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ERRATA CORRIGE TO: "APPROXIMATION BY MEANS OF NONLINEAR INTEGRAL OPERATORS IN THE SPACE OF FUNCTIONS WITH BOUNDED φ–VARIATION"

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1. Introduction

Here the authors want to point out that the term I_2, of Proposition 2 of the original paper ([2]), has to be estimated in a different way. Moreover, now the proof of Theorem 4 of the original paper holds with the new assumption (2) (involving $K_w.3)'$ mentioned below), instead of (6.2) of [2] (involving $K_w.3)$ of the original paper), while, since Theorem 3 (convergence theorem) can be proved with both assumptions $K_w.3)$ and $K_w.3)'$, we prefer here to use directly $K_w.3)'$, in analogy with condition (2). Let us notice that it is easy to see that the two conditions $K_w.3)$ and $K_w.3)'$ cannot be compared. Here we want also to point out that in the convergence theorem of [3] as well as in Lemma 2 of [3], a similar problem occurs and it is solved in the same way proving that $V_\varphi[\lambda(H_w \circ f - f)] \to 0$, as $w \to +\infty$ for sufficiently small $\lambda > 0$, using assumption $K_w.3)$ (see Remark below).

2. Convergence and rate of approximation

For the notation we refer to [2]. Assumption $K_w.3)$ of the original paper ([2]) has to be replaced by the following:

$K_w.3)'$ denoting by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$, for every $\gamma > 0$ there exists $\lambda > 0$ such that

$$\frac{V_\varphi[\lambda G_w, J]}{\varphi(\gamma\ m(J))} \to 0, \text{ as } w \to +\infty,$$

uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$; that is, in correspondence to each $\gamma > 0$ there exists $\lambda > 0$ such that

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for every $\varepsilon > 0$ there is $\overline{w} > 0$ for which $\frac{V_\varphi[\lambda G_w, J]}{\varphi(\gamma\ m(J))} < \varepsilon$, for every $w \geq \overline{w}$ and for every (proper) bounded interval $J \subset \mathbb{R}$.

It is not difficult to find examples of kernel functions which satisfy assumption $K_w.3')$, besides all the other assumptions of our theory: among them, for example, the kernel functions of Example 3 of [1], namely $K_w(t,u) = L_w(t)H_w(u)$, $t \in \mathbb{R}_0^+$, $u \in \mathbb{R}$, $w > 0$, where

$$H_w(u) = \begin{cases} u + \log\left(1 + \frac{u}{w}\right), & 0 \leq u < 1, \\ u + \log\left(1 + \frac{1}{wu}\right), & u \geq 1, \end{cases}$$

and we extend in the odd way the definition of $H_w(u)$ for $u < 0$. Then

$$G_w(u) = \begin{cases} \log\left(1 + \frac{u}{w}\right), & 0 \leq u < 1, \\ \log\left(1 + \frac{1}{wu}\right), & u \geq 1, \end{cases}$$

and so G_w is increasing in $[0,1]$ and decreasing in $[1, +\infty)$. Hence, for every $\gamma > 0$ and $[a,b] \subset [0,1]$, by convexity of φ, and since $\varphi(0) = 0$,

$$V_\varphi[\gamma G_w, [a,b]] = \varphi(\gamma(G_w(b) - G_w(a))) \leq \varphi\Big(\gamma\Big(\frac{b}{w} - \frac{a}{w}\Big)\Big) \leq \frac{1}{w}\varphi\left(\gamma(b-a)\right),$$

and so

$$\frac{V_\varphi[\gamma G_w, [a,b]]}{\varphi(\gamma(b-a))} \leq \frac{1}{w} \to 0,$$

as $w \to +\infty$. If $[a,b] \subset [1, +\infty)$, in a similar way it is easy to see that

$$\frac{V_\varphi[\gamma G_w, [a,b]]}{\varphi(\gamma(b-a))} \leq \frac{1}{wab} \leq \frac{1}{w} \to 0,$$

as $w \to +\infty$. Finally, if $0 \leq a < 1 < b$, by the properties of $\varphi-$variation (see [5]), $V_\varphi[\gamma G_w, [a,b]] \leq \frac{1}{2}V_\varphi[2\gamma G_w, [a,1]] + \frac{1}{2}V_\varphi[2\gamma G_w, [1,b]]$, hence $K_w.3)'$ holds with $\lambda = \frac{\gamma}{2}$.

About the results, the estimate of I_2 in the proof of Proposition 2 of the original paper has to be replaced by the following:

$$I_2 \leq A^{-1} \int_{\mathbb{R}} |L_w(t)| V_\varphi[2\mu A(H_w \circ f - f)]\, dt \leq V_\varphi[2\mu A(H_w \circ f - f)].$$

Hence the estimate of $V_\varphi[\mu(T_w f - f)]$ in the statement of Proposition 2 of the original paper has to be modified in the following way:

$$V_\varphi[\mu(T_w f - f)] \leq \frac{1}{2}\Big\{\omega_\varphi(2\mu A g_w, \delta) + A^{-1} V_\varphi\left[4\mu A g_w\right] \int_{|t|\geq\delta} |L_w(t)| dt + V_\varphi[2\mu A(H_w \circ f - f)]\Big\}. \tag{2.1}$$

About convergence results, we first notice that, in Lemma 1 of [2], the step function ν can be equally defined as

$$\nu(t) = \begin{cases} f(a), & t \le a, \\ f(t_i), & t_{i-1} < t \le t_i, \\ f(b), & t > b, \end{cases}$$

for $i = 1, \dots, n$.

Using the new assumption $K_w.3)'$ it is possible to prove the following result of convergence in variation for $(H_w \circ f - f)$.

Lemma 1. *Let $f \in BV_\varphi(\mathbb{R}) \cap C^0(\mathbb{R})$, where $C^0(\mathbb{R})$ denotes the space of continuous functions over the real line. If $K_w.3)'$ is satisfied, then there exists $\lambda > 0$ such that $V_\varphi[\lambda(H_w \circ f - f)] \to 0$, as $w \to +\infty$.*

Proof. Since $f \in BV_\varphi(\mathbb{R}) \cap C^0(\mathbb{R})$, f has at most countably infinitely many proper points of maximum/minimum ([4]). Let us consider the most general case in which f has countably infinitely many proper points of maximum/minimum, that we will denote $\overline{x}_i$, $i = 0, 1, 2, \dots$ Let us fix an increasing sequence $D = \{s_0, \dots, s_n\}$ in $\mathbb{R}$ and let $\widetilde{D} \equiv \{y_0, y_1, \dots\}$ be the (infinite) sequence obtained adding the points $\overline{x}_i$ to D. Then in each of the intervals $A_i := [\overline{x}_{i-1}, \overline{x}_i]$, $i = 1, 2, \dots$, f is monotone, and hence, for every $\mu > 0$,

$$V_\varphi[\mu(H_w \circ f - f), A_i] \le V_\varphi[\mu\, G_w, I_i], \quad i = 1, 2, \dots,$$

where $I_i := \big[\min\{f(\overline{x}_{i-1}), f(\overline{x}_i)\}, \max\{f(\overline{x}_{i-1}), f(\overline{x}_i)\}\big]$. Now, since $f \in BV_\varphi(\mathbb{R})$, there exists $\gamma > 0$ such that $V_\varphi[\gamma f] < +\infty$; then, by $K_w.3)'$, in correspondence to γ there exists $\lambda > 0$ such that, for a fixed $\varepsilon > 0$, there exists $\overline{w} > 0$ for which $V_\varphi[\lambda G_w, I_i] \le \varepsilon\, \varphi(\gamma m(I_i))$, for every $i = 1, 2, \dots$, and

$$\begin{aligned} &\sum_{i=1}^{+\infty} \varphi(\lambda|(H_w \circ f - f)(y_i) - (H_w \circ f - f)(y_{i-1})|) \\ &\le \sum_{i=1}^{+\infty} V_\varphi[\lambda(H_w \circ f - f), A_i] \\ &\le \sum_{i=1}^{+\infty} V_\varphi[\lambda G_w, I_i] \le \varepsilon \sum_{i=1}^{+\infty} \varphi(\gamma m(I_i)) \le \varepsilon V_\varphi[\gamma f]. \end{aligned}$$

Then, passing to the supremum over all the possible increasing sequences in $\mathbb{R}$, we obtain that, for every $w \ge \overline{w}$, $V_\varphi[\lambda(H_w \circ f - f)] \le \varepsilon V_\varphi[\gamma f]$, and so the thesis follows, since $V_\varphi[\gamma f] < +\infty$. □

Remark. We remark that, in order to obtain the previous convergence result, it is sufficient to assume condition $K_w.3)$ of the original paper (using a different proof). Since a condition of the form $K_w.3)'$ is needed in order to obtain the order of approximation, for the sake of simplicity we use directly condition $K_w.3)'$.

Using Lemma 1, the proof of Lemma 2 of the original paper immediately follows: indeed it is sufficient to notice that

$$V_\varphi[\lambda(H_w \circ f - \nu)] \leq \frac{1}{2}V_\varphi[2\lambda(H_w \circ f - f)] + \frac{1}{2}V_\varphi[2\lambda(f - \nu)]$$

and then to use the above Lemma 1 and Lemma 1 of [2].

About the main convergence result, using (2.1), in Theorem 3 of the original paper it is sufficient to replace the estimate of $V_\varphi[\mu(T_w f - f)]$ (page 15, lines 14-15 of [2]) as follows:

$$V_\varphi[\mu(T_w f - f)] \leq \frac{1}{2}\Big\{\omega_\varphi(2\mu A g_w, \delta) + A^{-1}V_\varphi\left[4\mu A g_w\right]\int_{|t|\geq\delta}|L_w(t)|dt + V_\varphi[2\mu A(H_w \circ f - f)]\Big\}.$$

Then the proof of the theorem can be completed, taking into account that, by Lemma 1, $V_\varphi[\lambda(H_w \circ f - f)] \to 0$ as $w \to +\infty$, for some $\lambda > 0$, and hence $V_\varphi[2\mu A(H_w \circ f - f)] \to 0$ as $w \to +\infty$, for sufficiently small $\mu > 0$.

About the order of approximation, in Theorem 4 of the original paper, instead of condition (6.2), we have to assume that, for every $\gamma > 0$, there exists $\lambda > 0$ such that

$$\frac{V_\varphi[\lambda G_w, J]}{\varphi(\gamma\ m(J))} = O(\xi(w^{-1})), \quad w \to +\infty, \tag{2.2}$$

uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$; i.e., there exists an absolute constant $M > 0$ such that, in correspondence to each $\gamma > 0$, there exists $\lambda > 0$ such that for every $\varepsilon > 0$ there is $\overline{w} > 0$ for which $\frac{V_\varphi[\lambda G_w, J]}{\varphi(\gamma\ m(J))} \leq M\xi(w^{-1})$, for every $w \geq \overline{w}$ and for every (proper) bounded interval $J \subset \mathbb{R}$. It is not difficult to provide examples of kernels which satisfy (2.2): among them, the family of kernel functions of Example 3 of [1] (i.e., the above example of this section) satisfy the above condition with $\xi(w) = w^\alpha$, $0 < \alpha \leq 1$. Moreover, in the estimate of $V_\varphi[\lambda(T_w f - f)]$ (page 20, line 1 of [2]), instead of $\frac{1}{2}V_\varphi[2\lambda A G_w, J]$ we now have $\frac{1}{2}V_\varphi[2\lambda A(H_w \circ f - f)]$. As concerns this term, following the same reasoning of Lemma 1, it is not difficult to see that, by (2.2), $V_\varphi[2\lambda A(H_w \circ f - f)] = O(\xi(w^{-1}))$, as $w \to +\infty$.

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