Differential and Integral Equations Volume 23, Numbers 3-4 (2010), 201–222

MULTIPLE POSITIVE SOLUTIONS FOR THE N-LAPLACE EQUATION WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. Let $u \in W^{1,N}(\Omega)$ be a weak solution of the following problem:

$$(P_{\mu,\lambda}) \qquad \begin{cases} \left.\begin{array}{rcl} -\mathrm{div}(|\nabla u|^{N-2}\nabla u) + |u|^{N-2}u &=& \mu h(u)e^{u^\alpha} \\ u &>& 0 \end{array}\right\} \text{ in } \Omega, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial \nu} = \lambda \psi u^q \text{ on } \partial\Omega, \end{cases}$$

where $\alpha \in (0, \frac{N}{N-1}], \lambda, \mu > 0, q \in [0, N-1)$ and ψ is a positive Hölder continuous function on $\overline{\Omega}$. Here, $h(u)$ is a "suitable" perturbation of e^{u^α} as $u \to \infty$ (see assumptions **(A1)-(A5)** in Section 1). In this article, we show that there exists a region $\Re \subset \{(\mu,\lambda) : \mu, \lambda > 0\}$ bounded by the graph of a map Λ such that $(P_{\mu,\lambda})$ admits at least two solutions for all $(\mu,\lambda) \in \Re$, at least one solution for any $(\mu,\lambda) \in \partial\Re$ and no solution for (μ,λ) outside $\overline{\Re}$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. In this work, we study the existence and multiplicity of weak solutions $u \in W^{1,N}(\Omega)$ of the following problem:

$$(P_{\mu,\lambda}) \qquad \begin{cases} \left.\begin{array}{rcl} -\mathrm{div}(|\nabla u|^{N-2}\nabla u) + |u|^{N-2}u &=& \mu h(u)e^{u^\alpha} \\ u &>& 0 \end{array}\right\} \text{ in } \Omega, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial \nu} = \lambda \psi u^q \quad \text{on } \partial\Omega, \end{cases}$$

Accepted for publication: June 2009.
AMS Subject Classifications: 34B15, 35J60.

where the nonlinearities satisfy the following assumptions:

(A1) $\alpha \in (0, \frac{N}{N-1}], \lambda, \mu > 0, q \in [0, N-1)$, and ψ is a positive Hölder continuous function on $\overline{\Omega}$;

(A2) $h : \mathbb{R} \to [0, \infty)$ is a C^1 odd map with $h(t) = 0$ if and only if $t = 0$ and $h(t) \sim t^k$ as $t \to 0$ for some $k \geq 1$;

(A3) the map $t \mapsto h(t)$ is nondecreasing for all large $t > 0$;

(A4) $\liminf_{t\to\infty} h(t)e^{\epsilon|t|^{N/N-1}} = \infty$, $\limsup_{t\to\infty} h(t)e^{-\epsilon|t|^{N/N-1}} = 0$, for all $\epsilon > 0$; and

(A5) if $\alpha = \frac{N}{N-1}$, we require that $\liminf_{t\to\infty} h(t)te^{\epsilon t^{\frac{1}{N-1}}} = \infty$, for all $\epsilon > 0$.

Let $g(u) = h(u)e^{|u|^\alpha}$ and $G(u) = \int_0^u g(s)ds$. Associated to the problem $(P_{\mu,\lambda})$ we have the energy functional $J_{\mu,\lambda} : W^{1,N}(\Omega) \to \mathbb{R}$ given by

$$J_{\mu,\lambda}(u) = \frac{1}{N}\int_\Omega (|\nabla u|^N + |u|^N) - \mu \int_\Omega G(u) - \frac{\lambda}{q+1}\int_{\partial\Omega} \psi |u|^{q+1}. \tag{1.1}$$

Clearly, any positive critical point of $J_{\mu,\lambda}$ is a weak solution of $(P_{\mu,\lambda})$. Also, thanks to **(A2)**, we see that $J_{\mu,\lambda}$ is an even functional. First, we prove the following simple facts about a critical point of $J_{\mu,\lambda}$.

Lemma 1.1. *Let $u_{\mu,\lambda}$ be a critical point of $J_{\mu,\lambda}$. Then $u_{\mu,\lambda} \in C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ and, if $u_{\mu,\lambda} \geq 0$, then in fact $u_{\mu,\lambda} > 0$ in $\overline{\Omega}$.*

Proof. By following the arguments in Lemma 7.2 (by taking $u_{\mu,\lambda} = 0, \mu_\epsilon = 0$), we obtain that any solution of $(P_{\mu,\lambda})$ is bounded and by Theorem 2 of [16], $u_{\mu,\lambda} \in C^{1,\alpha}(\overline{\Omega})$. If $u_{\mu,\lambda} \geq 0$, it follows from the strong maximum principle (see [29]) that $u_{\mu,\lambda} > 0$ in Ω. Supposing that $u_{\mu,\lambda}(x_0) = 0$ for some $x_0 \in \partial\Omega$, then by the Hopf lemma (again, [29]) we have $\frac{\partial u_{\mu,\lambda}}{\partial \nu}(x_0) < 0$, which contradicts the boundary condition. Hence, $u_{\mu,\lambda} > 0$ in $\overline{\Omega}$. □

The exponential nature of our nonlinearity g is motivated by the following Moser-Trudinger-type imbedding theorem [2]:

$$\exists C > 0 : \sup_{\|u\|_{W^{1,N}(\Omega)} \leq 1} \int_\Omega e^{\frac{\alpha_N}{2}|u|^{\frac{N}{N-1}}} \leq C, \quad (\alpha_N = N w_N^{1/N-1}, \;\; w_N = |S^{N-1}|). \tag{1.2}$$

The above imbedding immediately implies that the nonlinear map

$$W^{1,N}(\Omega) \ni u \mapsto e^{u^\alpha} \in L^1(\Omega) \tag{1.3}$$

is a continuous map for all $\alpha \in (0, \frac{N}{N-1}]$ and is compact if $\alpha \in (0, \frac{N}{N-1})$. The non-compactness of this imbedding for $\alpha = \frac{N}{N-1}$ can be shown by using a sequence of functions that are suitable truncations and dilations of $\log \frac{1}{|x|}$. These functions are commonly referred to as Moser functions in the literature (see [21] for details).

We prove the following global (in λ, μ) existence and multiplicity results up to the critical growth $\alpha = \frac{N}{N-1}$.

Theorem 1.2. *There exists a region $\Re \subset \{(\mu, \lambda) : \mu, \lambda > 0\}$ bounded by the graph of a map Λ such that $(P_{\mu,\lambda})$ admits a minimal solution $u_{\mu,\lambda}$.*

Theorem 1.3. *There exists a region $\Re \subset \{(\mu, \lambda) : \mu, \lambda > 0\}$ bounded by the graph of a map Λ such that $(P_{\mu,\lambda})$ admits at least two solutions for all $(\mu, \lambda) \in \Re$, at least one solution for any $(\mu, \lambda) \in \partial\Re$ and no solution for (μ, λ) outside $\overline{\Re}$.*

In fact, the first solution obtained in the above theorem will be a local minimum for $J_{\mu,\lambda}$ and the second one a solution of mountain-pass type about this local minimum.

At this point we briefly recall related results for the Laplacian operator with $N \geq 3$. We consider the following problem:

$$(\tilde{P}_\lambda) \qquad \begin{cases} -\Delta u + u &= u^\beta, \quad u > 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial \nu} &= \lambda \psi u^q \quad \text{ on } \partial\Omega. \end{cases}$$

Here, $1 < \beta \leq 2^* - 1 \triangleq \frac{N2}{N-2} - 1, 0 < q < 1$. In the work [12], Garcia-Peral-Rossi study the problem $(\tilde{P}_\lambda)$ and show the following full multiplicity result globally in λ.

Theorem 1.4. *There exists $\Lambda > 0$ such that $(\tilde{P}_\lambda)$ admits at least two solutions for all $\lambda \in (0, \Lambda)$, one solution when $\lambda = \Lambda$ and no solution if $\lambda > \Lambda$.*

A key differentiating technical fact that enables us to obtain the multiplicity result is that for a sequence $\{u_n\} \in W^{1,N}(\Omega)$ with a small norm in this space, the corresponding sequence $\{exp\{u_n^{\frac{N}{N-1}}\}$ is bounded in $L^r(\Omega)$ for larger values of $r > 1$. See Lemma 5.6 below. This enables us to get compactness of the semilinear part of the functional associated to (P_λ).

2. Existence of a local minimum for $J_{\mu,\lambda}$, $\lambda, \mu > 0$ small

In this section, we show that $J_{\mu,\lambda}$ possesses a local minimum in a small neighborhood of the origin in $W^{1,N}(\Omega)$.

Lemma 2.1. *There exists* $\lambda_0, \mu_0 > 0, R_0 \in (0, \frac{\alpha_N}{2})$, *and* $\delta > 0$ *such that* $J_{\mu,\lambda}(u) \geq \delta$ *for all* $\|u\|_{W^{1,N}(\Omega)} = R_0$ *and all* $(\lambda, \mu) \in \Re_{\lambda_0,\mu_0}$. *In fact we can take* $\mu_0 = \infty$ *if* k (*see* (**A2**)) *is bigger than* $N - 1$.

Proof. From assumptions **(A1)-(A4)** and simple pointwise estimates, we obtain that, for some $C > 0$,

$$\int_\Omega G(u) \leq C \int_\Omega |u|^{k+1} e^{2u^{\frac{N}{N-1}}} \leq C\|u\|^k_{L^{2k}(\Omega)} \Big(\int_\Omega e^{4\|u\|^{\frac{N}{N-1}} (\frac{|u|}{\|u\|_{1,N}})^{\frac{N}{N-1}}} \Big)^{1/2}.$$

Now choose $R_0 > 0$ such that $4R_0^{\frac{N}{N-1}} \leq \alpha_N$. Then, by (1.2) and a Sobolev imbedding, the last inequality gives, for some $C_1 > 0$,

$$\int_\Omega G(u) \leq C_1 \|u\|^{k+1}_{1,N}, \quad \forall \quad \|u\|_{1,N} \leq R_0. \tag{2.1}$$

Also, by Hölder's inequality and the trace imbedding $W^{1,N}(\Omega) \hookrightarrow L^N(\partial\Omega)$ we get, for some $C_2 > 0$,

$$\int_{\partial\Omega} \psi |u|^{q+1} \leq C_2 \|\psi\|_{L^\infty(\partial\Omega)} \|u\|^{q+1}_{L^{q+1}(\partial\Omega)} \leq C_2 \|u\|^{q+1}. \tag{2.2}$$

Thus, from (2.1) and (2.2) we have, for any $R_0^{N/(N-1)} \in (0, \frac{\alpha_N}{2})$,

$$J_{\mu,\lambda}(u) \geq \frac{1}{N} \|u\|^N_{1,N} - \mu C_1 \|u\|^{k+1}_{1,N} - \lambda C_2 \|u\|^{q+1}_{1,N}, \quad \forall \quad \|u\|_{1,N} = R_0. \tag{2.3}$$

Now we may fix R_0 as above and choose $\lambda_0, \mu_0 > 0$ small enough so that $\frac{1}{N} R_0^N - \mu C_1 R_0^{k+1} - \lambda C_2 R_0^{q+1} > 0$ for all $(\lambda, \mu) \in \Re_{\lambda_0,\mu_0}$. If $k > N - 1$ we see that by choosing R_0 in the range specified above but small enough we can achieve the same inequality for $\lambda_0 > 0$ small enough but any $\mu > 0$ and hence we can choose $\mu_0 = \infty$ in this case. With these choices of R_0, λ_0, μ_0, from (2.3), we get the conclusion of the lemma. □

Lemma 2.2. *Let* λ_0, μ_0 *be as in Lemma* 2.1. *Then* $J_{\mu,\lambda}$ *possesses a local minimum* (*which is positive in* $\overline{\Omega}$) *close to the origin in* $W^{1,N}(\Omega)$ *for all* $(\mu, \lambda) \in (0, \mu_0) \times (0, \lambda_0)$.

Proof. Let R_0, λ_0, μ_0 and δ be as in Lemma 2.1. Let $(\mu, \lambda) \in (0, \mu_0) \times (0, \lambda_0)$. We note that $J_{\mu,\lambda}(tu) < 0$ for $t > 0$ small enough and any $u \in W^{1,N}(\Omega)$. In particular, $\min_{\|u\|_{1,N} \leq R_0} J_{\mu,\lambda}(u) < 0$ and if this minimum is achieved at some $u_{\mu,\lambda}$ (clearly $u_{\mu,\lambda} \neq 0$), then from Lemma 2.1 necessarily $\|u_{\mu,\lambda}\|_{1,N} < R_0$ and hence $u_{\mu,\lambda}$ becomes a local minimum for $J_{\mu,\lambda}$. We show that indeed the above minimum is achieved. Let $\{u_n\} \subset \{\|u\|_{1,N} \leq R_0\}$ be a minimizing sequence and let $u_n \rightharpoonup u_{\mu,\lambda}$ in $W^{1,N}(\Omega)$. Clearly,

$$\|u_{\mu,\lambda}\|_{1,N} \leq \liminf_{n\to\infty} \|u_n\|_{1,N}.$$

Since $R_0 \in (0, \frac{\alpha_N}{2})$, by (1.2) and Vitali's convergence theorem we obtain

$$\int_\Omega G(u_n) \to \int_\Omega G(u_{\mu,\lambda}).$$

Also, by compactness of the trace imbedding $W^{1,N}(\Omega) \hookrightarrow L^N(\partial\Omega)$ we obtain that

$$\int_{\partial\Omega} \psi u_n^{q+1} \to \int_{\partial\Omega} \psi u_{\mu,\lambda}^{q+1}.$$

From these facts it is clear that $u_{\mu,\lambda}$ is a minimizer for $J_{\mu,\lambda}$ in $\{\|u\|_{1,N} \leq R_0\}$ and hence is a local minimum. Since $J_{\mu,\lambda}$ is an even functional we may in fact take $u_{\mu,\lambda} \geq 0$. Thanks to Lemma 1.1 we indeed have that $u_{\mu,\lambda} > 0$ in $\overline{\Omega}$. □

Definition 2.3. *Let* $\Pi = \{\mu > 0 : (P_{\mu,\lambda})$ has a solution for some $\lambda > 0\}$. Define $\Sigma = \sup \Pi$. *Finally, for every* $\mu \in \Pi$, *define* $\Lambda(\mu) = \sup\{\lambda > 0 : (P_{\mu,\lambda})$ has a solution$\}$.

Remark 2.4. Clearly, if $\Sigma < \infty$ we have $\Pi \subseteq (0, \Sigma]$.

Definition 2.5. *Define* $\Re = \{(\mu, \lambda) : \mu \in \Pi, 0 < \lambda < \Lambda(\mu)\}$.

Remark 2.6. From Lemmas 2.1 and 2.2, we see that there exist $\lambda_0, \mu_0 > 0$ such that $(0, \mu_0) \times (0, \lambda_0) \subset \Re$. If additionally $k > N - 1$, then $\Pi = (0, \infty)$ (and hence $\Sigma = \infty$) and we can conclude that $(0, \infty) \times (0, \lambda_0) \subset \Re$ for some $\lambda_0 > 0$. In fact, in subsequent sections we show that indeed $\Re$ is the domain of multiplicity required by Theorem 1.3.

3. $(P_{\mu,\lambda})$ ADMITS A SOLUTION FOR ALL $(\mu, \lambda) \in \Re$

In this section we describe the maximal region in the (μ, λ) plane for which $(P_{\mu,\lambda})$ admits a solution. First, we can easily show the following weak comparison results.

Lemma 3.1. *Let* $u, v \in W^{1,N}(\Omega)$ *be non-negative functions satisfying*

$$-\Delta_N u + u^{N-1} \geq -\Delta_N v + v^{N-1} \quad \text{in } \Omega,$$

$$|\nabla u|^{N-2} \frac{\partial u}{\partial \nu} \geq |\nabla v|^{N-2} \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega.$$

Then $u \geq v$ *in* $\overline{\Omega}$.

Proof. Taking $(u-v)^-$ as a test function in the equation and integrating by parts we get

$$0 \le \int_\Omega (|\nabla u|^{N-2}\nabla u - |\nabla v|^{N-2}\nabla v)\cdot\nabla(u-v)^- \\ + \int_\Omega (u^{N-1}-v^{N-1})(u-v)^- - \int_{\partial\Omega}(|\nabla u|^{N-2}\frac{\partial u}{\partial\nu} - |\nabla v|^{N-2}\frac{\partial v}{\partial\nu})(u-v)^-.$$

The conclusion follows in a standard way since the integral on $\partial\Omega$ is non-negative.

Lemma 3.2. *Let $\rho:\mathbb{R}\mapsto\mathbb{R}$ be such that $t^{1-N}\rho(t)$ is nonincreasing in t. Suppose $u,v\in C^{1,\theta}(\overline{\Omega})$, $0<\theta<1$, are two functions positive on $\overline{\Omega}$ such that*

$$\begin{aligned} -\Delta_N u + u^{N-1} &\ge 0 \quad \text{in } \Omega, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial\nu} &\ge \lambda\psi(x)\rho(u) \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -\Delta_N v + v^{N-1} &\le 0 \quad \text{in } \Omega, \\ |\nabla v|^{N-2}\frac{\partial v}{\partial\nu} &\le \lambda\psi(x)\rho(v) \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Then $u\ge v$ in $\overline{\Omega}$.

Proof. Consider the functions

$$w_1 = \frac{(v^N-u^N)^+}{u^{N-1}},\ \ w_2 = \frac{(v^N-u^N)^+}{v^{N-1}}.$$

Then $w_1,w_2\in W^{1,N}(\Omega)$. Define $E=\{x\in\overline{\Omega}: u(x)<v(x)\}$. Multiplying (3.1) by w_1 and (3.2) by w_2 and integrating by parts, we get

$$\int_{E\cap\Omega}|\nabla u|^{N-2}\nabla u.\nabla w_1 + \int_{E\cap\Omega}u^{N-1}w_1 \ge \int_{E\cap\partial\Omega}\psi(x)\rho(u)w_1, \tag{3.3}$$

$$\int_{E\cap\Omega}|\nabla v|^{N-2}\nabla v.\nabla w_2 + \int_{E\cap\Omega}v^{N-1}w_2 \le \int_{E\cap\partial\Omega}\psi(x)\rho(v)w_2. \tag{3.4}$$

Subtracting (3.3) from (3.4), we get

$$\int_{E\cap\Omega}|\nabla u|^{N-2}\nabla u.\nabla w_1 - |\nabla v|^{N-2}\nabla v.\nabla w_2 + \int_{E\cap\Omega}(u^{N-1}w_1 - v^{N-1}w_2) \\ \le \int_{E\cap\partial\Omega}\psi(x)\Big[\frac{\rho(u)}{u^{N-1}} - \frac{\rho(v)}{v^{N-1}}\Big](v^N-u^N).$$

Using in the above inequality arguments similar to those in [17] we get, for $N \geq 2$,

$$\text{the left-hand side } L.H.S. \geq \frac{1}{2^{N-1}-1} \int_{E\cap\Omega} \Big(\frac{1}{u^N + v^N}\Big)|u\nabla v - v\nabla u|^N \geq 0.$$

But the right-hand side $R.H.S. \leq 0$ since $\rho(t)t^{1-N}$ is non-increasing. Therefore, if E is non-empty, on each component E_i of E we may find constants $k_i > 0$ such that $u = k_i v$ on E_i. But $u = v$ on ∂E_i, which implies $u \equiv v$ on E_i, a contradiction to the definition of E. Therefore, $E = \varnothing$ which proves the theorem. □

Define $\theta \triangleq \min_{t>0} g(t)t^{1-N}$. In the following lemma we describe the set $\Re$ in more detail, in particular showing that $(P_{\mu,\lambda})$ has a solution for all $(\mu, \lambda) \in \Re$.

Lemma 3.3.

(i) *If* $k \leq N-1$, *then* $\Sigma < \frac{1}{\theta}$; *if* $k > N-1$, $\Sigma = \infty$.
(ii) $(0, \Sigma) \subseteq \Pi$.
(iii) $0 < \Lambda(\mu) < \infty$, *for any* $\mu \in (0, \Sigma)$.
(iv) *For all* $0 < \mu_0 < \Sigma$, $\sup_{[\mu_0,\Sigma)} \Lambda(\mu) < \infty$.
(v) $(P_{\mu,\lambda})$ *admits a solution for all* $(\mu, \lambda) \in \Re$.

Proof. *(i)* Let $u_{\mu,\lambda}$ be a solution of $(P_{\mu,\lambda})$. Recall from Lemma 1.1 that $u_{\mu,\lambda} > 0$ in $\overline{\Omega}$. Taking $\phi \equiv 1$ in Ω in the weak formulation of $(P_{\mu,\lambda})$ we get

$$\int_\Omega |u_{\mu,\lambda}|^{N-1} = \mu \int_\Omega g(u_{\mu,\lambda}) + \lambda \int_{\partial\Omega} \psi u_{\mu,\lambda}^q.$$

If $k \leq N-1$, thanks to (**A2**) and (**A4**) we have $\theta > 0$. Plugging this fact in the above inequality we immediately get that $\mu < \frac{1}{\theta}$.

(ii) Fix any $0 < \mu < \Sigma$ and choose $\mu < \overline{\mu} < \Sigma$ so that $(P_{\overline{\mu},\lambda})$ admits a solution $u_{\overline{\mu},\lambda}$ for some $\lambda > 0$. Then clearly, $u_{\overline{\mu},\lambda}$ is a super solution for $(P_{\mu,\lambda})$. And $\epsilon\phi_1$, where ϕ_1 is the eigenfunction corresponding to the principal eigenvalue of the problem

$$-\Delta_N u + u^{N-1} = 0 \quad \text{in } \Omega,$$
$$|\nabla u|^{N-2}\frac{\partial u}{\partial \nu} = \lambda u^{N-1} \quad \text{on } \partial\Omega,$$

is a subsolution to $(P_{\mu,\lambda})$. We may choose ϵ small so that $\epsilon\phi_1 \leq u_{\overline{\mu},\lambda}$ in $\overline{\Omega}$. Now, by monotone iteration and the above weak comparison results, we obtain a solution $u_{\mu,\lambda}$ of $(P_{\mu,\lambda})$. Hence, $\mu \in \Pi$.

$(iii)-(iv)$ From Definition 2.3 we obtain that $\Lambda(\mu) > 0$ for all $\mu \in \Pi$. Due to the exponential growth of $g(t)$ as $t \to \infty$, the above equation implies that $u_{\mu,\lambda}$ is uniformly bounded in $L^{N-1}(\Omega)$ with respect to $\lambda > 0$ and $\mu \in [\mu_0, \infty) \cap \Pi$ for any fixed $\mu_0 > 0$. Now, taking $\phi = u_{\mu,\lambda}^{-q}$ in the weak formulation of $(P_{\mu,\lambda})$ again, we get

$$-q\int_\Omega u_{\mu,\lambda}^{-1-q}|\nabla u_{\mu,\lambda}|^N + \int_\Omega u_{\mu,\lambda}^{N-1-q} = \mu\int_\Omega u_{\mu,\lambda}^{-q} g(u_{\mu,\lambda}) + \lambda\int_{\partial\Omega}\psi.$$

From the last equation,

$$\int_{\Omega\cap\{u_{\mu,\lambda}\geq 1\}} u_{\mu,\lambda}^{N-1} + \int_{\Omega\cap\{u_{\mu,\lambda}<1\}} u_{\mu,\lambda}^{N-1-q} \geq \lambda\int_{\partial\Omega}\psi.$$

Since $u_{\mu,\lambda}$ is uniformly bounded in $L^{N-1}(\Omega)$ with respect to $\lambda > 0, \mu \in [\mu_o, \infty) \cap \Pi$ and $N-1-q > 0$, the above inequality implies that $\Lambda(\mu)$ is finite for all $\mu > 0$ and $\sup_{\mu\in[\mu_0,\infty)\cap\Pi} \Lambda(\mu) < \infty$.

(v) Fix any $(\mu, \lambda) \in \Re$. Choose $\overline{\lambda}$ such that $\lambda < \overline{\lambda} < \Lambda(\mu)$ and $(P_{\mu,\overline{\lambda}})$ admits a solution $u_{\mu,\overline{\lambda}}$. Then clearly, $u_{\mu,\overline{\lambda}}$ is a super solution for $(P_{\mu,\lambda})$. Following similar arguments as in case (ii) above, we obtain that $(P_{\mu,\lambda})$ has a solution for any $(\mu, \lambda) \in \Re$. □

4. $J_{\mu,\lambda}$ HAS A LOCAL MINIMUM FOR ALL $(\mu, \lambda) \in \Re$

Lemma 4.1. *There exists a minimal solution for $(P_{\mu,\lambda})$ for all $(\mu, \lambda) \in \Re$.*

Proof. Fix $(\mu, \lambda) \in \Re$. Choose $\tilde{\lambda} \in (\lambda, \Lambda(\mu))$ such that $(P_{\mu,\tilde{\lambda}})$ has a solution, say $u_{\mu,\tilde{\lambda}}$. Let v_λ denote the unique solution of

$$\left.\begin{array}{rcl} -\Delta_N u + |u|^{N-2}u &=& 0 \\ u &>& 0 \end{array}\right\} \text{ in } \Omega, \qquad |\nabla u|^{N-2}\frac{\partial u}{\partial\nu} = \lambda\psi u^q \text{ on } \partial\Omega. \tag{4.1}$$

Clearly, $u_{\mu,\tilde{\lambda}}$ is a super solution of (4.1) and hence, by Lemma 3.2, $v_\lambda \leq u_{\mu,\tilde{\lambda}}$ in $\overline{\Omega}$. Define the following monotone iteration:

$$\begin{aligned} u_1 &= v_\lambda, \\ -\Delta_N u_{n+1} + u_{n+1}^{N-1} &= \mu g(x, u_n) \quad \text{in } \Omega, \\ |\nabla u_{n+1}|^{N-2}\frac{\partial u_{n+1}}{\partial\nu} &= \lambda\psi u_n^q \text{ on } \partial\Omega \quad \text{for } n = 1, 2, 3, ... \end{aligned}$$

By the comparison theorem in Lemma 3.1, we get that the sequence $\{u_n\}$ is monotone; i.e., $v_\lambda \equiv u_1 \leq u_2 \leq ... \leq u_n \leq u_{n+1} \leq .... \leq u_{\mu,\tilde{\lambda}}$. By standard monotonicity arguments we obtain a solution $u_{\mu,\lambda}$ of $(P_{\mu,\lambda})$ which will be also the minimal solution. □

Lemma 4.2. *Let* $(\mu,\lambda) \in \Re$ *and* $u_{\mu,\lambda}$ *be the minimal solution obtained in the last lemma. Then* $u_{\mu,\lambda}$ *is a local minimum for* $J_{\mu,\lambda}$ *restricted to the space* $C^1(\overline{\Omega})$.

Proof. Choose $\lambda_1, \lambda_2 \in (0, \Lambda(\mu))$ such that $\lambda_1 < \lambda < \lambda_2$ and let $u_{\mu,\lambda_1}, u_{\mu,\lambda_2}$ be the corresponding minimal solutions of $(P_{\mu,\lambda_1}), (P_{\mu,\lambda_2})$. Then, by the definition of minimal solutions, $u_{\mu,\lambda_1} \le u_{\mu,\lambda_2}$ in $\overline{\Omega}$.

Claim: $\mathcal{Z} = \{x \in \Omega : u_{\mu,\lambda_1}(x) = u_{\mu,\lambda_2}(x)\}$ is an empty set.

Proof of Claim. First, we show that $\mathcal{Z}$ is a compact subset of Ω. Suppose not. Then there exists a sequence $\{x_n\} \subset \mathcal{Z}$ such that $x_n \to x_0 \in \partial\Omega$. Since each x_n is a global minimum for $u_{\mu,\lambda_2} - u_{\mu,\lambda_1}$ we get that $u_{\mu,\lambda_1}(x_n) = u_{\mu,\lambda_2}(x_n)$ and $\nabla u_{\mu,\lambda_1}(x_n) = \nabla u_{\mu,\lambda_2}(x_n)$. Since the two functions are in $C^1(\overline{\Omega})$, letting $n \to \infty$ in the last two equalities, we get a contradiction to the boundary condition satisfied by $u_{\mu,\lambda_1}, u_{\mu,\lambda_2}$. Thus, we can find a subdomain $\Omega' \subset \Omega$ such that $u_{\mu,\lambda_1} < u_{\mu,\lambda_2}$ on the boundary of Ω'. Now by standard strong comparison theorems (see [27]), we get that $\mathcal{Z}$ is empty. This proves the claim.

Hence, $u_{\mu,\lambda_1} < u_{\mu,\lambda_2}$ in Ω. Define the following cut-off functions:

$$(x \in \Omega, t \in \mathbb{R}) \quad \overline{g}(x,t) = \begin{cases} g(v_\lambda(x)) & t < v_\lambda(x), \\ g(t) & v_\lambda(x) \le t \le u_{\mu,\lambda_2}(x), \\ g(u_{\mu,\lambda_2}(x)) & t > u_{\mu,\lambda_2}(x), \end{cases}$$

$$(x \in \partial\Omega, t \in \mathbb{R}) \quad \overline{h}(x,t) = \begin{cases} \psi(x) v_\lambda^q & t < v_\lambda(x), \\ \psi(x) t^q & v_\lambda(x) \le t \le u_{\mu,\lambda_2}(x), \\ \psi(x) u_{\mu,\lambda_2}^q(x) & t > u_{\mu,\lambda_2}(x). \end{cases}$$

Let

$$\overline{G}(x,u) = \int_0^u \overline{g}(x,t)\,dt \qquad (x \in \Omega),$$
$$\overline{H}(x,u) = \int_0^u \overline{h}(x,t)\,dt \qquad (x \in \partial\Omega).$$

Then the functional $I : W^{1,N}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{N} \int_\Omega (|\nabla u|^N + |u|^N) - \mu \int_\Omega \overline{G}(x,u) - \lambda \int_{\partial\Omega} \overline{H}(x,u)$$

is coercive and bounded below. Let $u_{\mu,\lambda}$ denote the global minimum of I on $W^{1,N}(\Omega)$. Clearly $u_{\mu,\lambda}$ solves the equation

$$\left.\begin{array}{rcl} -\Delta_N u_{\mu,\lambda} + u_{\mu,\lambda}^{N-1} & = & \mu\overline{g}(x,u_{\mu,\lambda}) \\ u_{\mu,\lambda} & > & 0 \end{array}\right\} \text{ in } \Omega,$$
$$|\nabla u_{\mu,\lambda}|^{N-2}\frac{\partial u_{\mu,\lambda}}{\partial \nu} = \lambda\overline{h}(x,u_{\mu,\lambda}) \text{ on } \partial\Omega.$$

Hence, by regularity (see Lemma 1.1) we have $u_{\mu,\lambda} \in C^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$. Since $u_{\mu,\lambda}$ is a super solution for (4.1), $\overline{g}(x,u_{\mu,\lambda}(x)) \le g(u_{\mu,\lambda_2}(x))$ for all $x \in \Omega$ and $\overline{h}(x,u_{\mu,\lambda}(x)) \le \psi(x)u^q_{\mu,\lambda_2}(x)$ for all $x \in \partial\Omega$, using Lemma 3.1, we get that $u_{\mu,\lambda_1} \le u_{\mu,\lambda} \le u_{\mu,\lambda_2}$ in $\overline{\Omega}$. Now, as in the above claim, we can show that

$$u_{\mu,\lambda_1} < u_{\mu,\lambda} < u_{\mu,\lambda_2} \quad \text{in} \quad \Omega. \tag{4.2}$$

Let

$$\delta = \inf_{x\in\overline{\Omega}}\{|(u_{\mu,\lambda_2} - u_{\mu,\lambda_1})(x)| + |(\nabla u_{\mu,\lambda_2} - \nabla u_{\mu,\lambda_1})(x)|\}.$$

We claim that $\delta > 0$. Indeed, if $\delta = 0$, thanks to (4.2) there exists at least one $x_0 \in \partial\Omega$ where $u_{\mu,\lambda_1}(x_0) = u_{\mu,\lambda_2}(x_0)$ and $\nabla u_{\mu,\lambda_1}(x_0) = \nabla u_{\mu,\lambda_2}(x_0)$ which again gives a contradiction to the boundary data of $u_{\mu,\lambda_1}, u_{\mu,\lambda_2}$. Hence, if

$$u \in B \triangleq \{u \in C^1(\overline{\Omega}) | \|u - u_{\mu,\lambda}\|_{C^1(\overline{\Omega})} < \delta/2\},$$

then we have that $u_{\mu,\lambda_1} < u < u_{\mu,\lambda_2}$ in Ω and hence $\overline{g}_\lambda(x,u(x)) = g_\lambda(u(x))$ for all $x \in \Omega$. Hence, $J_{\mu,\lambda} = I$ on B. Hence, $u_{\mu,\lambda}$ is a local minimum for $J_{\mu,\lambda}$ restricted to the space $C^1(\overline{\Omega})$.

Lemma 4.3. *Let $(\mu,\lambda) \in \Re$ and $u_{\mu,\lambda}$ be the minimal solution of $(P_{\mu,\lambda})$ given by Lemma* 4.1. *Then $u_{\mu,\lambda}$ is a local minimum for $J_{\mu,\lambda}$ in $W^{1,N}(\Omega)$.*

The proof of this lemma follows from Theorem 1 in [13].

5. Existence of a mountain-pass type solution

Throughout this section, we fix $(\mu,\lambda) \in \Re$ and let $u_{\mu,\lambda}$ denote the minimal solution for $J_{\mu,\lambda}$ obtained in Lemma 4.3. We also know that $u_{\mu,\lambda}$ is also a local minimum for $J_{\mu,\lambda}$. Fix $0 < \tilde{\lambda} < \lambda$ and let $u_{\mu,\tilde{\lambda}}$ be the corresponding minimal solution. Define $\tilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{g}(x,s) = \begin{cases} g(s) & s > u_{\mu,\tilde{\lambda}}(x), \\ g(u_{\mu,\lambda}(x)) & s \le u_{\mu,\tilde{\lambda}}(x), \end{cases}$$

and $\tilde{h} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{h}(x,s) = \begin{cases} \psi(x)s^q & s > u_{\mu,\tilde{\lambda}}(x), \\ \psi(x)u^q_{\mu,\tilde{\lambda}}(x) & s \le u_{\mu,\tilde{\lambda}}(x). \end{cases}$$

Let

$$\tilde{G}(x,s) = \int_0^s \tilde{g}(x,t)dt, \qquad \tilde{H}(x,s) = \int_0^s \tilde{h}(x,t)dt.$$

Define now the functional $\tilde{J}_{\mu,\lambda} : W^{1,N}(\Omega) \to \mathbb{R}$ by

$$\tilde{J}_{\mu,\lambda}(v) = \frac{1}{N}\int_\Omega (|\nabla v|^N + |v|^N) - \mu \int_\Omega \tilde{G}(x,v)dx - \lambda \int_{\partial\Omega} \tilde{H}(x,v)dx. \quad (5.1)$$

Then, as in the previous section, we can check that if $\|v - u_{\mu,\lambda}\|_{C^1(\overline{\Omega})}$ is small enough then $\tilde{J}_{\mu,\lambda}(v) = J_{\mu,\lambda}(v)$ and, hence, as in that section we can conclude that $u_{\mu,\lambda}$ is a local minimum for $\tilde{J}_{\mu,\lambda}$ in $W^{1,N}_0(\Omega)$. Thanks to Lemma 3.1, any critical point $v_{\mu,\lambda} \ge 0$ of $\tilde{J}_{\mu,\lambda}$ solves $(P_{\mu,\lambda})$. Hence, to show multiplicity, it is enough to find a non-negative function $v_{\mu,\lambda}$ different from $u_{\mu,\lambda}$ which is a critical point of $\tilde{J}_{\mu,\lambda}$. This we can do by using a generalized version of a mountain-pass theorem about the local minimum $u_{\mu,\lambda}$.

First,we give the following generalized definition of a Palais-Smale sequence around a closed set.

Definition 5.1. *Let $F \subset W^{1,N}(\Omega)$ be a closed set. We say that a sequence $\{v_n\} \subset W^{1,N}(\Omega)$ is a Palais-Smale sequence for $\tilde{J}_{\mu,\lambda}$ at the level ρ around F (a $(P.S)_{F,\rho}$ sequence, for short) if*

$$\lim_{n\to\infty} dist(v_n, F) = 0, \lim_{n\to\infty} \tilde{J}_{\mu,\lambda}(v_n) = \rho; \textit{and} \lim_{n\to\infty} \|\tilde{J}'_{\mu,\lambda}(v_n)\|_{(W^{1,N}(\Omega))^*} = 0.$$

Remark 5.2. Note that when $F = W^{1,N}(\Omega)$, the above definition reduces to the usual definition of a Palais-Smale sequence at the level ρ.

We can show the following "compactness result:"

Lemma 5.3. *Let $F \subset W^{1,N}(\Omega)$ be a closed set, $\rho \in \mathbb{R}$. Let $\{v_n\} \subset W^{1,N}(\Omega)$ be a $(P.S)_{F,\rho}$ sequence for $\tilde{J}_{\mu,\lambda}$. Then (up to a subsequence), $v_n \rightharpoonup v_0$ in $W^{1,N}(\Omega)$,*

$$\lim_{n\to\infty} \int_\Omega \tilde{g}(x,v_n) = \int_\Omega \tilde{g}(x,v_0), \ \lim_{n\to\infty} \int_{\partial\Omega} \tilde{h}(x,v_n) = \int_{\partial\Omega} \tilde{h}(x,v_0),$$

$$\lim_{n\to\infty} \int_\Omega \tilde{G}(x,v_n) = \int_\Omega \tilde{G}(x,v_0), \ \lim_{n\to\infty} \int_{\partial\Omega} \tilde{H}(x,v_n) = \int_{\partial\Omega} \tilde{H}(x,v_0).$$

Proof. Since $\{v_n\}$ is a $(P.S)_{F,\rho}$ sequence for $\tilde{J}_{\mu,\lambda}$, we have the following relations as $n \to \infty$:

$$\frac{1}{N}\int_\Omega (|\nabla v_n|^N + v_n^N) - \int_\Omega \tilde{G}(x,v_n) - \int_{\partial\Omega} \tilde{H}(x,v_n) = \rho + o_n(1), \tag{5.2}$$

$$\Big|\int_\Omega |\nabla v|^{N-2}\nabla v_n.\nabla\phi + \int_\Omega |v_n|^{N-2}v_n\phi - \int_\Omega \tilde{g}(x,v_n)\phi - \int_{\partial\Omega}\tilde{h}(x,v_n)\phi\Big| \\ \le o(1)\|\phi\|_{1,N}, \tag{5.3}$$

for all $\phi \in W^{1,N}(\Omega)$.

Step 1:

$$\sup_n \|v_n\|_{1,N} < \infty, \quad \sup_n \int_\Omega \tilde{g}(x,v_n)v_n < \infty, \quad \sup_n \int_{\partial\Omega}\tilde{h}(x,v_n)v_n < \infty.$$

From **(A2)**, given $\epsilon > 0$, there exists $s_\epsilon > 0$ such that $\tilde{G}(x,s) \le \epsilon\tilde{g}(x,s)s$, for all $s \ge s_\epsilon$. Using (5.2) together with this relation, we get

$$\begin{aligned}\frac{1}{N}\|v_n\|_{1,N}^N &\le \int_{\Omega\cap\{v_n\le s_\epsilon\}} \tilde{G}(x,v_n) + \epsilon\int_\Omega \tilde{g}(x,v_n)v_n + \int_{\partial\Omega}\tilde{h}(x,v_n)v_n + \rho + o_n(1) \\ &\le C_\epsilon + \epsilon\int_\Omega \tilde{g}(x,v_n)v_n + \int_{\partial\Omega}\tilde{h}(x,v_n)v_n + \rho + o_n(1).\end{aligned} \tag{5.4}$$

From (5.3) with $\phi = v_n$ we obtain

$$\int_\Omega \tilde{g}(x,v_n)v_n + \int_{\partial\Omega}\tilde{h}(x,v_n)v_n \le \|v_n\|_{1,N}^N + o_n(1)\|v_n\|_{1,N}. \tag{5.5}$$

From the definition of $\tilde{h}$ and trace imbedding we have

$$\int_{\partial\Omega}\tilde{h}(x,v_n)v_n \le O_n(1)\|v_n\|_{L^{q+1}(\partial\Omega)}^{q+1} \le O_n(1)\|v_n\|_{1,N}^{q+1}.$$

Hence, plugging the last two inequalities into (5.4) we get

$$(\frac{1}{N} - \epsilon)\|v_n\|_{1,N}^N - O_n(1)\|v_n\|_{1,N}^{q+1} + \epsilon o_n(1)\|v_n\|_{1,N} \le C_\epsilon + \rho + o_n(1).$$

This shows that $\sup_n \|v_n\|_{1,N} < \infty$ and hence by (5.5) the claim in Step 1 follows.

Since $\{v_n\} \subset W^{1,N}(\Omega)$ is bounded, up to a subsequence, $v_n \rightharpoonup v_0$ in $W^{1,N}(\Omega)$ for some $v_0 \in W^{1,N}(\Omega)$.

Step 2:

$$\lim_{n\to\infty}\int_\Omega \tilde{g}(x,v_n) = \int_\Omega \tilde{g}(x,v_0), \ \lim_{n\to\infty}\int_{\partial\Omega}\tilde{h}(x,v_n) = \int_{\partial\Omega}\tilde{h}(x,v_0), \tag{5.6}$$

$$\lim_{n\to\infty}\int_\Omega \tilde G(x,v_n)=\int_\Omega \tilde G(x,v_0),\ \lim_{n\to\infty}\int_{\partial\Omega}\tilde H(x,v_n)=\int_{\partial\Omega}\tilde H(x,v_0). \quad (5.7)$$

Let $|A|_N$ denote the N-dimensional Hausdorff measure of a set $A\subset\Omega$ and $|B|_{N-1}$ denote the $(N-1)$-dimensional Hausdorff measure of a set $B\subset\partial\Omega$. We first show that $\{\tilde g(.,v_n)\}$ and $\{\tilde h(.,v_n)\}$ are equi-integrable families in $L^1(\Omega)$ and $L^1(\partial\Omega)$ respectively; i.e., given $\epsilon>0$, there exists a $\delta>0$ such that for any $A\subset\Omega, B\subset\partial\Omega$, with $|A|_N+|B|_{N-1}<\delta$, we have

$$\sup_n\int_A|\tilde g(x,v_n)|+\int_B|\tilde h(x,v_n)|\le\epsilon.$$

Once this is shown, (5.6) follows from Vitali's convergence theorem. Relation (5.7) follows from (5.6) by noting that, for all $(x,s)\in\overline\Omega\times\mathbb R$, $|\tilde G(x,s)|\le C|\tilde g(x,s)|$, the trace imbedding $W^{1,N}(\Omega)\hookrightarrow L^N(\partial\Omega)$ is compact, and appealing to Lebesgue's dominated convergence theorem and Vitali's convergence theorem once more.

Let

$$\tilde C=\sup_n\left(\int_\Omega\tilde g(x,v_n)v_n+\int_{\partial\Omega}\tilde h(x,v_n)v_n\right).$$

By Step 1, $\tilde C<\infty$. Given $\epsilon>0$, define

$$\mu_\epsilon=\max_{x\in\overline\Omega,|s|\le\frac{4\tilde C}{\epsilon}}|\tilde g(x,s)s|+\max_{x\in\partial\Omega,|s|\le\frac{4\tilde C}{\epsilon}}|\tilde h(x,s)s|.$$

Then for any $A\subset\Omega, B\subset\partial\Omega$ with $|A|_N\le\frac{\epsilon}{4\mu_\epsilon}, |B|_{N-1}\le\frac{\epsilon}{4\mu_\epsilon}$ we get

$$\begin{aligned}&\int_A|\tilde g(x,v_n)|+\int_B|\tilde h(x,v_n)|\\ &\le\int_{A\cap\{|v_n|\ge\frac{4\tilde C}{\epsilon}\}}\frac{|\tilde g(x,v_n)v_n|}{|v_n|}+\int_{A\cap\{|v_n|\le\frac{4\tilde C}{\epsilon}\}}|\tilde g(x,v_n)|\\ &\quad+\int_{B\cap\{|v_n|\ge\frac{4\tilde C}{\epsilon}\}}\frac{|\tilde h(x,v_n)v_n|}{|v_n|}+\int_{B\cap\{|v_n|\le\frac{4\tilde C}{\epsilon}\}}|\tilde h(x,v_n)|\\ &\le\frac{\epsilon}{2}+\mu_\epsilon(|A|_N+|B|_{N-1})\le\epsilon.\end{aligned}$$

This finishes Step 2 and the proof of the lemma. □

We already noted that $u_{\mu,\lambda}$ is a local minimum for $\tilde J_{\mu,\lambda}$. Also,

$$\lim_{t\to\infty}\tilde J_{\mu,\lambda}(tv)=-\infty\quad\text{for any } v\in W^{1,N}(\Omega)\backslash\{0\}.$$

Hence, we may fix $e \in W^{1,N}(\Omega)$ such that $\|e\|_{1,N} > \|u_{\mu,\lambda}\|_{1,N}$ and $\tilde{J}_{\mu,\lambda}(e) < \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})$. Let $\Gamma = \{\gamma : [0,1] \to W^{1,N}(\Omega) : \gamma$ is continuous, $\gamma(0) = u_{\mu,\lambda}$, $\gamma(1) = e\}$. We define the mountain-pass level

$$\rho_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \tilde{J}_{\mu,\lambda}(\gamma(t)).$$

It follows that $\rho_0 \geq \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})$. Let $R_0 = \|u_{\mu,\lambda}\|_{1,N}, R_1 = \|e\|_{1,N}$. If $\rho_0 = \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})$ we obtain that $\inf\{\tilde{J}_{\mu,\lambda}|\|v\|_{1,N} = R\} = \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})$ for all $R \in (R_0, R_1)$. We now let $F = W^{1,N}(\Omega)$ if $\rho_0 > \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})$ and $F = \{\|v\|_{1,N} = \frac{(R_0+R_1)}{2}\}$ if $\rho_0 = \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})$. Let us assume without loss of generality, that $0 \in \partial\Omega$. The next lemma is taken from [25] where we have replaced l by $\frac{1}{n}, n = 1, 2, 3, ...$

Lemma 5.4. *There exists $R > 0$ such that for any integer $n \geq \frac{1}{R}$ there exists a function $v_n \in W^{1,N}(\Omega)$ satisfying*

(i) $v_n \geq 0$, *support*$(v_n) \subset B_R(0) \cap \overline{\Omega}$;

(ii) $\|v_n\|_{1,N} = 1$,

$$\int_\Omega v_n^N = O_n((\log n)^{-1});$$

(iii) *for all* $x \in B_{\frac{1}{n}}(0), v_n$ *is a constant which satisfies*

$$v_n(x) = (\frac{2}{w_N})^{1/N}(\log(Rn))^{(N-1)/N} + O_n(1);$$

(iv)

$$\int_\Omega |\nabla v_n|^{N-1} + |v_n|^{N-1} = O_n((\log n)^{\frac{1-N}{N}}),$$

$$\int_\Omega |\nabla v_n|^2 + |v_n|^2 = O_n((\log n)^{-\frac{2}{N}}).$$

It is easy to show the above lemma by essentially taking v_n to be an appropriate truncated and dilated version of the fundamental solution $\log(|x|)$. We can now prove the following upper bound for ρ_0 when $\alpha = \frac{N}{N-1}$.

Lemma 5.5. *Let $\alpha = \frac{N}{N-1}$. Then, $\rho_0 < \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}) + \frac{\alpha_N^{N-1}}{2N}$.*

Proof. Let $\delta_n \to 0^+$ which will be chosen explicitly later. Set $\psi_n(x) = v_n(\frac{x}{\delta_n}), x \in B_{R\delta_n}(0) \cap \Omega$. It is clear from the last lemma that $\|v_n\|_{1,N} = 1 + o_n(1)$. We prove the lemma by a contradiction argument. Suppose

$$\rho_0 \geq \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}) + \frac{1}{2N}(\alpha_N)^{N-1}.$$

Then, by definition of ρ_0, for each n there exists $t_n > 0$ such that

$$\sup_{t>0} \tilde{J}_{\mu,\lambda}(t\psi_n + u_{\mu,\lambda}) = \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda} + t_n\psi_n) \geq \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}) + \frac{\alpha_N^{N-1}}{2N}. \tag{5.8}$$

From (5.8) we get that $\{t_n\}$ is a bounded sequence since otherwise

$$\tilde{J}_{\mu,\lambda}(t_n\psi_n + u_{\mu,\lambda}) \to -\infty.$$

We have the inequality

$$|\nabla(u_{\mu,\lambda} + t\psi_n)|^N = (|\nabla u_{\mu,\lambda}|^2 + 2t\nabla u_{\mu,\lambda}\nabla\psi_n + t^2|\nabla\psi_n|^2)^{N/2}.$$

Since $u_{\mu,\lambda} \in L^\infty$, from the above equation and the one-dimensional inequality

$$(1 + t^2 + 2t\cos\alpha)^{N/2} \leq 1 + t^N + Nt\cos\alpha + C(t^2 + t^{N-1}),$$

for $t \geq 0$, uniformly in α, we estimate $\tilde{J}_{\mu,\lambda}(t_n\psi_n + u_{\mu,\lambda})$:

$$\begin{aligned}
\tilde{J}_{\mu,\lambda}(t_n\psi_n + u_{\mu,\lambda}) &\leq \frac{t_n^N}{N} + \frac{1}{N}\int_\Omega (|\nabla u_{\mu,\lambda}|^N + |u_{\mu,\lambda}|^N) \\
&\quad + t_n \int_\Omega (|\nabla u_{\mu,\lambda}|^{N-2}\nabla u_{\mu,\lambda}\nabla\psi_n + u_{\mu,\lambda}^{N-2}u_{\mu,\lambda}\psi_n) + \\
&\quad + O\Big(t_n^2\int_\Omega (|\nabla\psi_n|^2 + |\psi_n|^2) + t_n^{N-1}\int_\Omega (|\nabla\psi_n|^{N-1} + |\psi_n|^{N-1})\Big) \\
&\quad - \mu\int_\Omega \tilde{G}(x, u_{\mu,\lambda} + t_n\psi_n) - \lambda\int_{\partial\Omega} \tilde{H}(x, u_{\mu,\lambda} + t_n\psi_n) \\
&= \frac{t_n^N}{N} + \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}) + O_n\left(t_n^2(\log n)^{-2/N} + t_n^{N-1}(\log n)^{-(N-1)/N}\right) \\
&\quad - \mu\int_\Omega [\tilde{G}(x, u_{\mu,\lambda} + t_n\psi_n) - \tilde{G}(x, u_{\mu,\lambda}) - t_n\tilde{g}(x, u_{\mu,\lambda})\psi_n] \\
&\quad - \lambda\int_{\partial\Omega} [\tilde{H}(x, u_{\mu,\lambda} + t_n\psi_n) - \tilde{H}(x, u_{\mu,\lambda}) - t_n\tilde{h}(x, u_{\mu,\lambda})\psi_n] \\
&\leq \frac{t_n^N}{N} + \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}) + t_n^2 O_n\left(\delta_n^{N-2}(\log n)^{-2/N}\right) + t_n^{N-1}O_n\left(\delta_n(\log n)^{\frac{1-N}{N}}\right).
\end{aligned}$$

In the above computations, we have used the fact that $\tilde{g}(x,.), \tilde{h}(x,.)$ are nondecreasing. Therefore, for some $C > 0$,

$$t_n^N \geq \frac{\alpha_N^{N-1}}{2} - C\left(\delta_n^{N-2}(\log n)^{-2/N} + \delta_n(\log n)^{-(N-1)/N}\right). \tag{5.9}$$

Now, since t_n is a point of maximum for the one-dimensional map $t \mapsto \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda} + t\psi_n)$, we get

$$\frac{d}{dt}\tilde{J}_{\mu,\lambda}(t\psi_n + u_{\mu,\lambda})\Big|_{t=t_n} = 0.$$

That is,

$$\int_\Omega |\nabla(u_{\mu,\lambda} + t_n\psi_n)|^{N-2}\nabla(u_{\mu,\lambda} + t_n\psi_n)\nabla\psi_n \tag{5.10}$$

$$+ \int_\Omega |u_{\mu,\lambda} + t_n\psi_n|^{N-2}(u_{\mu,\lambda} + t_n\psi_n)\psi_n$$

$$= \int_\Omega \tilde{g}_{\mu,\lambda}(x, u_{\mu,\lambda} + t_n\psi_n)\psi_n + \int_{\partial\Omega} \tilde{h}_{\mu,\lambda}(x, u_{\mu,\lambda} + t_n\psi_n)\psi_n. \tag{5.11}$$

We estimate the first term on the right-hand side of (5.10) from below. Let $B_n = \{x \in \Omega : |x| \le n^{-\frac{1}{2}}\delta_n\}$, $c_n = \min_{x\in B_n} u_{\mu,\lambda}(x)$. Then,

$$\int_\Omega \tilde{g}_{\mu,\lambda}(x, u_{\mu,\lambda} + t_n\psi_n)\psi_n \ge \int_{B_n} h(t_n\psi_n(0))\psi_n(0)e^{(c_n+t_n\psi_n(0))^{\frac{N}{N-1}}}$$

$$= h(t_n\psi_n(0))\psi_n(0)e^{(c_n+t_n\psi_n(0))^{\frac{N}{N-1}}}|B_n|$$

$$\ge C(n^{-\frac{1}{2}}\delta_n)^N h(t_n\psi_n(0))\psi_n(0)e^{(c_n+t_n\psi_n(0))^{\frac{N}{N-1}}} \tag{5.12}$$

for some constant $C > 0$. Using Taylor's expansion we estimate:

$$(c_n + t_n\psi_n(0))^{\frac{N}{N-1}} \ge (t_n\psi_n(0))^{\frac{N}{N-1}} + \frac{c_nN}{N-1}(t_n\psi_n(0))^{\frac{1}{N-1}}.$$

Now, using (5.9), the explicit value of $\psi_n(0)$ and the fact that $c_n \to u_{\mu,\lambda}(0)$ as $n \to \infty$, the above inequality becomes, for all large n,

$$(c + t_n\psi_n(0))^{\frac{N}{N-1}}$$
$$\ge \frac{N}{2}(1 - C(\delta_n^{N-2}(\log n)^{-2/N} + \delta_n(\log n)^{-(N-1)/N}))\log n + K_0(\log n)^{1/N},$$

for some $C, K_0 > 0$. Hence, from (5.12) and choosing $\delta_n = (\log n)^{-1/N}$ we get from the last inequality, for some $\eta, \epsilon > 0$,

$$\int_\Omega \tilde{g}_{\mu,\lambda}(x, u_{\mu,\lambda} + t_n\psi_n)\psi_n$$
$$\ge C\left(n^{-\frac{1}{2}}(\log n)^{-\frac{1}{N}}\right)^N h(t_n\psi_n(0))\psi_n(0)e^{\frac{N}{2}\log n + \eta(\log n)^{1/N}}$$
$$= C(\log n)^{-1}h(t_n\psi_n(0))t_n\psi_n(0)e^{\eta(\log n)^{1/N}} \ge Ch(t_n\psi_n(0))t_n\psi_n(0)e^{\epsilon t_n\psi_n(0)}.$$

By assumption **(A5)** we get that the right-hand side of (5.12) tends to ∞ as $n \to \infty$. It is easy to see that the left-hand side of (5.12) is bounded as $n \to \infty$. This gives the required contradiction and proves the lemma. □

Next, we prove the following form of Lions' theorem (Theorem xx in [18].)

Lemma 5.6. *Let $\{u_n\} \subset W^{1,N}(\Omega)$ be a sequence such that $\|u_n\| = 1$, for all n and $u_n \rightharpoonup u$ in $W^{1,N}(\Omega)$ for some $u \not\equiv 0$. Furthermore, assume that $-\Delta u_n = f_n + g_n$ in Ω where $f_n \to f$ in $L^1(\Omega), h_n \to 0$ in $W^{-1,\frac{N}{N-1}}(\Omega)$. Then, for any $1 < p < (1 - \|u\|)^{-\frac{1}{(N-1)}}$,*

$$\sup_{n\geq 1} \int_\Omega e^{p\frac{\alpha_N}{2}|u|^{\frac{N}{N-1}}} < \infty. \tag{5.13}$$

Proof. Under the above hypothesis, it follows from the work of Boccardo-Murat ([3]) that $\nabla u_n \to \nabla u$ pointwise almost everywhere in Ω. Clearly, $u_n \to u$ pointwise almost everywhere on Ω. Therefore,

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^N - \int_\Omega |\nabla(u_n - u)|^N \\
&= \int_\Omega \Big(\int_0^1 \frac{d}{dt}[|t\nabla u_n + (1-t)\nabla(u_n - u)|^N]dt \Big)\, dx \\
&= \int_\Omega N \Big[\int_0^1 |t\nabla u_n + (1-t)\nabla(u_n - u)|^{N-2} \\
&\qquad \times (t\nabla u_n + (1-t)\nabla(u_n-u)).\nabla u \Big] dt\, dx.
\end{aligned}$$

Since $\sup_{n\geq 1} \int_\Omega |\nabla u_n|^N dx < \infty$, by Vitali's convergence theorem, we obtain from the last equation,

$$\begin{aligned}
\int_\Omega |\nabla u_n|^N - \int_\Omega |\nabla(u_n - u)|^N &\to \Big(\int_\Omega |\nabla u|^N dx \Big) \Big(N \int_0^1 t^{N-1} dt \Big) \\
&= \int_\Omega |\nabla u|^N dx.
\end{aligned} \tag{5.14}$$

A similar argument gives that

$$\int_\Omega |u_n|^N - \int_\Omega |u_n - u|^N \to \int_\Omega |u|^N. \tag{5.15}$$

Hence, from (5.14) and (5.15) we get

$$\|u_n\|^N_{W^{1,N}(\Omega)} - \|u_n - u\|^N_{W^{1,N}(\Omega)} \to \|u\|^N_{W^{1,N}(\Omega)}.$$

Since $\|u\|_{W^{1,N}(\Omega)} = 1$, for all n, we get

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,N}} = (1 - \|u\|_{W^{1,N}})^{\frac{1}{N}}. \tag{5.16}$$

Using Taylor's theorem, we get

$$\begin{aligned} |u_n(x)|^{\frac{N}{N-1}} &\leq (|(u_n - u)(x)| + |u(x)|)^{\frac{N}{N-1}} \\ &= |(u_n - u)(x)|^{\frac{N}{N-1}} + (\frac{N}{N-1})(\xi(x))^{\frac{1}{N-1}}|u(x)|, \end{aligned} \tag{5.17}$$

where $\xi_n(x)$ lies between $|(u - u_n)(x)|$ and $|u(x)|$ for all $x \in \Omega$. Clearly, $|\xi(x)| \leq |u_n(x)| + |u(x)|$ for all $x \in \Omega$. Therefore, from (5.17) we obtain

$$\begin{aligned} &\int_\Omega e^{p(\frac{\alpha_N}{2})|u_n|^{\frac{N}{N-1}}} \\ &\leq \int_\Omega e^{p(\frac{\alpha_N}{2})|u_n-u|^{\frac{N}{N-1}}} e^{(\frac{pN}{N-1})(\frac{\alpha_N}{2})|u_n|^{\frac{1}{N-1}}|u|} e^{(\frac{pN}{N-1})(\frac{\alpha_N}{2})|u|^{\frac{N}{N-1}}} dx. \end{aligned} \tag{5.18}$$

We note that for small $\epsilon > 0$ we can write

$$|u_n|^{\frac{1}{N-1}}|u| \leq \frac{\epsilon^N}{N}|u_n|^{\frac{N}{N-1}} + \Big(\frac{N}{N-1}\Big)\epsilon^{-(\frac{N}{N-1})}|u|^{\frac{N}{N-1}},$$

which gives that

$$\sup_{n\geq 1} \int_\Omega e^{(\frac{pN}{N-1})(\frac{\alpha_N}{2})|u_n|^{\frac{1}{N-1}}|u|} dx < \infty.$$

Applying a generalized Hölder inequality to the right-hand side of (5.18) and using the above observation we get, if $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$,

$$\int_\Omega e^{p\frac{\alpha_N}{2}|u_n|^{\frac{N}{N-1}}} \leq C\Big(\int_\Omega e^{(pr_1(\frac{\alpha_N}{2})\|u_n-u\|^{\frac{N}{N-1}}_{W^{1,N}(\Omega)})\Big(\frac{u_n-u}{\|u_n-u\|_{W^{1,N}(\Omega)}}\Big)^{\frac{N}{N-1}}} dx\Big)^{\frac{1}{r_1}}. \tag{5.19}$$

If now $p < (1 - \|u\|_{W^{1,N}})^{\frac{-1}{N-1}}$, by choosing $r_1 > 1$ close to 1 we can ensure that also $pr_1 < (1 - \|u\|_{W^{1,N}})^{\frac{-1}{N-1}}$. Using (5.16) we therefore obtain that $pr_1 < \frac{1}{\|u_n-u\|^{\frac{N}{N-1}}_{W^{1,N}(\Omega)}}$ for all large $n \geq 1$. That is, for all large n, $pr_1\|u_n - u\|^{\frac{N}{N-1}}_{W^{1,N}(\Omega)} < 1$. Thus, the right-hand side (and hence the left-hand side) of (5.19) is bounded independent of n by the Trudinger-Moser imbedding as long as $1 < p < (1 - \|u\|_{W^{1,N}})^{\frac{-1}{N-1}}$. □

We can now prove the following.

Lemma 5.7. *Let $(\mu,\lambda)\in\Re$ and $u_{\mu,\lambda}$ be the local minimum for $\tilde{J}_{\mu,\lambda}$ obtained in Lemma 5.3. Then there exists another solution $v_{\mu,\lambda}\in W^{1,N}(\Omega)$ of mountain-pass type.*

Proof. From Lemma 5.5, we know that the mountain-pass level $\rho_0<\tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})+\frac{\alpha_N^{N-1}}{2N}$. Let $\{v_n\}\subset W^{1,N}(\Omega)$ be a Palais-Smale sequence for $\tilde{J}_{\mu,\lambda}$ at the level ρ_0 around F (such a sequence always exists by the result in [14]). Then, by Lemma 5.3, there exists $v_{\mu,\lambda}$ such that $v_n\rightharpoonup v_{\mu,\lambda}$ in $W^{1,N}(\Omega)$,

$$\begin{aligned}
&\int_\Omega \tilde{g}_{\mu,\lambda}(x,v_n)\to\int_\Omega \tilde{g}_{\mu,\lambda}(x,v_{\mu,\lambda}),\\
&\int_\Omega \tilde{G}_{\mu,\lambda}(x,v_n)\to\int_\Omega \tilde{G}_{\mu,\lambda}(x,v_{\mu,\lambda}),\\
&\int_{\partial\Omega} \tilde{H}_{\mu,\lambda}(x,v_n)\to\int_{\partial\Omega} \tilde{H}_{\mu,\lambda}(x,v_{\mu,\lambda}).
\end{aligned}$$

It follows that $v_{\mu,\lambda}$ is a critical point of $\tilde{J}_{\mu,\lambda}$ and, as noted in the beginning of this section, this implies $v_{\mu,\lambda}$ solves $(P_{\mu,\lambda})$. It only remains to show that $v_{\mu,\lambda}\not\equiv u_{\mu,\lambda}$. We consider the following cases:

Case I: $\rho_0=\tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}), v_{\mu,\lambda}\equiv u_{\mu,\lambda}$. We recall that

$$F=\Big\{v\in W^{1,N}(\Omega):\|v-u_{\mu,\lambda}\|=\frac{R_0+R_1}{2}\Big\}.$$

Also,

$$\begin{aligned}
&\tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})=\rho_0=\lim_{n\to\infty}\tilde{J}_{\mu,\lambda}(v_n)\\
&=\lim_{n\to\infty}\frac{1}{N}\int_\Omega(|\nabla v_n|^N+|v_n|^N)-\mu\int_\Omega\tilde{G}_{\mu,\lambda}(x,v_{\mu,\lambda})-\lambda\int_{\partial\Omega}\tilde{H}_{\mu,\lambda}(x,v_{\mu,\lambda})\\
&=\lim_{n\to\infty}\frac{1}{N}\|v_n\|_{1,N}^N+\tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})-\frac{1}{N}\|u_{\mu,\lambda}\|_{1,N}^N.
\end{aligned}$$

Therefore, $v_n\to u_{\mu,\lambda}$ strongly in $W^{1,N}(\Omega)$, which gives a contradiction to the fact that $\{v_n\}$ is a $(PS)_{F,\rho_0}$ sequence.

Case II: $\rho_0>\tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}),\ v_{\mu,\lambda}\equiv u_{\mu,\lambda}$. First, we show that

$$\int_\Omega\tilde{g}_{\mu,\lambda}(x,v_n)v_n\to\int_\Omega\tilde{g}_{\mu,\lambda}(x,u_{\mu,\lambda})u_{\mu,\lambda}. \tag{5.20}$$

Since $\rho_0<\tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})+\frac{1}{2N}(\alpha_N)^{N-1}$, there exists $\epsilon>0$ small enough such that

$$0<(\rho_0-\tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}))(1+\epsilon)^{N-1}<\frac{1}{2N}(\alpha_N)^{N-1}. \tag{5.21}$$

Set

$$\beta_0 := \int_\Omega \tilde{G}_{\mu,\lambda}(x, u_{\mu,\lambda}) + \int_{\partial\Omega} \tilde{H}_{\mu,\lambda}(x, u_{\mu,\lambda}).$$

Then we have

$$\lim_{n\to\infty} \|v_n\|_{1,N}^N = N(\rho_0 + \lim_{n\to\infty} \mu \int_\Omega \tilde{G}_{\mu,\lambda}(x, v_n) + \lambda \int_{\partial\Omega} \tilde{H}_{\mu,\lambda}(x, v_n) = N(\rho_0 + \beta_0)). \tag{5.22}$$

Since $\frac{v_n}{\|v_n\|} \rightharpoonup \frac{u_{\mu,\lambda}}{(N(\rho_0+\beta_0))^{1/N}}$, choosing $1 < p < \left(1 - \frac{\|u_{\mu,\lambda}\|^N}{N(\rho_0+\beta_0)}\right)^{\frac{1}{1-N}}$, from Proposition 5.6 we get

$$\sup \int_\Omega \exp\left(p \frac{\alpha_N}{2} (\frac{v_n}{\|v_n\|})^{\frac{N}{N-1}}\right) < \infty.$$

From this it follows that $\sup_n \int_\Omega |\tilde{g}_{\mu,\lambda}(x, v_n) v_n|^q < \infty$ for some $q > 1$. Now Vitali's convergence theorem gives the conclusion in (5.20). We note that

$$\begin{aligned}
\rho_0 &= \lim_{n\to\infty} \left[\tilde{J}_{\mu,\lambda}(v_n) - \frac{1}{N} \langle \tilde{J}'_{\mu,\lambda}, (v_n), v_n \rangle\right] \\
&= \lim_{n\to\infty} \Big[\mu \int_\Omega (\tilde{G}_{\mu,\lambda}(x, v_n) - \frac{1}{N} \tilde{g}_{\mu,\lambda}(x, v_n)) \\
&\qquad + \lambda \int_{\partial\Omega} (\tilde{H}_{\mu,\lambda}(x, v_n) - \frac{1}{N} \tilde{h}_{\lambda,\mu}(x, v_n))\Big] \\
&= \Big[\mu \int_\Omega (\tilde{G}_{\mu,\lambda}(x, u_{\mu,\lambda}) - \frac{1}{N} \tilde{g}_{\mu,\lambda}(x, u_{\mu,\lambda})) \\
&\qquad + \lambda \int_{\partial\Omega} (\tilde{H}_{\mu,\lambda}(x, u_{\mu,\lambda}) - \frac{1}{N} \tilde{h}_{\mu,\lambda}(x, u_{\mu,\lambda}))\Big] \\
&= \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}) - \frac{1}{N} \langle \tilde{J}'_{\mu,\lambda}(u_{\mu,\lambda}), u_{\mu,\lambda} \rangle = \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda}).
\end{aligned}$$

This contradicts the assumption that $\rho_0 > \tilde{J}_{\mu,\lambda}(u_{\mu,\lambda})$. In either case the assumption $u_{\mu,\lambda} \equiv v_\lambda$ leads to a contradiction, thereby proving the lemma.

6. Proof of Theorems 1.2 and 1.3

The assertion in Theorem 1.2 follows readily from Lemma 4.1. From Lemmas 4.3 and 5.7 we obtain that apart from the local minimum $u_{\mu,\lambda}$ we obtain a mountain-pass type solution $v_{\mu,\lambda}$ for all $(\mu, \lambda) \in \Re$. From Lemma 3.3, $(P_{\mu,\lambda})$ has no solution for (λ, μ) outside $\overline{\Re}$. From Lemma 4.3, it is clear that $J_{\mu,\lambda}(u_{\mu,\lambda}) < 0$. Suppose $\{(\mu_n, \lambda_n)\}$ is a sequence such that

$\lim_{n\to\infty}(\mu_n, \lambda_n) = (\mu, \lambda) \in \partial\Re$ and let u_{μ_n,λ_n} be the corresponding minimal solution to (P_{μ_n,λ_n}). Then it is easy to see that

$$\limsup_{n\to\infty} J_{\mu_n,\lambda_n}(u_{\mu_n,\lambda_n}) < \infty, \qquad J'_{\mu_n,\lambda_n}(u_{\mu_n,\lambda_n}) = 0. \tag{6.1}$$

By following the same arguments as in step 1 of Lemma 5.3 we get that $\{u_{\mu_n,\lambda_n}\}$ is a bounded sequence in $W^{1,N}(\Omega)$. Now, it is easy to verify that $u_{\lambda,\mu}$, the weak limit of a subsequence of $\{u_{\mu_n,\lambda_n}\}$, is a solution of $(P_{\mu,\lambda})$ thereby proving Theorem 1.3. □

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