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# STABLE AND UNSTABLE MANIFOLDS FOR NONLINEAR PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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**Abstract.** The aim of this work is to investigate the asymptotic behavior of solutions near hyperbolic equilibria for nonlinear partial neutral functional differential equations. We suppose that the linear part $A$ satisfies the Hille-Yosida condition on a Banach space and is not necessarily densely defined; the delayed part is assumed to be Lipschitz. We show the existence of stable and unstable manifolds near hyperbolic equilibria when the neutral operator is stable and the semigroup generated by the part of $A$ in $\overline{D(A)}$ is compact. Local stable and unstable manifolds are also obtained when the undelayed part is a $C^1$ function in a neighborhood of the equilibria.

## 1. Introduction

We will investigate the asymptotic behavior of solutions near hyperbolic equilibria for partial neutral functional differential equations. More precisely, we study the existence of the stable and unstable manifolds for the equation

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + g(u_t) \quad \text{for} \quad t \geq \sigma, \\ u_\sigma = \varphi \in C = C([-r,0]\,;X), \end{cases} \tag{1.1}$$

where $A$ is a closed linear operator on a Banach space $X$ and satisfies the following well-known Hille-Yosida condition:

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($\mathbf{H_0}$) there exist $M_0 \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\left\|(\lambda - A)^{-n}\right\| \leq \frac{M_0}{(\lambda-\omega)^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \omega, \tag{1.2}$$

where $\rho(A)$ is the resolvent set of $A$. $C$ is the space of continuous functions from $[-r, 0]$ to $X$ endowed with the uniform norm topology. $\mathcal{D} : C \to X$ is a bounded linear operator which has the form

$$\mathcal{D}\varphi = \varphi(0) - \int_{-r}^{0} [d\eta(\theta)]\, \varphi(\theta) \text{ for } \varphi \in C,$$

for a mapping $\eta : [-r, 0] \to \mathcal{L}(X)$ of bounded variation nonatomic at zero, which means that there exists a continuous nondecreasing function $\delta : [0, r] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\left\| \int_{-s}^{0} [d\eta(\theta)]\, \varphi(\theta) \right\| \leq \delta(s) \sup_{-r \leq \theta \leq 0} \|\varphi(\theta)\| \text{ for } \varphi \in C \text{ and } s \in [0, r],$$

where $\mathcal{L}(X)$ denotes the space of bounded linear operators from $X$ to $X$. For every $t \geq \sigma$, the history function $u_t \in C$ is defined by

$$u_t(\theta) = u(t + \theta) \text{ for } \theta \in [-r, 0].$$

$L$ is a bounded linear operator from $C$ into $X$ and $g$ is a continuous function from $C$ to $X$.

Partial neutral functional differential equations have many applications in the context of physical systems; in [24] and [25], the authors proposed and studied a system of partial neutral functional differential-difference equations defined on the unit circle $S$. This is a model for a continuous circular array of resistively coupled transmission lines with mixed initial-boundary conditions and is given by

$$\frac{\partial}{\partial t}[u(.,t) - qu(.,t-r)] = k\frac{\partial^2}{\partial x^2}[u(.,t) - qu(.,t-r)] + \zeta(u_t) \text{ for } t \geq 0, \tag{1.3}$$

where $x \in S$, $k$ is a positive constant, $\zeta$ is a continuous function and $0 \leq q < 1$. The phase space is $C([-r, 0], H^1(S))$. In [13] and [14], the author investigated the qualitative behavior of solutions to equation (1.3) and obtained several results about stability, attractiveness of solutions, and bifurcation of solutions near an equilibria.

In [1], [4], [6], [7], [21] and [23], the authors investigated the fundamental theory of partial functional differential equations. Particularly in [4], the authors studied the asymptotic behavior of the solution semigroup to the homogeneous linear equation

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t), \quad u_0 = \varphi \in C, \quad t \geq 0. \tag{1.4}$$

Moreover, the authors established a new variation of constants formula for the nonhomogeneous linear partial neutral functional differential equation

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), \quad u_\sigma = \varphi \in C, \quad t \geq \sigma, \tag{1.5}$$

where $f$ is a continuous function from $[\sigma, +\infty)$ into $X$; for more details, we refer to [4]. The stability of stationary solutions, which plays an important role in the qualitative analysis of differential equations. Many results on the existence of stable and unstable manifolds are developed in the context of the class of differential equations

$$\frac{dv}{dt}(t) = Cv(t) + \chi(v(t)), \quad v(0) = v_0, \quad t \geq 0, \tag{1.6}$$

where $C$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space and $\chi$ is a smooth function. The solutions to equation (1.6) can be expressed by using the variation of constants formula

$$v(t) = T(t)v_0 + \int_0^t T(t-s)\chi(v(s)))ds \text{ for } t \geq 0. \tag{1.7}$$

Formula (1.7) and fixed-point theory are powerful tools to deal with equation (1.6). In [16] and [17], the authors studied the center manifolds for ordinary differential equations. The book [15] contains more detailed analysis of this problem. In [12], the author proved the existence of stable and unstable manifolds near stationary solutions for functional differential equations of neutral type in the finite-dimensional case. Recently, in [5], the authors established the existence of stable and unstable manifolds for partial functional differential equations with infinite delay; for more details about invariant manifolds in the context of partial functional differential equations, we refer to [8], [18], [19] and [20].

Here, we will use the variation of constants formula obtained in [4] to investigate the asymptotic behavior of solutions near a hyperbolic equilibrium. First, we study the existence of global stable and unstable manifolds when $g$ is Lipschitz continuous. When $g$ is not Lipschitz continuous and is $C^1$ in a neighborhood of the equilibria, then the existence of the local stable and unstable manifolds is established and the estimation of solutions in each manifold are given. An application is given to show the local stability of the equilibria when the unstable space associated to the linearized equation is reduced to zero.

The paper is organized as follows: In section 2, we recall the variation of constants formula that will be used in this work and we give the spectral decomposition of the phase space when the solution semigroup to the linear equation is hyperbolic. In section 3, we investigate the existence of bounded solutions of equation (1.5), respectively on $\mathbb{R}^-$, $\mathbb{R}^+$ and $\mathbb{R}$. In section 4, we establish the existence of stable and unstable manifolds for equation (1.1) near hyperbolic equilibria when $g$ is globally Lipschitz. In section 5, we study the existence of local stable and unstable manifolds when $g$ is a $C^1$ function in a small neighborhood of the equilibria. Finally, we apply the basic theory of this work to study the local stability of stationary solutions for the model (1.3).

## 2. Variation of constants formula and spectral decomposition

We suppose that the closed operator $A$ satisfies the Hille-Yosida condition $(\mathbf{H_0})$. We need to recall some preliminary results.

**Definition 2.1.** [2] A continuous function $u$ from $[-r+\sigma,+\infty)$ into $X$ is an integral solution to equation (1.1), if

(**i**) $\displaystyle\int_\sigma^t \mathcal{D}u_s ds \in D(A)$ for $t \geq \sigma$,

(**ii**) $\mathcal{D}u_t = \mathcal{D}\varphi + A\displaystyle\int_\sigma^t \mathcal{D}u_s ds + \int_\sigma^t [L(u_s)+g(u_s)]\, ds$ for $t \geq \sigma$, (**iii**) $u_\sigma = \varphi$.

From (**i**), we can see that if $u$ is an integral solution to equation (1.5), then $\mathcal{D}u_t \in \overline{D(A)}$ for all $t \geq 0$; in particular $\mathcal{D}\varphi \in \overline{D(A)}$.

Let us introduce the part $A_0$ of the operator $A$ in $\overline{D(A)}$ which is defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D\,(A)}\}, \qquad A_0x = Ax \text{ for } x \in D(A_0).$$

**Lemma 2.2.** [4] *$A_0$ generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

For simplicity, throughout the paper, integral solutions will be called solutions.

**Theorem 2.3.** [2] *Assume that $(\mathbf{H_0})$ holds and $g$ is Lipschitz continuous. Then, for all $\varphi \in C$ such that $\mathcal{D}\varphi \in \overline{D(A)}$, equation* (1.1) *has a unique* *solution u on* $[-r+\sigma,+\infty)$. *Moreover, u is given by*

$$\mathcal{D}u_t = T_0(t-\sigma)\mathcal{D}\varphi + \lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)B_\lambda[L(u_s)+g(u_s)]ds \text{ for } t\geq\sigma,$$

*where* $B_\lambda = \lambda R(\lambda,A)$ *and* $R(\lambda,A)$ *is the resolvent operator* $(\lambda I - A)^{-1}$ *for* $\lambda > \omega$.

**Corollary 2.4.** *Assume that* $(\mathbf{H_0})$ *holds. Then, for all* $\varphi \in C$ *such that* $\mathcal{D}\varphi \in \overline{D(A)}$, *equation* (1.5) *has a unique solution u on* $[-r+\sigma,+\infty)$. *Moreover, u is given by*

$$\mathcal{D}u_t = T_0(t-\sigma)\mathcal{D}\varphi + \lim_{\lambda\to+\infty}\int_\sigma^t T_0(t-s)B_\lambda[L(u_s)+f(s)]ds \text{ for } t\geq\sigma.$$

Denote by $u_t(.,\sigma,\varphi,f)$ the solution to equation (1.5). The phase space $C_0$ of equation (1.1) is given by $C_0 = \{\varphi \in C : \mathcal{D}\varphi \in \overline{D(A)}\}$. For each $t \geq 0$, we define the linear operator $V(t)$ on $C_0$ by $V(t)\varphi = u_t(.,\varphi)$, where $u(.,\varphi)$ is the solution to the homogeneous equation (1.4).

**Proposition 2.5.** [4] $(V(t))_{t\geq 0}$ *is a strongly continuous semigroup on* $C_0$*; that is,*

(**i**) *for all* $t \geq 0$, $V(t)$ *is a bounded linear operator on* $C_0$,

(**ii**) $V(0) = I$,

(**iii**) $V(t+s) = V(t)V(s)$ *for all* $t, s \geq 0$,

(**iv**) *for all* $\varphi \in C_0$, $V(t)\varphi$ *is a continuous function of* $t \geq 0$ *with values in* $C_0$. *Moreover,*

(**v**) $(V(t))_{t\geq 0}$ *satisfies, for* $t \geq 0$ *and* $\theta \in [-r,0]$, *the translation property*

$$(V(t)\varphi)(\theta) = \begin{cases} (V(t+\theta)\varphi)(0) \text{ if } t+\theta \geq 0, \\ \varphi(t+\theta) \text{ if } t+\theta \leq 0. \end{cases}$$

**Theorem 2.6.** [4, Theorem 3] *The operator* $\mathcal{A}$ *defined on* $C_0$ *by*

$$\begin{cases} D(\mathcal{A}) = \Big\{\varphi \in C^1([-r,0];X) : \mathcal{D}\varphi \in D(A),\ \mathcal{D}\varphi' \in \overline{D(A)} \\ \qquad\qquad \text{and } \mathcal{D}\varphi' = A\mathcal{D}\varphi + L(\varphi)\Big\}, \\ \mathcal{A}\varphi = \varphi' \text{ for } \varphi \in D(\mathcal{A}) \end{cases}$$

*is the infinitesimal generator of the semigroup* $(V(t))_{t\geq 0}$ *on* $C_0$.

In order to determine the asymptotic behavior of the semigroup $(V(t))_{t\geq 0}$, we need to introduce some preliminary results.

**Definition 2.7.** [4, Theorem 16] The operator $\mathcal{D}$ is said to be stable if there exist positive constants $\eta$ and $\mu$ such that the solution to the homogeneous difference equation

$$\mathcal{D}u_t = 0 \quad \text{for } t \geq 0, \qquad u_0 = \varphi,$$

where $\varphi \in \{\psi \in C : \ \mathcal{D}\psi = 0\}$, satisfies $\|u_t(.,\varphi)\| \leq \mu e^{-\eta t} \|\varphi\|$ for $t \geq 0$.

**Example 2.8.** The operator $\mathcal{D}$ defined by $\mathcal{D}\varphi = \varphi(0) - q\varphi(-r)$ is stable if and only if $|q| < 1$.

Assume that
($\mathbf{H_1}$) The semigroup $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$ for $t > 0$.
($\mathbf{H_2}$) The operator $\mathcal{D}$ is stable.
Then, we have the following fundamental result on the semigroup $(V(t))_{t\geq 0}$.

**Theorem 2.9.** [4, lemma 10] *Assume that* ($\mathbf{H_0}$), ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold. Then the semigroup* $(V(t))_{t\geq 0}$ *is decomposed on* $C_0$ *as*

$$V(t) = V_1(t) + V_2(t) \text{ for } t \geq 0,$$

*where* $(V_1(t))_{t\geq 0}$ *is an exponentially stable semigroup on* $C_0$*, which means that there are positive constants* $\alpha_0$ *and* $N_0$ *such that*

$$||V_1(t)\varphi|| \leq N_0 e^{-\alpha_0 t} ||\varphi|| \text{ for } t \geq 0 \text{ and } \varphi \in C_0,$$

*and* $V_2(t)$ *is compact for every* $t > 0$.

Consequently, by Theorem 2.9, we have that

$$\omega_{ess}(V) < 0, \tag{2.1}$$

where $\omega_{ess}(V)$ is the essential growth bound of the semigroup $(V(t))_{t\geq 0}$; for more details about this notion, we refer to [22].

Let $\sigma^+(\mathcal{A}) = \{\lambda \in \sigma(\mathcal{A}) : \mathcal{R}e(\lambda) \geq 0\}$. As an immediate consequence of (2.1), we get the following spectral property of $\mathcal{A}$.

**Lemma 2.10.** *Assume that* ($\mathbf{H}_0$), ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *hold. Then,* $\sigma^+(\mathcal{A})$ *is a finite set of the eigenvalues of* $\mathcal{A}$ *which are not in the essential spectrum. Moreover,* $\lambda \in \sigma^+(\mathcal{A})$ *if and only if there exists* $x \in D(A)\backslash\{0\}$ *solution to the characteristic equation*

$$\Delta(\lambda)x = \lambda\mathcal{D}(e^{\lambda\cdot}x) - A\mathcal{D}(e^{\lambda\cdot}x) - L(e^{\lambda\cdot}x) = 0,$$

*where* $e^{\lambda\cdot}x$ *is the element of* $C$ *defined for all* $\theta \in [-r, 0]$ *by* $(e^{\lambda\cdot}x)(\theta) = e^{\lambda\theta}x$.

**Proof.** The following formulae hold: $r_{ess}(V(t)) = e^{t\omega_{ess}(V)} < 1$, and $e^{t\sigma_{ess}(\mathcal{A})} \subset \sigma_{ess}(V(t))$ for $t > 0$, since $\omega_{ess}(V) < 0$. Consequently, we obtain $\sigma_{ess}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$. Since $\sigma(\mathcal{A}) - \sigma_{ess}(\mathcal{A})$ is a part of the point spectrum, it then follows that $\sigma^+(\mathcal{A})$ is a subset of the point spectrum $\sigma_p(\mathcal{A})$. Moreover, it is well known that $\sigma^+(\mathcal{A})$ is a finite subset. Let $\lambda \in \sigma^+(\mathcal{A})$. Then, there exists $\varphi \in D(\mathcal{A})$ with $\varphi \neq 0$ such that $\mathcal{A}\varphi = \lambda\varphi$. Therefore, $\varphi(\theta) = e^{\lambda\theta}\varphi(0)$ with $\varphi(0) \neq 0$. Since $\varphi \in D(\mathcal{A})$, it follows that $\Delta(\lambda)\varphi(0) = 0$. Conversely, let $x \neq 0$ be such that $\Delta(\lambda)x = 0$; if we take the function $\varphi = e^{\lambda \cdot}x$, then $\varphi \in D(\mathcal{A})$ and $\mathcal{A}\varphi = \lambda\varphi$. □

**Definition 2.11.** The semigroup $(V(t))_{t\geq 0}$ is said to be hyperbolic if $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

**Theorem 2.12.** [4, Theorem 16] *Assume that* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. If the semigroup* $(V(t))_{t\geq 0}$ *is hyperbolic, then* $C_0$ *is decomposed as follows:* $C_0 = \mathcal{S} \oplus \mathcal{V}$, *where* $\mathcal{S}$ *and* $\mathcal{V}$ *are two closed subspaces of* $C_0$ *invariant under* $V(t)$. $\mathcal{V}$ *is a finite-dimensional space and the restriction of* $V(t)$ *on* $\mathcal{V}$ *becomes a group on* $\mathbb{R}$. *Moreover, there exist positive constants* $\delta$ *and* $\mu$ *such that the following estimations hold:*

$$\begin{aligned} \|V(t)\varphi\| &\leq \delta e^{-\mu t}\|\varphi\| \quad \text{for } \varphi \in \mathcal{S} \quad \text{and } t \geq 0, \\ \|V(t)\varphi\| &\leq \delta e^{\mu t}\|\varphi\| \quad \text{for } \varphi \in \mathcal{V} \quad \text{and } t \leq 0. \end{aligned} \tag{2.2}$$

$\mathcal{S}$ and $\mathcal{V}$ are called respectively the stable and unstable subspaces of $(V(t))_{t\geq 0}$. Let $P^-$ and $P^+$ denote the projection operators respectively on $\mathcal{S}$ and $\mathcal{V}$.

In order to give a variation of constants formula associated to equation (1.1), we introduce the space $\langle X_0 \rangle$ defined by $\langle X_0 \rangle = \{X_0 c : c \in X\}$, where the function $X_0 c$ is given by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if} \quad \theta \in [-r, 0), \\ c & \text{if} \quad \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$, equipped with the norm $\|\phi + X_0 c\| = ||\phi|| + \|c\|$ for $(\phi, c) \in C_0 \times X$, is a Banach space. Consider the extension $\widetilde{\mathcal{A}}$ of the operator $\mathcal{A}$ on $C_0 \oplus \langle X_0 \rangle$ defined by

$$\begin{cases} D\left(\widetilde{\mathcal{A}}\right) &= \left\{\varphi \in C^1([-r, 0]; X) : \mathcal{D}\varphi \in D(A) \text{ and } \mathcal{D}\varphi' \in \overline{D(A)}\right\}, \\ \widetilde{\mathcal{A}}\phi &= \varphi' + X_0 (A\mathcal{D}\varphi + L\varphi - \mathcal{D}\varphi'). \end{cases}$$

In order to estimate $R(\lambda, \widetilde{\mathcal{A}}) = (\lambda I - \widetilde{\mathcal{A}})^{-1}$, we assume the following assumption:

($\mathbf{H_3}$) $\mathcal{D}(e^{\lambda \cdot} c) \in D(A)$ for all $c \in D(A)$ and all complex $\lambda$.

**Lemma 2.13.** [4, Theorem 13] *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_3)$ *hold. Then* $\widetilde{\mathcal{A}}$ *satisfies the Hille-Yosida condition on* $C_0 \oplus \langle X_0 \rangle$ : *there exist* $\widetilde{N} \geq 1$ *and* $\widetilde{\omega} \in \mathbb{R}$ *such that* $(\widetilde{\omega}, +\infty) \subset \rho(\widetilde{\mathcal{A}})$ *and*

$$\left\| R(\lambda, \widetilde{\mathcal{A}})^n \right\| \leq \frac{\widetilde{N}}{(\lambda - \widetilde{\omega})^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

**Theorem 2.14.** [4, Theorem 16] *Assume that* $(\mathbf{H_0})$ *and* $(\mathbf{H_3})$ *hold. Then, for all* $\varphi \in C_0$, *the solution* $u$ *to equation* (1.5)*is given by the variation of constants formula*

$$u_t = V(t-\sigma)\varphi + \lim_{n \to +\infty} \int_\sigma^t V(t-s)\,\Theta^n f(s) ds \text{ for } t \geq \sigma,$$

*where* $\Theta^n c$ *is defined for any* $c \in X$ *by* $\Theta^n c = nR(n, \widetilde{\mathcal{A}})(X_0 c)$ *for* $n > \widetilde{\omega}$.

## 3. Formulas for bounded solutions on $\mathbb{R}^-$, $\mathbb{R}^+$ and $\mathbb{R}$

Let $Y$ be a Banach space and $J \in \{\mathbb{R}^-, \mathbb{R}^+, \mathbb{R}\}$; the space $BC(J, Y)$ denote the space of bounded continuous functions from $J$ to $Y$ provided with the uniform norm topology.

In the sequel, we assume that the semigroup $(V(t))_{t \geq 0}$ is hyperbolic.

**Theorem 3.1.** *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H}_3)$ *hold. If* $f$ *is bounded on* $(-\infty, 0]$ *and* $u$ *is a solution to equation* (1.5) *on* $(-\infty, 0]$, *then* $u \in BC((-\infty, 0]; X)$ *if and only if*

$$P^- u_0 = \lim_{n \to +\infty} \int_{-\infty}^0 V(-s) P^- \left(\Theta^n f(s)\right) ds. \tag{3.1}$$

*Moreover,* $u$ *is given by*

$$u_t = V(t) P^+ u_0 + (\mathcal{K}^- f)(t) \quad \text{for } t \leq 0,$$

*where* $\mathcal{K}^- : BC((-\infty, 0]; X) \to BC((-\infty, 0]; C_0)$ *is the bounded linear operator defined for each* $f \in BC((-\infty, 0]; X)$ *and* $t \leq 0$ *by*

$$\begin{aligned}(\mathcal{K}^- f)(t) = &\lim_{n \to +\infty} \int_0^t V(t-s) P^+ \left(\Theta^n f(s)\right) ds \\ &+ \lim_{n \to +\infty} \int_{-\infty}^t V(t-s) P^- \left(\Theta^n f(s)\right) ds.\end{aligned}$$

**Proof.** Let $u$ be a bounded solution to equation (1.5) on $(-\infty, 0]$. Then, for $a < t \leq 0$, we have

$$u_t = V(t-a) u_a + \lim_{n \to +\infty} \int_a^t V(t-s) \Theta^n f(s) ds,$$

and it follows, for $a < t \le 0$, that

$$P^- u_t = V(t-a)P^- u_a + \lim_{n\to+\infty}\int_a^t V(t-s)P^-\left(\Theta^n f(s)\right)ds. \tag{3.2}$$

Since $u$ is bounded on $(-\infty, 0]$, we deduce that $t \to u_t$ is bounded on $(-\infty, 0]$. Letting $a \to -\infty$ in (3.2), we get for $t \le 0$

$$P^- u_t = \lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n f(s)\right)ds.$$

Taking $t = 0$, we obtain formula (3.1). On the other hand, for $t \le 0$, we have

$$u_t = P^+u_t + P^-u_t = V(t)P^+u_0 + \lim_{n\to+\infty}\int_0^t V(t-s)P^+\left(\Theta^n f(s)\right)ds$$
$$+ \lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n f(s)\right)ds.$$

Conversely, assume that formula (3.1) holds and consider the $C_0$-valued function $v$ defined for $t \le 0$ by

$$v(t) = V(t)P^+u_0 + (\mathcal{K}^- f)(t).$$

Then, $v$ is a bounded function on $(-\infty, 0]$. Moreover, for $a < t \le 0$ we have

$$v(t) = V(t)P^+u_0 + \lim_{n\to+\infty}\int_0^a V(t-s)P^+\left(\Theta^n f(s)\right)ds$$
$$+ \lim_{n\to+\infty}\int_a^t V(t-s)P^+\left(\Theta^n f(s)\right)ds$$
$$+ \lim_{n\to+\infty}\int_{-\infty}^a V(t-s)P^-\left(\Theta^n f(s)\right)ds + \lim_{n\to+\infty}\int_a^t V(t-s)P^-\left(\Theta^n f(s)\right)ds.$$

Therefore, for $a < t \le 0$, we have

$$v(t) \ = \ V(t-a)v(a) + \lim_{n\to+\infty}\int_a^t V(t-s)\Theta^n f(s)ds.$$

Moreover, $v(0) = P^+u_0 + (\mathcal{K}^- f)(0) = P^+u_0 + P^-u_0 = u_0$, which implies that $u_t = v(t)$ for $t \le 0$, thus $u \in BC((-\infty, 0]; X)$. □

**Theorem 3.2.** *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H}_3)$ *hold. If* $f$ *is bounded on* $[0, +\infty)$ *and* $u$ *is a solution to equation* (1.5) *on* $[0, +\infty)$*, then,* $u$ *is bounded on* $[0, +\infty)$ *if and only if*

$$P^+u_0 = -\lim_{n\to+\infty}\int_0^{+\infty} V(-s)P^+\left(\Theta^n f(s)\right)ds. \tag{3.3}$$

*If* (3.3) *holds, then* $u$ *is given for* $t \geq 0$ *by*

$$u_t = V(t)P^- u_0 + (\mathcal{K}^+ f)(t),$$

*where* $\mathcal{K}^+ : BC([0,+\infty);X) \longrightarrow BC([0,+\infty);C_0)$ *is the bounded linear operator defined, for* $f \in BC([0,+\infty);X)$ *and* $t \geq 0$ , *by*

$$(\mathcal{K}^+ f)(t) = \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n f(s)\right) ds$$
$$- \lim_{n\to+\infty}\int_t^{+\infty} V(t-s)P^+\left(\Theta^n f(s)\right) ds.$$

**Proof.** Assume that $u$ is a bounded solution to equation (1.5) on $[0,+\infty)$. Then, for $t > a \geq 0$,

$$P^+ u_t \;=\; V(t-a)P^+ u_a + \lim_{n\to+\infty}\int_a^t V(t-s)P^+\left(\Theta^n f(s)\right) ds.$$

Since $(V(t))_{t\geq 0}$ becomes a group on $\mathcal{V}$, we get for $t > a \geq 0$ that

$$P^+ u_a = V(a-t)P^+ u_t - \lim_{n\to+\infty}\int_a^t V(a-s)P^+\left(\Theta^n f(s)\right) ds. \tag{3.4}$$

Since $u$ is a bounded function on $\mathbb{R}^+$, it follows that $t \to u_t$ is bounded on $[0,+\infty)$. Letting $t \to +\infty$ in (3.4), we have

$$P^+ u_a = -\lim_{n\to+\infty}\int_a^{+\infty} V(a-s)P^+\left(\Theta^n f(s)\right) ds.$$

Taking $a = 0$, we get formula (3.3). Moreover, for $t \geq 0$,

$$u_t = P^- u_t + P^+ u_t = V(t)P^- u_0 + \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n f(s)\right) ds$$
$$- \lim_{n\to+\infty}\int_t^{+\infty} V(t-s)P^+\left(\Theta^n f(s)\right) ds.$$

Therefore, for $t \geq 0$, we have

$$u_t = V(t)P^- u_0 + (\mathcal{K}^+ f)(t).$$

Conversely, assume that formula (3.3) holds and consider the $C_0$-valued function $v$ defined, for $t \geq 0$, by $v(t) = V(t)P^- u_0 + (\mathcal{K}^+ f)(t)$. Then, $v$ is a bounded function and, for all $t > a \geq 0$, we have

$$v(t) = V(t)P^- u_0 + \lim_{n\to+\infty}\int_0^a V(t-s)P^-\left(\Theta^n f(s)\right) ds$$
$$+ \lim_{n\to+\infty}\int_a^t U(t-s)P^-\left(\Theta^n f(s)\right) ds$$

$$-\lim_{n\to+\infty}\int_a^{+\infty}V(t-s)P^+(\Theta^n f(s))ds-\lim_{n\to+\infty}\int_t^a V(t-s)P^+(\Theta^n f(s))ds.$$

Consequently,

$$v(t) \;=\; V(t-a)v(a)+\lim_{n\to+\infty}\int_t^a V(t-s)\Theta^n f(s)ds.$$

On the other hand, $v(0)=P^-u_0+(\mathcal{K}^+f)(0)=P^-u_0+P^+u_0=u_0$, which implies for $t\geq 0$ that $u_t=v(t)$. Consequently, $u\in BC([0,+\infty);X)$. □

**Theorem 3.3.** *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H}_3)$ *hold. Suppose* $f\in BC(\mathbb{R};X)$. *Then, equation* (1.5) *has a unique bounded solution* $w$ *on* $\mathbb{R}$ *which is given by*

$$\begin{aligned} w_t=&\lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n f(s)\right)ds\\ &-\lim_{n\to+\infty}\int_t^{+\infty}V(t-s)P^+\left(\Theta^n f(s)\right)ds \text{ for } t\in\mathbb{R}. \end{aligned} \tag{3.5}$$

**Proof.** Let $w$ be given by formula (3.5). Then, for $t\in\mathbb{R}$,

$$\begin{aligned} \|w_t\| &\leq \|P^-\|\,\delta\widetilde{N}\,\|f\|_\infty\int_{-\infty}^t e^{-\mu(t-s)}ds+\|P^+\|\,\delta\widetilde{N}\,\|f\|_\infty\int_t^{+\infty}e^{\mu(t-s)}ds\\ &\leq \frac{\delta\widetilde{N}\,\|f\|_\infty}{\mu}(\|P^-\|+\|P^+\|), \end{aligned}$$

which implies that $w$ is bounded on $\mathbb{R}$. Moreover, for all $t\geq a$ in $\mathbb{R}$, we have

$$\begin{aligned} w_t=&\lim_{n\to+\infty}\int_{-\infty}^a V(t-s)P^-\left(\Theta^n f(s)\right)ds+\int_a^t V(t-s)P^-\left(\Theta^n f(s)\right)ds\\ &-\lim_{n\to+\infty}\int_t^a V(t-s)P^+\left(\Theta^n f(s)\right)ds-\int_a^{+\infty}V(t-s)P^+\left(\Theta^n f(s)\right)ds\\ &=V(t-a)w_a+\int_a^t V(t-s)\Theta^n f(s)ds. \end{aligned}$$

Therefore, $w$ is a solution to equation (1.5) on $\mathbb{R}$. For the uniqueness, we suppose that there is another bounded solution $v$ to equation (1.5). Then $v-w$ is a bounded solution to the homogeneous equation

$$\frac{d}{dt}\mathcal{D}u\,(t)=A\mathcal{D}u(t)+L(u_t) \text{ for } t\in\mathbb{R}.$$

It follows, for $t>a$, that

$$v_t-w_t=V(t-a)(v_a-w_a),$$

and

$$P^+(v_t - w_t) = V(t-a)P^+(v_a - w_a) \text{ for } t,\, a \in \mathbb{R},$$
$$P^-(v_t - w_t) = V(t-a)P^-(v_a - w_a) \text{ for } t > a.$$

Thus,

$$\|P^+(v_t - w_t)\| \le \delta e^{\mu(t-a)} \left\|P^+\right\| \sup_{s\in\mathbb{R}} \|v_s - w_s\| \text{ for } t < a, \tag{3.6}$$

$$\|P^-(v_t - w_t)\| \le \delta e^{-\mu(t-a)} \left\|P^-\right\| \sup_{s\in\mathbb{R}} \|v_s - w_s\| \text{ for } t > a. \tag{3.7}$$

Letting $a$ go to $+\infty$ in (3.6), we have that $P^+(v_t - w_t) = 0$. Consequently, $(v_t - w_t) \in \mathcal{S}$; that is, $v_t - w_t = P^-(v_t - w_t)$. Passing to the limit as $a$ goes to $-\infty$ in (3.7), we have that $v_t = w_t$ for $t \in \mathbb{R}$ and conclude that formula (3.5) determines a unique bounded solution on $\mathbb{R}$. □

## 4. STABLE AND UNSTABLE MANIFOLDS

In this section, we study the existence of stable and unstable manifolds near a hyperbolic equilibria of equation (1.1). Without loss of generality, we assume that zero is an equilbria of equation (1.1), which is equivalent to saying that $g(0) = 0$. We suppose that $(\mathbf{H}_4)$ $g : C \to X$ is a Lipschitz continuous function and $g$ is differentiable at 0 with $g'(0) = 0$.

Then the linearized equation of equation (1.1) at 0 is given by

$$\frac{d}{dt}\mathcal{D}u\,(t) = A\mathcal{D}u(t) + L(u_t) \text{ for } t \ge 0, \qquad u_0 = \varphi \in C. \tag{4.1}$$

The Lipschitz constant $Lip(g)$ of $g$ is defined by

$$Lip(g) = \sup_{\varphi_1 \ne \varphi_2} \frac{\|g(\varphi_1) - g(\varphi_2)\|}{\|\varphi_1 - \varphi_2\|}.$$

Equilibria 0 is said to be hyperbolic if the solution semigroup to the linear equation (4.1) is hyperbolic.

The stable manifold $\mathcal{S}(g)$ and unstable manifold $\mathcal{V}(g)$ associated to the stationary solution 0 to equation (1.1) are defined respectively by

$$\mathcal{S}(g) = \{\varphi \in C_0 : u_t(.,\varphi,g) \underset{t\to+\infty}{\longrightarrow} 0\}, \quad \mathcal{V}(g) = \{\varphi \in C_0 : u_t(.,\varphi,g) \underset{t\to-\infty}{\longrightarrow} 0\},$$

where $u(.,\varphi,g)$ denotes the solution to equation (1.1) in $(-\infty, 0]$ or $[0, +\infty)$ with $u_0(.,\varphi,g) = \varphi$.

**Theorem 4.1.** *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H}_3)$ *and* $(\mathbf{H_4})$ *hold. Then, there exists* $\varepsilon > 0$ *such that, for* $Lip(g) < \varepsilon$,

$$\mathcal{S}(g) = \{\varphi \in C_0 : \sup_{t\geq 0}\|u_t(.,\varphi,g)\| < \infty\},$$
$$\mathcal{V}(g) = \{\varphi \in C_0 : \sup_{t\leq 0}\|u_t(.,\varphi,g)\| < \infty\},$$

*and*

$$\|u_t(.,\varphi,g)\| \leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}(g),$$
$$\|u_t(.,\varphi,g)\| \leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{V}(g),$$

*where* $\delta$ *and* $\mu$ *are the positive constants defined previously in* (2.2). *Moreover,* $\mathcal{S}(g)$ *and* $\mathcal{V}(g)$ *are respectively positively and negatively invariant.*

**Proof.** Let $\varphi \in \mathcal{S}(g)$ and $u = u(.,\varphi,g)$ be the solution to equation (1.1) on $[0,\infty)$ with $u_0(.,\varphi,g) = \varphi$. Then $u$ is bounded on $[0,\infty)$ and $\sup_{t\geq 0}\|g(u_t)\| < \infty$. By Theorem 3.2, we deduce that $u$ is given for $t \geq 0$ by

$$\begin{aligned} u_t = \quad & V(t)P^-\varphi + \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n\left(g(u_s)\right)\right)ds \\ & - \lim_{n\to+\infty}\int_t^\infty V(t-s)P^+\left(\Theta^n\left(g(u_s)\right)\right)ds. \end{aligned} \tag{4.2}$$

This implies that

$$\begin{aligned} \|u_t\| \leq \quad & \delta e^{-\mu t}\|P^-\varphi\| + \delta\left\|P^-\right\| \widetilde{N}Lip(g)\int_0^t e^{-\mu(t-s)}\|u_s\|ds \\ & + \delta\left\|P^+\right\|\widetilde{N}Lip(g)\int_t^\infty e^{\mu(t-s)}\|u_s\|ds. \end{aligned} \tag{4.3}$$

To complete the proof, we need the following fundamental lemma.

**Lemma 4.2.** [11, page 110] *Let* $\alpha'$, $\nu'$, $K', l'$ *and* $N'$ *be positive constants and* $v$ *be a nonnegative bounded continuous solution to either the inequality*

$$i)\ v(t) \leq K'e^{-\alpha' t} + l'\int_0^t e^{-\alpha'(t-s)}v(s)ds + N'\int_0^{+\infty} e^{-\nu' s}v(t+s)ds \text{ for } t \geq 0,$$

*or the inequality*

$$ii)\ v(t) \leq K'e^{\alpha' t} + l'\int_t^0 e^{\alpha'(t-s)}v(s)ds + N'\int_{-\infty}^0 e^{\nu' s}v(t+s)ds \quad \text{for } t \leq 0.$$

*If* $\beta' = \frac{l'}{\alpha'} + \frac{N'}{\nu'} < 1$, *then, in either case,*

$$v(t) \leq (1-\beta')^{-1}K'e^{-(\left[\alpha'-(1-\beta')^{-1}l'\right]|t|}.$$

Let $\varepsilon > 0$ be chosen such that

$$\frac{4\varepsilon\delta\widetilde{N}}{\mu}\max(\left\|P^+\right\|, \left\|P^-\right\|) < 1.$$

Then, for $Lip(g) < \varepsilon$, we obtain for $t \geq 0$ that

$$\begin{aligned}\|u_t\| \leq \delta\|P^-\varphi\|e^{-\mu t} + \delta\left\|P^-\right\|\widetilde{N}\varepsilon\int_0^t e^{-\mu(t-s)}\|u_s\|ds \\ + \delta\left\|P^+\right\|\widetilde{N}\varepsilon\int_t^\infty e^{\mu(t-s)}\|u_s\|ds.\end{aligned}$$

Using Lemma 4.2, we get that

$$\|u_t\| \leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0. \tag{4.4}$$

Conversely, let $u$ be a bounded solution to equation (1.1) on $[0,\infty)$. Then, by Theorem 3.2, we have that $u$ is given for $t \geq 0$ by formula (4.2) and using Lemma 4.2 we get the estimation (4.4).

We use the same approach in the case of the unstable manifold. In fact, let $\varphi \in \mathcal{V}(g)$ and $u = u(.,\varphi,g)$ be the solution to equation (1.1) in $(-\infty,0]$ with $u_0(.,\varphi,g) = \varphi$. Then $u$ is bounded on $(-\infty,0]$ and $\sup_{t\leq 0}\|g(u_t)\| < \infty$. By Theorem 3.1, we deduce that $u$ is given for $t \leq 0$ by

$$\begin{aligned}u_t = \;& V(t)P^+\varphi + \lim_{n\to+\infty}\int_0^t V(t-s)P^+\left(\Theta^n\left(g(u_s)\right)\right)ds \\ &+ \lim_{n\to+\infty}\int_{-\infty}^t V(t-s)P^-\left(\Theta^n\left(g(u_s)\right)\right)ds.\end{aligned} \tag{4.5}$$

It follows that

$$\begin{aligned}\|u_t\| \leq \;& \delta e^{\mu t}\|P^+\varphi\| + \delta\|P^+\|\ \widetilde{N}Lip(g)\int_t^0 e^{\mu(t-s)}\|u_s\|ds \\ &+ \delta\|P^-\|\ \widetilde{N}Lip(g)\int_{-\infty}^t e^{-\mu(t-s)}\|u_s\|ds.\end{aligned} \tag{4.6}$$

Let $\varepsilon > 0$ be chosen such that

$$\frac{4\varepsilon\delta\widetilde{N}}{\mu}\max(\left\|P^+\right\|, \left\|P^-\right\|) < 1.$$

Then, for $Lip(g) < \varepsilon$, we obtain for $t \leq 0$ that

$$\begin{aligned}\|u_t\| \leq \delta\|P^+\varphi\|e^{\mu t} + \delta\left\|P^+\right\|\widetilde{N}\varepsilon\int_t^0 e^{\mu(t-s)}\|u_s\|ds \\ + \delta\left\|P^-\right\|\widetilde{N}\varepsilon\int_{-\infty}^t e^{-\mu(t-s)}\|u_s\|ds.\end{aligned}$$

Using Lemma 4.2, we have that

$$\|u_t\| \leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0. \tag{4.7}$$

Conversely, let $u$ be a bounded solution to the equation on $(-\infty, 0]$. Then, by Theorem 3.1, we have that $u$ is given for $t \leq 0$ by formula (4.5) and using Lemma 4.2 we get the estimation (4.7). About the invariance principle, let $u_t(., \varphi, g)$ be a bounded solution to equation (1.1) on $\mathbb{R}^+$ such that $\sup_{t\geq 0}\|u_t(., \varphi, g)\| < \infty$. Since equation (1.1) is autonomous, it follows that

$$u_t(., u_s(., \varphi, g), g) = u_{t+s}(., \varphi, g) \text{ for } t, s \geq 0,$$

which implies that $u_s(., \varphi, g) \in \mathcal{S}(g)$ for $s \geq 0$. The same argument can be used to show that the unstable manifold is negatively invariant. □

## 5. Local stable and unstable manifolds

In this section, we establish the existence of the local stable and unstable manifolds where $g$ is not globally Lipschitz. More precisely, besides the assumption $(\mathbf{H_4})$, we suppose the following assumption.

$(\mathbf{H_5})$ $g$ is continuously differentiable in $B(0, \rho_0)$ for some $\rho_0 > 0$ with $g(0) = 0$ and $g'(0) = 0$.

For $\rho < \rho_0$, we define respectively the local stable and unstable manifolds associated to the zero stationary solution of (1.1) by

$$\mathcal{S}_{loc}(g) = \{\varphi \in B(0, \rho) : \|u_t(., \varphi, g)\| < \rho \text{ for } t \geq 0\},$$
$$\mathcal{V}_{loc}(g) = \{\varphi \in B(0, \rho) : \|u_t(., \varphi, g)\| < \rho \text{ for } t \leq 0\}.$$

**Theorem 5.1.** *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H}_3)$ *and* $(\mathbf{H_5})$ *hold. Then, there exists* $\rho < \rho_0$ *such that*

$$\|u_t(., \varphi, g)\| \leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}_{loc}(g),$$
$$\|u_t(., \varphi, g)\| \leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{V}_{loc}(g).$$

*Moreover, any solution, bounded by* $\rho$ *on* $\mathbb{R}^+$ *, to equation* (1.1) *lies in* $\mathcal{S}_{loc}(g)$ *and any solution, bounded by* $\rho$ *on* $\mathbb{R}^-$*, to equation* (1.1) *lies in* $\mathcal{V}_{loc}(g)$.

**Proof.** Without loss of generality, one can choose $\rho_0$ such that $\sup_{\|\varphi\|<\rho_0}\|g'(\varphi)\| < \infty$. For $0 < \rho < \rho_0$, we introduce the function $g_\rho$ defined on $C$ by

$$g_\rho(\varphi) = \begin{cases} g(\varphi) \text{ for } \|\varphi\| \leq \rho, \\ g\left(\rho\frac{\varphi}{\|\varphi\|}\right) \text{ for } \|\varphi\| > \rho. \end{cases}$$

Then, we have the following Lipschitz property of the function $g_\rho$.

**Lemma 5.2.** [22] *For all $\rho < \rho_0$, the function $g_\rho$ is Lipschitz continuous and*

$$Lip(g_\rho) \leq 2 \sup_{\|\varphi\|<\rho} \left\| g'(\varphi) \right\|.$$

We consider the partial functional differential equation

$$\frac{d}{dt}\mathcal{D}u\,(t) = A\mathcal{D}u(t) + L(u_t) + g_\rho(u_t) \text{ for } t \geq 0, \qquad u_0 = \varphi. \tag{5.1}$$

Let $\varepsilon$ be given in Theorem 4.1. Since $\sup_{\|\varphi\|<\rho} \|g'(\varphi)\| \to 0$ as $\rho \to 0$, there exists $\rho < \rho_0$ such that $Lip(g_\rho) < \varepsilon$. Applying Theorem 4.1 to equation (5.1), we obtain the estimations

$$\begin{aligned} \|u_t(.,\varphi,g_\rho)\| &\leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}(g_\rho), \\ \|u_t(.,\varphi,g_\rho)\| &\leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{V}(g_\rho). \end{aligned}$$

Since $g = g_\rho$ in $B(0,\rho)$, then $u(.,\varphi,g) = u(.,\varphi,g_\rho)$ for $\varphi \in \mathcal{S}_{loc}(g)$, $u(.,\varphi,g) = u(.,\varphi,g_\rho)$ for $\varphi \in \mathcal{V}_{loc}(g)$. Consequently, we get the asymptotic behavior of solutions

$$\begin{aligned} \|u_t(.,\varphi,g)\| &\leq 2\delta\|P^-\varphi\|e^{-\frac{\mu}{2}t} \text{ for } t \geq 0 \text{ and } \varphi \in \mathcal{S}_{loc}(g), \\ \|u_t(.,\varphi,g)\| &\leq 2\delta\|P^+\varphi\|e^{\frac{\mu}{2}t} \text{ for } t \leq 0 \text{ and } \varphi \in \mathcal{V}_{loc}(g). \end{aligned} \tag{5.2}$$

Let $u(.,\varphi,g)$ be a solution to equation (1.1) which is bounded by $\rho$. Then $u_t(.,\varphi,g) = u_t(.,\varphi,g_\rho)$ for $t \geq 0$. By Theorem 4.1 we have that $\varphi \in \mathcal{S}(g_\rho)$. Since $\mathcal{S}(g_\rho)$ is positively invariant, $u_t(.,\varphi,g_\rho) \in \mathcal{S}(g_\rho)$ for $t \geq 0$. Therefore, $u_t(.,\varphi,g) \in \mathcal{S}_{loc}(g)$ for $t \geq 0$. The same argument can be used in the case of the unstable manifold. □

**Definition 5.3.** Let $\mathcal{O}$ be a subset of $C_0$ which contains the origin 0. We say that $\mathcal{O}$ is tangent to $\mathcal{S}$ (respectively $\mathcal{V}$) at 0 if

$$\frac{\|P^+\varphi\|}{\|P^-\varphi\|} \longrightarrow 0 \text{ (respectively } \frac{\|P^-\varphi\|}{\|P^+\varphi\|} \longrightarrow 0) \text{ as } \varphi \to 0 \text{ in } \mathcal{O}.$$

**Theorem 5.4.** *$\mathcal{S}_{loc}(g)$ (respectively $\mathcal{V}_{loc}(g)$) is tangent to $\mathcal{S}$ (respectively to $\mathcal{V}$) at 0.*

**Proof.** Let $\varphi \in \mathcal{S}_{loc}(g)$. Then the corresponding solution $u(.,\varphi,g)$ to equation (1.1) is bounded on $[0,\infty)$. From Theorem 3.2, we have that

$$\varphi - P^-\varphi = P^+\varphi = -\lim_{n\to+\infty} \int_0^\infty V(-s)P^+\left(\Theta^n\left(g(u_s)\right)\right) ds. \tag{5.3}$$

Thus,

$$\left\|P^+\varphi\right\| \leq \frac{\delta\widetilde{N}\,\|P^+\|}{\mu} \sup_{\|\psi\|<\rho} \left\|g'(\psi)\right\| \sup_{t\geq 0} \|u_t\|.$$

By estimation (5.2), we obtain that

$$\|P^+\varphi\| \le 2\frac{\delta^2 \widetilde{N}\,\|P^+\|}{\mu} \sup_{\|\psi\|<\rho} \left\|g'(\psi)\right\| \|P^-\varphi\|.$$

We can choose $\rho$ sufficiently small such that

$$\frac{\delta^2 \widetilde{N}\,\|P^+\|}{\mu} \sup_{\|\psi\|<\rho} \left\|g'(\psi)\right\| < \frac{1}{4}.$$

Then,

$$\left\|P^+\varphi\right\| \le \frac{1}{2}\|P^-\varphi\|. \tag{5.4}$$

Consequently, $\|\varphi\| \ge \frac{1}{2}\|P^-\varphi\|$. Let $\varphi \in \mathcal{S}_{loc}(g)$ such that $\varphi \ne 0$. Then, (5.3) and (5.4) yield $\|u_t(.,\varphi,g)\| \le 2\delta(2\,\|\varphi\|) = 4\delta\|\varphi\|$. Hence,

$$\|P^+\varphi\| \le \frac{2\delta^2 \widetilde{N}\,\|P^+\|}{\mu} \sup_{\|\psi\|<4\delta\|\varphi\|} \left\|g'(\psi)\right\| \|P^-\varphi\|,$$

or

$$\frac{\|P^+\varphi\|}{\|P^-\varphi\|} \le \frac{2\delta^2 \widetilde{N}\,\|P^+\|}{\mu} \sup_{\|\psi\|<4\delta\|\varphi\|} \left\|g'(\psi)\right\|.$$

Therefore,

$$\frac{\|P^+\varphi\|}{\|P^-\varphi\|} \to 0 \text{ as } \|\varphi\| \to 0 \text{ in } \mathcal{S}_{loc}(g);$$

that is, $\mathcal{S}_{loc}(g)$ is tangent to $\mathcal{S}$ at 0. One can use the same reasoning to show that $\mathcal{V}_{loc}(g)$ is tangent to $V$ at 0. □

For $\rho < \rho_0$, we define the sets

$$\widetilde{\mathcal{S}}_\rho(g) = \{\varphi \in B(0,\rho) : \|P^-\varphi\| < \frac{\rho}{2\delta} \text{ and } \|u_t(.,\varphi,g)\| < \rho \text{ for } t \ge 0\},$$

and

$$\widetilde{\mathcal{V}}_\rho(g) = \{\varphi \in B(0,\rho) : \|P^+\varphi\| < \frac{\rho}{2\delta} \text{ and } \ \|u_t(.,\varphi,g)\| < \rho, \text{ for } t \le 0\}.$$

Without loss of generality we assume that $g'$ is bounded in $B(0,\rho_0)$, otherwise, since $g'$ is continuous, we can choose $\rho_0$ small enough such that $g'$ is bounded in $B(0,\rho_0)$.

**Theorem 5.5.** *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H}_3)$ *and* $(\mathbf{H_5})$ *hold. Then, there exists* $\rho < \rho_0$ *such that* $P^-$ *is a diffeomorphism from* $\widetilde{\mathcal{S}}_\rho(g)$ *to* $B(0,\frac{\rho}{2\delta}) \cap \mathcal{S}$ *and* $P^+$ *is a diffeomorphism from* $\widetilde{\mathcal{V}}_\rho(g)$ *to* $B(0,\frac{\rho}{2\delta}) \cap V$.

**Proof.** Let $\rho < \rho_0$ be obtained in Theorem 5.1. Set

$$B_{BC(\mathbb{R}^+,C_0)}(0,\rho) = \{y \in BC(\mathbb{R}^+, C_0) : \sup_{t\geq 0} \|y(t)\| < \rho\}.$$

Consider the Nemitsky operator $G$ defined, for $u \in B_{BC(\mathbb{R}^+,C_0)}(0,\rho)$ and $t \geq 0$, by $(Gu)(t) = g(u(t))$. Let $H$ be the operator defined, from $B_{BC(\mathbb{R}^+,C_0)}(0,\rho)$ $\times(B(0, \frac{\rho}{2\delta}) \cap \mathcal{S})$ to $BC(\mathbb{R}^+, C_0)$ and for $t \geq 0$, by

$$H(u,\varphi)(t) = u(t) - V(t)\varphi - [\mathcal{K}^+(Gu)](t).$$

Using the same argument as in Lemma 1.1, Appendix IV in [10], one can show that $G$ has the same properties as $g$. More precisely, we have the following interesting result.

**Lemma 5.6.** *$G$ is continuously differentiable. Moreover, for* $z \in B_{BC(\mathbb{R}^+,C_0)}(0,\rho)$, $h \in BC(\mathbb{R}^+, C_0)$ *and* $t \geq 0$ $(G'(z)h)(t) = g'(z(t))h(t)$.

Consequently, $H$ is continuously differentiable in a neighborhood of $(0,0)$. Moreover, $H(0,0) = 0$ and $\frac{\partial H}{\partial u}(0,0) = I$. By the implicit function theorem, we deduce that there exists $\rho < \rho_0$ such that, for $\varphi \in B(0, \frac{\rho}{2\delta}) \cap \mathcal{S}$, there exists a unique $u^*(\varphi) \in B_{BC(\mathbb{R}^+,C_0)}(0,\rho)$ satisfying

$$H(u^*(\varphi),\varphi) = 0. \tag{5.5}$$

Moreover, the mapping $\varphi \to u^*(\varphi)$ is a diffeomorphism from $B(0, \frac{\rho}{2\delta}) \cap \mathcal{S}$ to $B_{BC(\mathbb{R}^+,C_0)}(0,\rho)$. On the other hand, for $t \geq 0$, $u^*(\varphi)$ satisfies

$$u^*(\varphi)(t) = V(t)\varphi + \lim_{n\to+\infty}\int_0^t V(t-s)P^-\left(\Theta^n\left(g(u^*(\varphi)(s))\right)\right)ds$$
$$- \lim_{n\to+\infty}\int_t^\infty V(t-s)P^+\left(\Theta^n\left(g(u^*(\varphi)(s))\right)\right)ds.$$

We introduce the function

$$v^*(t,\varphi) = \begin{cases} [u^*(\varphi)(t)](0) \text{ for } t \geq 0, \\ \varphi(t) - \left(\lim_{n\to+\infty}\int_0^\infty V(-s)P^+\left(\Theta^n\left(g(u^*(\varphi)(s))\right)\right)ds\right)(t) \\ \text{ for } t \leq 0. \end{cases}$$

Then $v^*(.,\varphi)$ is a bounded solution to equation (1.1) on $\mathbb{R}^+$ and $P^-v_0^*(.,\varphi) = \varphi$. If $t \geq 0$, then, by Theorem 5.1, we get $\|v_t^*(.,\varphi)\| \leq 2\delta e^{-\frac{\mu}{2}t}\|P^-\varphi\|$. Since $\|\varphi\| < \frac{\rho}{2\delta}$, we deduce that $\|v_t^*(.,\varphi)\| < \rho$. Let $Q(\varphi) = v_0^*(.,\varphi)$, for $\varphi \in B(0, \frac{\rho}{2\delta}) \cap \mathcal{S}$. Then $Q$ is a diffeomorphism from $B(0, \frac{\rho}{2\delta}) \cap \mathcal{S}$ to $\widetilde{\mathcal{S}}_\rho(g)$. Furthermore, for $\varphi \in B(0, \frac{\rho}{2\delta}) \cap \mathcal{S}$, we have $P^-Q(\varphi) = \varphi$; that is, $P^- = Q^{-1}$.

Thus, $P^-$ is a diffeomorphism from $\widetilde{\mathcal{S}}_\rho(g)$ to $B(0, \frac{\rho}{2\delta}) \cap \mathcal{S}$. The same reasoning can be used in the case of the local unstable manifold. □

As an immediate consequence of Theorem 5.1, we have the following well-known linearized stability principle which was established in [3].

**Corollary 5.7.** *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H}_3)$ *and* $(\mathbf{H_5})$ *hold. If* $\sigma^+(\mathcal{A}) = \emptyset$, *then the zero solution to equation* (1.1) *is locally exponentially stable; that is, there exist* $\rho_1,\ \mu_1,\ M_1 > 0$ *such that*

$$\|u_t(.,\varphi,g)\| \le M_1 e^{-\mu_1 t}\|\varphi\| \text{ for } t \ge 0 \text{ and } \varphi \in B(0,\rho_1). \tag{5.6}$$

**Proof.** If $\sigma^+(\mathcal{A}) = \emptyset$, then the unstable manifold is reduced to zero and the estimation (5.6) is obtained from Theorem 5.1. □

## 6. Application

For illustration, we propose to study the stability of the model (1.3)

$$\begin{cases} \dfrac{\partial}{\partial t}\left[w(t,x) - qw(t-r,x)\right] = \dfrac{\partial^2}{\partial x^2}\left[w(t,x) - qw(t-r,x)\right] \\ \quad + \displaystyle\int_{-r}^{0} \gamma(\theta) w(t+\theta,x) d\theta + h\left(w(t-r)\right) \text{ for } t \ge 0 \text{ and } x \in [0,\pi], \\ w(t,x) - qw(t-r,x) = 0 \text{ for } x = 0,\ \pi \text{ and } t \ge 0, \\ w(\theta,x) = \psi(\theta,x) \text{ for } \theta \in [-r,0] \text{ and } x \in [0,\pi], \end{cases} \tag{6.1}$$

where $\gamma : [-r,0] \to \mathbb{R}$, $\psi : [-r,0] \times [0,\pi] \to \mathbb{R}$ (the initial value function) and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions and $q$ is a positive constant in $(0,1)$. We give sufficient conditions to show that the local unstable manifold is reduced to 0.

In order to write equation (6.1) in an abstract form, let $X = C([0,\pi];\mathbb{R})$ be the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology. Define the operator $A : D(A) \subset X \to X$ by

$$D(A) = \left\{ y \in C^2([0,\pi];\mathbb{R}) : y(0) = y(\pi) = 0 \right\}, \qquad Ay = y''.$$

**Lemma 6.1.** [9] *The operator* $A$ *satisfies the Hille-Yosida condition on* $X$ :

$$(0,+\infty) \subset \rho(A) \text{ and } \left\| (\lambda I - A)^{-1} \right\| \le \frac{1}{\lambda} \text{ for } \lambda > 0.$$

This lemma implies that condition $(\mathbf{H}_0)$ is satisfied and we also have $\overline{D(A)} = \{y \in X : y(0) = y(\pi) = 0\}$. Let $A_0$ be the part of the operator $A$ in $\overline{D(A)}$. Then $A_0$ is given by

$$\begin{cases} D(A_0) = \left\{ y \in C^2([0,\pi];\mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0 \right\}, \\ A_0 y = y'' \text{ for } y \in D(A_0). \end{cases}$$

Moreover, $A_0$ generates a strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$, which implies that $(\mathbf{H_2})$ holds. Consider the bounded linear operator $\mathcal{D} : C = C([-r,0]\,;X) \to X$ defined by $\mathcal{D}\phi = \phi(0) - q\phi(-r)$. Since $0 < q < 1$, $\mathcal{D}$ is stable and hypothesis $(\mathbf{H}_3)$ holds. Let $L : C \to X$ be the linear operator defined by

$$L(\phi)(x) = \int_{-r}^{0} \gamma(\theta)\phi(\theta)(x)d\theta \text{ for } x \in [0,\pi] \text{ and } \phi \in C.$$

Let also $g : C \longrightarrow X$ be the mapping defined by $g(\phi)(x) = h(\phi(-r)(x))$ for $x \in [0,\pi]$. The initial data $\varphi \in C$ is given by $\varphi(\theta)(x) = \psi(\theta,x)$ for $\theta \in [-r,0]$ and $x \in [0,\pi]$. Thus $L$ is a bounded linear operator from $C$ to $X$. Moreover, if $h$ is a $C^1$ function in a neighborhood of 0 with $h(0) = h'(0) = 0$, it follows that $g$ is a $C^1$ function from a neighborhood of 0 with $g(0) = 0$ and $g'(0) = 0$.

Let $u(t) = w(t,.)$. Then equation (6.1) takes the abstract form

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + g(u_t) \text{ for } t \geq 0, \qquad u_0 = \varphi \in C. \tag{6.2}$$

By Theorem 2.3, for any $\varphi \in C$ such that $\mathcal{D}\varphi \in \{y \in X : y(0) = y(\pi) = 0\}$, there exists a unique solution of equation (6.2) on $[0,+\infty)$. The linearized equation at 0 is then given by

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) \text{ for } t \geq 0, \qquad u_0 = \varphi \in C. \tag{6.3}$$

**Proposition 6.2.** *Under the above condition, if*

$$\int_{-r}^{0} |\gamma(\theta)|\, d\theta < 1 - q,$$

*then, the solution semigroup to equation* (6.3) *is exponentially stable. Consequently, the zero solution of equation* (6.2) *is locally exponentially stable.*

**Proof.** By Corollary 5.7, it is enough to show that $\sigma^+(\mathcal{A}) = \emptyset$. To achieve this goal, we need to compute the characteristic equation. In fact, for $\lambda \in \sigma^+(\mathcal{A})$, the operator $\Delta(\lambda)$ associated to the linear equation (6.3) is defined, for $\vartheta \in D(A)$, by

$$\Delta(\lambda)\vartheta = \lambda(1 - qe^{-\lambda r})\vartheta - A(1 - qe^{-\lambda r})\vartheta - \Big(\int_{-r}^{0} \gamma(\theta)e^{\lambda\theta}d\theta\Big)\vartheta = 0.$$

We proceed by contradiction and suppose that there exists $\lambda \in \sigma^+(\mathcal{A})$ such that $\mathrm{Re}(\lambda) \geq 0$. By Lemma 2.10, there exists $\vartheta \in D(A)\backslash\{0\}$ such that

$\Delta(\lambda)\vartheta = 0$, which is equivalent to

$$A\vartheta = \Big(\lambda - \frac{1}{1-qe^{-\lambda r}}\Big(\int_{-r}^{0}\gamma(\theta)e^{\lambda\theta}d\theta\Big)\Big)\vartheta = 0. \tag{6.4}$$

Therefore,

$$\lambda - \frac{1}{1-qe^{-\lambda r}}\int_{-r}^{0}\gamma(\theta)e^{\lambda\theta}d\theta \in \sigma_p(A),$$

since the spectrum $\sigma(A)$ is reduced to the point spectrum $\sigma_p(A)$ and $\sigma_p(A) = \{-n^2 : n \in \mathbb{N}^*\}$. Thus $\lambda \in \sigma^+(\mathcal{A})$ if and only if $\lambda$ satisfies

$$\lambda - \frac{1}{1-qe^{-\lambda r}}\int_{-r}^{0}\gamma(\theta)e^{\lambda\theta}d\theta = -n^2 \text{ for some } n \in \mathbb{N}^*. \tag{6.5}$$

It follows that

$$\mathrm{Re}(\lambda) \le \frac{1}{|1-qe^{-\lambda r}|}\int_{-r}^{0}|\gamma(\theta)|\, e^{\mathrm{Re}(\lambda)\theta}d\theta - 1,$$

and

$$\mathrm{Re}(\lambda) \le \frac{1}{1-q}\int_{-r}^{0}|\gamma(\theta)|\, e^{\mathrm{Re}(\lambda)\theta}d\theta - 1 < 0,$$

which is impossible since $\mathrm{Re}(\lambda) \ge 0$. Consequently, $\sigma^+(\mathcal{A}) = \emptyset$, the unstable manifold is reduced to 0 and Corollary 5.7 guarantees that 0 is locally exponentially stable for equation (6.2).

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