf{u}(0)\|_{\mathrm{H}}^2. \tag{4.20}$$

Replacing the right-hand side of (4.20) by the energy equality (4.13) and thus (4.19) becomes

$$\int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}(\mathbf{v}(t)) + \mathbf{r}'(t)\mathbf{v}(t), \mathbf{v}(t) - \mathbf{u}(t) > dt \tag{4.21}$$

$$\geq \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}_0(t) + \mathbf{r}'(t)\mathbf{u}(t), \mathbf{v}(t) > dt$$

$$- \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}_0(t) + \mathbf{r}'(t)\mathbf{u}(t), \mathbf{u}(t) > dt$$
$$= \int_0^T e^{-\mathbf{r}(t)} < 2\mathrm{F}_0(t) + \mathbf{r}'(t)\mathbf{u}(t), \mathbf{v}(t) - \mathbf{u}(t) > dt.$$

This estimate holds for any $\mathbf{v} \in \mathrm{L}^\infty(0,T;\mathrm{H}_m)$ and any $m \in \mathbb{N}$. It is clear by a density argument that the above inequality remains true for any $\mathbf{v} \in \mathrm{L}^\infty(0,T;\mathrm{H}) \cap \mathrm{L}^2(0,T;\mathrm{V})$. Indeed, for any $\mathbf{v} \in \mathrm{L}^\infty(0,T;\mathrm{H}) \cap \mathrm{L}^2(0,T;\mathrm{V})$, there exists a strongly convergent sequence $\mathbf{v}_m \in \mathrm{L}^\infty(0,T;\mathrm{H}) \cap \mathrm{L}^2(0,T;\mathrm{V})$ that satisfies inequality (4.21).

Now, take $\mathbf{v} = \mathbf{u} + \lambda\mathbf{w}, \lambda > 0$ where $\mathbf{w} \in \mathrm{L}^\infty(0,T;\mathrm{H}) \cap \mathrm{L}^2(0,T;\mathrm{V})$, and substitute $\mathbf{v}$ into (4.21); we have

$$\int_0^T e^{-\mathbf{r}(t)} < \mathrm{F}(\mathbf{u} + \lambda\mathbf{w}) - F_0 + \frac{1}{2}\mathbf{r}'(t)\lambda\mathbf{w}, \lambda\mathbf{w} > dt \geq 0. \tag{4.22}$$

The left-hand side becomes

$$\int_0^T e^{-\mathbf{r}(t)} < \mathrm{A}(\mathbf{u} + \lambda\mathbf{w}) + \mathrm{B}(\mathbf{u} + \lambda\mathbf{w}) - \mathbf{f} - \mathrm{F}_0 + \mathbf{r}'(t)\lambda\mathbf{w}, \lambda\mathbf{w} > dt \tag{4.23}$$
$$= \int_0^T e^{-\mathbf{r}(t)} < \mathrm{A}\mathbf{u} + \lambda\mathrm{A}\mathbf{w} + \mathrm{B}(\mathbf{u}) + \lambda(\mathrm{B}(\mathbf{u},\mathbf{w}) + \mathrm{B}(\mathbf{w},\mathbf{u}))$$
$$+ \lambda^2\mathrm{B}(\mathbf{w}) - \mathbf{f} - \mathrm{F}_0 + \frac{1}{2}\mathbf{r}'(t)\lambda\mathbf{w}, \lambda\mathbf{w} > dt.$$

Now, dividing this inequality by λ and letting $\lambda \to 0$, we pick up

$$\int_0^T e^{-\mathbf{r}(t)} < \mathrm{A}\mathbf{u} + \mathrm{B}(\mathbf{u}) - \mathbf{f} - \mathrm{F}_0, \mathbf{w} > dt = \int_0^T e^{-\mathbf{r}(t)} < \mathrm{F}(\mathbf{u}) - \mathrm{F}_0, \mathbf{w} > dt.$$

Hence,

$$\int_0^T e^{-\mathbf{r}(t)} < \mathrm{F}(\mathbf{u}(t)) - \mathrm{F}_0(t), \mathbf{w}(t) > dt \geq 0, \tag{4.24}$$

for any $\mathbf{w} \in \mathrm{L}^\infty(0,T;\mathrm{H}) \cap \mathrm{L}^2(0,T;\mathrm{V})$. Thus, we have $\mathrm{F}(\mathbf{u}(t)) = \mathrm{F}_0$. Therefore, $\mathbf{u}$ is the weak solution of the 2-D Navier-Stokes equation. □

Remark 1. In the proof, we used the fact that

$$\int_0^T e^{-\mathbf{r}(t)}\mathbf{r}'(t)\lambda < \mathbf{w}, \mathbf{w} > dt \to 0 \quad as \quad \lambda \to 0. \tag{4.25}$$

Actually, we can use Lemma 2.2 to control the term

$$\int_0^T e^{-\mathbf{r}(t)}\mathbf{r}'(t) < \mathbf{w}, \mathbf{w} > dt = \int_0^T e^{-\mathbf{r}(t)}\|\mathbf{v}(s,\cdot)\|^4_{\mathrm{L}^4(\mathbb{R}^2)}\|\mathbf{w}\|^2_{\mathrm{H}}dt$$

$$\leq \int_0^T \|\mathbf{v}(s,\cdot)\|^4_{\mathrm{L}^4(\mathbb{R}^2)}\|\mathbf{w}\|^2_{\mathrm{H}}dt \tag{4.26}$$
$$\leq 2\Big(\sup_{t\leq T}\|\mathbf{v}(t,\cdot)\|^2_{\mathrm{H}}\Big)\Big(\sup_{t\leq T}\|\mathbf{w}\|^2_{\mathrm{H}}\Big)\int_0^T \|\mathbf{v}\|^2_{\mathrm{V}}dt<\infty.$$

Remark 2. In the proof we defined

$$\mathbf{r}(t)=\frac{16}{\nu^3}\int_0^T \|\mathbf{v}(s,\cdot)\|^4_{\mathrm{L}^4(\mathbb{R}^2)}ds. \tag{4.27}$$

It turns out that we can also choose

$$\mathbf{r}(t)=\frac{C}{\nu^3}\int_0^T \|\mathbf{v}(s,\cdot)\|^{\frac{2p}{p-2}}_{\mathrm{L}^p(\mathbb{R}^2)}ds, \tag{4.28}$$

for some suitable constant C. The idea of the proof still works and we can use Theorem 3.1 on L^p-monotonicity. In fact, we can control the term $\int_0^T \|\mathbf{v}(s,\cdot)\|^{\frac{2p}{p-2}}_{\mathrm{L}^p(\mathbb{R}^2)}ds$ in the space $\mathrm{L}^\infty(0,T;\mathrm{H})\cap\mathrm{L}^2(0,T;\mathrm{V})$. Suppose $\mathbf{v}\in\mathrm{L}^\infty(0,T;\mathrm{H})\cap\mathrm{L}^2(0,T;\mathrm{V})$; by the Gagliardo-Nirenberg inequality, we can show

$$\|\mathbf{v}\|_{\mathrm{L}^p}\leq C\|\mathbf{v}\|^{\frac{p-2}{p}}_{\mathrm{V}}\|\mathbf{v}\|^{\frac{2}{p}}_{\mathrm{H}}. \tag{4.29}$$

Raising both sides to the power $\frac{2p}{p-2}$ and integrating over time from 0 to t gives

$$\int_0^t \|\mathbf{v}(s,\cdot)\|^{\frac{2p}{p-2}}_{\mathrm{L}^p(\mathbb{R}^2)}ds\leq\int_0^t \|\mathbf{v}\|^2_{\mathrm{V}}\|\mathbf{v}\|^{\frac{4}{p-2}}_{\mathrm{H}}ds\leq\sup_{0\leq s\leq t}\|\mathbf{v}\|^{\frac{4}{p-2}}_{\mathrm{H}}\int_0^t\|\mathbf{v}\|^2_{\mathrm{V}}ds<\infty.$$

5. Appendix: 3D Local Monotonicity

The general L^p-ball result can be stated as follows.

Lemma 5.1. *Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be in the space $\mathrm{L}^p(\mathbb{R}^3)$ and V respectively; then*

$$|<\mathrm{B}(\boldsymbol{w}),\boldsymbol{v}>|\leq C\|\boldsymbol{w}\|^{\frac{p+3}{p}}_{\mathrm{V}}\|\boldsymbol{w}\|^{\frac{p-3}{p}}_{\mathrm{H}}\|\boldsymbol{v}\|_{\mathrm{L}^p(\mathbb{R}^3)},\quad (p\geq 3). \tag{5.1}$$

Proof. By Hölder's inequality,

$$|<\mathrm{B}(\mathbf{w}),\mathbf{v}>|\leq\sum_{i,j}\int_{\mathbb{R}^3}|\mathbf{w}_i(\partial_i\mathbf{w}_j)\mathbf{v}_j|dx \tag{5.2}$$
$$\leq\sum_{i,j}\|\mathbf{w}_i\|_{\mathrm{L}^{\frac{2p}{p-2}}}\|\mathbf{v}_j\|_{\mathrm{L}^p}\|\partial_i\mathbf{w}_j\|_{\mathrm{L}^2}\leq\|\mathbf{w}\|_{\mathrm{L}^{\frac{2p}{p-2}}}\|\mathbf{v}\|_{\mathrm{L}^p}\|\mathbf{w}\|_{\mathrm{V}}.$$

By the Gagliardo-Nirenberg inequality,

$$\|\mathbf{w}\|_{\mathrm{L}^{\frac{2p}{p-2}}} \leq C\|\mathbf{w}\|_{\mathrm{V}}^{\frac{3}{p}}\|\mathbf{w}\|_{\mathrm{H}}^{1-\frac{3}{p}}, \tag{5.3}$$

thus

$$| < \mathrm{B}(\mathbf{w}), \mathbf{v} > | \leq C\|\mathbf{w}\|_{\mathrm{V}}^{\frac{p+3}{p}}\|\mathbf{w}\|_{\mathrm{H}}^{\frac{p-3}{p}}\|\mathbf{v}\|_{\mathrm{L}^p(\mathbb{R}^3)}. \tag{5.4}$$

As in the 2-D case, we have the following result in 3D.

Lemma 5.2. *For given $r > 0$ and $p > 3$, consider the L^p-ball $\mathbb{B}_r$ in* V*:*

$$\mathbb{B}_r := \{\boldsymbol{v} \in \mathrm{V} : \ \|\boldsymbol{v}\|_{\mathrm{L}^p(\mathbb{R}^3)} \leq r\}; \tag{5.5}$$

then $\boldsymbol{u} \to \mathrm{A}\boldsymbol{u} + \mathrm{B}(\boldsymbol{u})$ is monotone in the convex ball $\mathbb{B}_r$; i.e., for any $\boldsymbol{u} \in \mathrm{V}$, $\boldsymbol{v} \in \mathrm{V}_r$ and $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, we have

$$< \mathrm{A}\boldsymbol{w}, \boldsymbol{w} > + < \mathrm{B}(\boldsymbol{u}) - \mathrm{B}(\boldsymbol{v}), \boldsymbol{w} > + C_4\|\boldsymbol{w}\|_{\mathrm{H}}^2 \geq \frac{\nu}{2}\|\boldsymbol{w}\|_{\mathrm{V}}^2. \tag{5.6}$$

Similarly, if $\boldsymbol{r}(t)$ is a positive and measurable real function and $\mathbb{B}_{\boldsymbol{r}}(t)$ is the time-variable L^p-ball of $\mathrm{L}^2(0, T; \mathrm{V})$

$$\mathbb{B}_{\boldsymbol{r}}(t) := \{\boldsymbol{v}(\cdot) \in \mathrm{L}^2(0, T; \mathrm{V}) : \ \|\boldsymbol{v}(t)\|_{\mathrm{L}^p(\mathbb{R}^3)} \leq \boldsymbol{r}(t)\}, \tag{5.7}$$

then, for any $\boldsymbol{u}(\cdot)$ in $\mathrm{L}^2(0, T; \mathrm{V})$, $\boldsymbol{v}(t)$ in $\mathbb{B}_{\boldsymbol{r}}(t)$, $\boldsymbol{w}(\cdot) = \boldsymbol{u}(\cdot) - \boldsymbol{v}(\cdot)$ and any measurable real function $\rho(t)$, we have

$$\int_0^T [< \mathrm{A}\boldsymbol{w}, \boldsymbol{w} > + < \mathrm{B}(\boldsymbol{u}) - \mathrm{B}(\boldsymbol{v}), \boldsymbol{w} >] e^{\rho(t)} dt$$
$$+ C_4 \int_0^T \|\boldsymbol{w}(t)\|_{\mathrm{H}}^2 \boldsymbol{r}^{\frac{2p}{p-3}}(t) e^{\rho(t)} dt \geq \frac{\nu}{2} \int_0^T \|\boldsymbol{w}\|_{\mathrm{V}}^2 e^{\rho(t)} dt,$$

where C_4 is a constant which depends on ν and p.

Proof. This can be proved similarly to the proof of Theorem 3.1. □

Remark. In three dimensions the Minty-Browder technique appears to fail since $\{\mathrm{F}(\mathbf{u}^n)\}$ is only known to exist in $\mathrm{L}^{\frac{4}{3}}(0, T; \mathrm{V}')$. Hence, the energy equality does not hold as the term $< \mathrm{F}_0(t), \mathbf{u}(t) >$ does not make sense any more. Also, we can not find a suitable bound for

$$\mathbf{r}(t) = \frac{C}{\nu^3} \int_0^T ||\mathbf{v}(s, \cdot)||_{\mathrm{L}^p(\mathbb{R}^3)}^{\frac{3p}{p-3}} ds \quad p > 3. \tag{5.8}$$

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