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# SQUARE ROOT OF ABSOLUTE VALUE OF DERIVATIVE ON A SEGMENT

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**Abstract.** We consider the initial-boundary-value problem for a nonlinear pseudodifferential equation on a segment. We are interested in the case of a nonanalytic symbol of the pseudodifferential operator $K(p) = \sqrt{|p|}$. We study traditionally important problems of the theory of nonlinear partial differential equations, such as global-in-time existence of solutions to the initial-boundary-value problem and the asymptotic behavior of solutions for large time.

## 1. Introduction

In this paper we study the initial-boundary-value problem for a nonlinear equation with a square root of the absolute value of the derivative on a segment

$$\begin{cases} u_t + |u|^{\sigma} u + |\partial_x|^{\frac{1}{2}} u = 0, \quad x \in (0,1), \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in (0,1), \end{cases} \tag{1.1}$$

where $\sigma > 0$ and the operator $|\partial_x|^{\frac{1}{2}}$ on a segment is defined as

$$|\partial_x|^{\frac{1}{2}} \phi = \theta(x) \mathcal{L}^{-1} \left\{ |p|^{\frac{1}{2}} \mathcal{L}\phi \right\}. \tag{1.2}$$

Here and below $\theta(x) = 1$ for $x \in [0,1]$ and $\theta(x) = 0$ for $x \notin [0,1]$, $p^{\beta}$ is the main branch of the complex analytic function in the complex half-plane $\mathrm{Re}(p) \geq 0$ . The direct Laplace transformation $\mathcal{L}$ is

$$\widehat{u}(\xi) \equiv \mathcal{L}u = \int_0^1 e^{-\xi x} u(x)\, dx$$

and the inverse Laplace transformation $\mathcal{L}^{-1}$ is defined by

$$u(x) \equiv \mathcal{L}^{-1}\widehat{u} = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} e^{\xi x} \widehat{u}(\xi)\, d\xi.$$

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We study traditionally important problems of a theory of nonlinear partial differential equations, such as global-in-time existence of solutions to the initial-boundary-value problem (1.1) and the asymptotic behavior of solutions for large time.

Mathematical theory of nonlinear model equations of the form

$$u_t + \mathbb{N}(u) + \mathbb{K}u = 0, \tag{1.3}$$

plays an important role in modern mathematical physics. Here the nonlinear term $\mathbb{N}(u)$ depends on an unknown function $u$ and it's derivatives and the pseudodifferential operator $\mathbb{K}u$ on the whole line $x \in \mathbb{R}$ is given by the inverse Fourier transformation

$$\mathbb{K}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx} K(p)\widehat{u}(p,t)dp;$$

here $\widehat{u}(p,t)$ is the direct Fourier transform of $u(x,t)$. Note that the symbol $K(p)$ usually is a nonanalytic nonhomogeneous function in the complex plane; this fact is acceptable for the Fourier theory. For example, Ott, Sudan and Ostrovskiy ([18], [19]) proposed the following equation in the case of the whole line $x \in \mathbb{R}$ :

$$u_t + uu_x + \alpha u_{xxx} + \int_{-\infty}^{+\infty} \frac{\operatorname{sign}(x-y)u_y\,(y,t)}{\sqrt{|x-y|}}dy = 0;$$

here we choose $K(p) = p\,|p|^{-\frac{1}{2}}$ .

Recently the Cauchy problem for nonlinear evolution equations of the form (1.3) was studied extensively. For example, the existence, uniqueness and some qualitative properties of the solutions to the Cauchy problem for some classes of nonlinear nonlocal dissipative equations were studied in [4] - [6], [10] - [12], [14] - [17] . Large-time asymptotic behavior of solutions to the Cauchy problem for dissipative and dispersive nonlinear nonlocal equations was investigated in the book [2].

The general theory of nonlinear nonlocal equations on a half-line was developed in the book [3], where the pseudodifferential operator $\mathbb{K}$ was defined by the symbol $K(p)$, which is analytic in the right half of the complex plane. Further development of the theory of the book [3] was obtained in the paper [7], where problem (1.1) was studied on a half-line .

In the present paper we study the initial-boundary-value problem (1.1) on a segment. The boundary-value problems on a segment are more natural for applications, however their rigorous mathematical investigation is more complicated (see [13]). For example, it is necessary to answer the question of the well-posedness of the problem, in particular, to determine how many boundary values should be posed in the problem for its solvability and the uniqueness of the solution. After that it is also interesting to study the influence of the boundary data on the qualitative properties of the solution. In spite of the importance and actuality there are few results for the initial-boundary-value problems with pseudodifferential operators on a segment. For example, the papers [8] and [9] considered the equation of the type (1.3) with a symbol $K(p) = p^{\alpha}$, which is analytic in the right half of the complex plane, so the Laplace transformation with some modifications could be applied. As far as we know the case of a nonanalytic symbol $K(p)$ was not studied previously. In the present paper we fill this gap, considering as an example equation (1.1) with a nonanalytic symbol $K(p) = |p|^{\frac{1}{2}}$. Note that the methods of the book [3] and papers [8], [9] and [7] cannot be applied to our problem (1.1) directly. In order to overcome the main difficulty, that the symbol $K(p) = |p|^{\frac{1}{2}}$ is nonanalytic, we introduce some special operators of Cauchy type. Then comparing with the case of the half-line we now need to check that the Laplace transform of the solution should be analytic in the complex plane $\mathbf{C}$ with some special growth restriction at infinity, which leads to a system of two coupled nonhomogeneous Riemann problems. We believe that the method proposed in this paper can be applied also to a wide class of local equations with nonself-adjoint differential operators, where due to the lack of completeness it is difficult to apply the usual Fourier methods if we want to take into account the boundary data.

To state precisely the results of the present paper we give some notation. We denote $\langle t\rangle = \sqrt{1+t^2}$ ,$\{t\} = \frac{t}{\langle t\rangle}$. The weighted Lebesgue space is

$$\mathbf{L}^{q,a}(\mathbf{R}^+) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{\mathbf{L}^{q,a}} < \infty\},$$

where

$$\|\varphi\|_{\mathbf{L}^{q,a}} = \Big( \int_0^1 x^{aq} \left|\varphi\left(x\right)\right|^q dx \Big)^{\frac{1}{q}}$$

for $a > 0$, $1 \le q < \infty$ and $\|\varphi\|_{\mathbf{L}^{\infty}} = \operatorname{ess.sup}_{x\in[0,1]} \left|\varphi\left(x\right)\right|$. We define a linear functional $f$ :

$$f(\phi) = \int_0^1 \phi(y)dy. \tag{1.4}$$

Now we state the main results.

**Theorem 1.1.** *Suppose that the initial data* $u_0 \in \mathbf{L}^{\infty}(\mathbf{0},\mathbf{1})$ *are such that the* $\mathbf{L}^{\infty}$ *norm is sufficiently small,* $\|u_0\|_{\mathbf{L}^{\infty}} \le \varepsilon$. *Then there exists a unique* *global solution* $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^1(\mathbf{0},\mathbf{1})\right) \cap \mathbf{C}\left((0,\infty);\mathbf{L}^\infty(\mathbf{0},\mathbf{1})\right)$ *to the initial-boundary-value problem* (1.1). *Moreover, the following asymptotic is valid in* $\mathbf{L}^\infty(\mathbf{0},\mathbf{1})$ :

$$u = t^{-2}A\Lambda + O\left(t^{-2-\gamma}\right), \tag{1.5}$$

*for* $t \to \infty$, *small* $\gamma > 0$. *The constant* $\Lambda$ *is defined below by the formula* (2.53) *and the constant*

$$A = f(u_0) - \int_0^{+\infty} f\left(\mathcal{N}(u)\right) d\tau, \qquad \mathcal{N}(u) = |u|^\sigma u.$$

## 2. Preliminaries

In subsequent consideration we shall have frequently to use certain theorems of the theory of functions of a complex variable, the statements of which we now quote. The proofs can be found in [1].

**Theorem 2.1.** *Let* $\phi(q)$ *be a complex function, which obeys the Hölder condition for all finite* $q$ *and tends to a definite limit* $\phi_\infty$ *as* $|q| \to \infty$, *such that for large* $q$ *the following inequality holds:*

$$|\phi(q) - \phi_\infty| \le C\,|q|^{-\mu}, \ \mu > 0.$$

*Then the Cauchy-type integral*

$$F(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z}dq$$

*constitutes a function analytic in the left and right semi-planes. Here and below these functions will be denoted* $F^+(z)$ *and* $F^-(z)$, *respectively. These functions have the limiting values* $F^+(p)$ *and* $F^-(p)$ *at all points of the imaginary axis* $Re(p) = 0$, *on approaching the contour from the left and from the right, respectively. These limiting values are expressed by the Sokhotski-Plemelj formula*

$$F^+(p) = \lim_{z\to p, Rez<0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z}dq = \frac{1}{2\pi i}PV\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-p}dq + \frac{1}{2}\phi(p), \tag{2.1}$$

$$F^-(p) = \lim_{z\to p, Rez>0} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z}dq = \frac{1}{2\pi i}PV\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-p}dq - \frac{1}{2}\phi(p). \tag{2.2}$$

Subtracting and adding the formula (2.1) we obtain the following two equivalent formulas:

$$F^+(p) - F^-(p) = \phi(p), \quad F^+(p) + F^-(p) = \frac{1}{\pi i} PV \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-p} dq, \qquad (2.3)$$

which will be frequently employed hereafter.

**Theorem 2.2.** *An arbitrary function $\phi(p)$ given on the contour $Re(p) = 0$, satisfying the Hölder condition, can be uniquely represented in the form*

$$\phi(p) = U^+(p) - U^-(p)$$

*where $U^{\pm}(p)$ are the boundary values of the analytic functions $U^{\pm}(z)$ and the condition $U^{\pm}_{\infty} = 0$ holds. These functions are determined by the formula*

$$U(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z} dq.$$

**Theorem 2.3.** *An arbitrary function $\varphi(p)$ given on the contour $Re(p) = 0$, satisfying the Hölder condition, and having zero index,*

$$\text{ind}\varphi(t) := \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d \ln \varphi(p) = 0,$$

*is uniquely representable as the ratio of the functions $X^+(p)$ and $X^-(p)$, constituting the boundary values of functions, $X^+(z)$ and $X^-(z)$, analytic in the left and right complex semi-plane and having in these domains no zero. These functions are determined to within an arbitrary constant factor and given by the formula*

$$X^{\pm}(z) = e^{\Gamma^{\pm}(z)}, \quad \Gamma(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \ln \varphi(q) dq.$$

Now we consider the following linear initial-boundary-value problem on the half-line

$$\begin{cases} u_t + |\partial_x|^{\frac{1}{2}} = 0, x \in (0,1), \quad t > 0, \\ u(x,0) = u_0(x), \ x \in (0,1), \end{cases} \qquad (2.4)$$

where the fractional derivative operator $|\partial_x|^{\frac{1}{2}}$ is defined in formula (1.2).

Setting

$$K(q) = |q|^{\frac{1}{2}}, \ K_1(q) = q^{\frac{1}{2}}, \qquad (2.5)$$

we define the Green's operator $\mathcal{G}$

$$\mathcal{G}(t)\phi = \theta(x) \int_0^1 G(x,y,t)\phi(y)dy, \ t > 0, \qquad (2.6)$$

where the function $G(x,y,t)$ is given by the formula, for $x \in (0,1)$ , $t>0$,

$$G(x,y,t) = \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} dp e^{p(x-1)}\frac{1}{Y^{+}(p,\xi)} \times \lim_{z\to p, \mathrm{Re} z<0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{q(1-y)}}{q-z}\frac{Y^{+}(q,\xi)}{K(q)+\xi}dq. \tag{2.7}$$

Here and below

$$Y^{\pm} = e^{\Gamma^{\pm}}w^{\pm}, \tag{2.8}$$

$\Gamma^{+}(p,\xi)$ and $\Gamma^{-}(p,\xi)$ are left and right limiting values of a sectionally analytic function $\Gamma(z,\xi)$ given by the formula

$$\Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\Big\{\Big(\frac{K(q)+\xi}{K_1(q)+\xi}\Big)\frac{w^{-}(q)}{w^{+}(q)}\Big\}dq, \tag{2.9}$$

where for some fixed point $z_0$ ($\mathrm{Re}(z_0)>0$)

$$w^{-}(z) = \Big(\frac{z}{z+z_0}\Big)^{\frac{1}{4}}, \quad w^{+}(z) = \Big(\frac{z}{z-z_0}\Big)^{\frac{1}{4}}.$$

All the integrals are understood in the sense of the principal values. Now we prove the following proposition.

**Proposition 2.4.** *Let the initial data* $u_0 \in \mathbf{L}^1(\mathbf{0},\mathbf{1})$. *Then there exists a unique solution* $u(x,t)$ *of the initial-boundary-value problem* (2.4), *which has the integral representation*

$$u(x,t) = \mathcal{G}(t)u_0. \tag{2.10}$$

**Proof.** To derive an integral representation for the solutions of the problem (2.4) we suppose that there exists a solution $u(x,t)$ of problem (2.4 ), which is continued by zero outside of $x \in [0,1]$ .

Let $\phi(p)$ be a function of the complex variable $p$, which obeys the Hölder condition for all finite $p$ and tends to 0 as $p \to \pm i\infty$. We define the operator

$$\mathbb{P}\phi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-z)}-1}{q-z}\phi(q)dq.$$

It is readily observed that $\mathbb{P}\phi(z)$ constitutes a function analytic in the complex plane $z \in \mathbb{C}$ . Also, note that

$$\mathcal{L}^{-1}\mathbb{P}\phi = \theta(x)\mathcal{F}^{-1}\phi. \tag{2.11}$$

We have for the Laplace transform

$$\mathcal{L}\big\{\, |\partial_x|^{\frac{1}{2}}\,\big\} = \mathbb{P}\big\{\, |p|^{\frac{1}{2}}\,\mathcal{L}\{u\}\,\big\}, \qquad \widehat{u}(q,t) = \mathcal{L}\{u\} = \mathbb{P}\widehat{u}(p,t).$$

Therefore, applying the Laplace transform with respect to $x$ to problem (2.4) we obtain for $t > 0$

$$\mathbb{P}\left\{\widehat{u}_t + K(p)\widehat{u}(p,t)\right\} = 0, \qquad \widehat{u}(p,0) = \widehat{u}_0(p), \tag{2.12}$$

where $K(p) = |p|^{\frac{1}{2}}$. We rewrite (2.12) in the form

$$\widehat{u}_t + K(p)\widehat{u}(p,t) = \Phi(p,t), \qquad \widehat{u}(p,0) = \widehat{u}_0(p), \tag{2.13}$$

with some function $\Phi(p,t)$ such that, for all $\mathrm{Re}(p) > 0$,

$$\mathbb{P}\left\{\Phi(p,t)\right\} = 0, \tag{2.14}$$

and for $|p| > 1$, $|\Phi(p,t)| \leq C|p|^{-\frac{1}{2}}$. Applying the Laplace transformation with respect to the time variable to problem (2.13) we find, for $\mathrm{Re}(p) = 0$,

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p)+\xi}\left(\widehat{u}_0(p) + \widehat{\Phi}(p,\xi)\right). \tag{2.15}$$

Here the functions $\widehat{\widehat{u}}(p,\xi)$ and $\widehat{\Phi}(p,\xi)$ are the Laplace transforms for $\widehat{u}(p,t)$ and $\Phi(p,t)$ with respect to time, respectively. We will find the function $\widehat{\Phi}(p,\xi)$ using the analytic properties of the function $\widehat{\widehat{u}}$ in the entire complex plane $p$ and $\mathrm{Re}(\xi) > 0$, such that

$$\widehat{\widehat{u}}(p,\xi) = \mathbb{P}\widehat{\widehat{u}}(q,\xi). \tag{2.16}$$

We introduce $\Phi_1(p,t)$ and $\Phi_2(p,t)$ such that

$$\Phi(p,t) = e^{-p}\Phi_1^+(p,t) - \Phi_2^-(p,t), \tag{2.17}$$

where $\Phi_1^+(p,\xi)$ and $\Phi_2^-(p,\xi)$ are left and right limiting values of a sectionally analytic functions $\Phi_1(z,t)$, $\Phi_2(z,t)$ given by the formulas

$$\Phi_1(z,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^q}{q-z}\Phi(q,\xi)dq,$$
$$\Phi_2(z,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\Phi(q,\xi)dq.$$

In view of the Sokhotski-Plemelj formula via (2.15) the condition (2.16 ) can be written in terms of the limiting boundary values as

$$\Theta_1^-(p,\xi) = -\Lambda_1^-(p,\xi), \qquad \Theta_2^+(p,\xi) = -\Lambda_2^+(p,\xi), \tag{2.18}$$

where the sectionally analytic functions $\Theta_1(z,\xi)$, $\Theta_2(z,\xi)$, $\Lambda_1(z,\xi)$, and $\Lambda_2(z,\xi)$ are given by Cauchy-type integrals

$$\Theta_1(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^q}{q-z}\frac{1}{K(q)+\xi}\widehat{\Phi}(q,\xi)dq, \tag{2.19}$$

$$\Theta_2(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{K(q)+\xi}\widehat{\Phi}(q,\xi)dq$$
$$\Lambda_1(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^q}{q-z}\frac{1}{K(q)+\xi}\widehat{u}_0(q)dq, \tag{2.20}$$
$$\Lambda_2(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{K(q)+\xi}\widehat{u}_0(q)dq.$$

To perform the condition (2.18) in the form of the system of nonhomogeneous Riemann problems we also introduce the sectionally analytic functions

$$\Psi_1(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^q}{q-z}\frac{K(q)}{K(q)+\xi}\widehat{\Phi}(q,\xi)dq, \tag{2.21}$$
$$\Psi_2(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{K(q)}{K(q)+\xi}\widehat{\Phi}(q,\xi)dq.$$

Taking into account the assumed condition (2.14) and making use of the Sokhotski-Plemelj formula (2.1) we get for limiting values of the functions $\Psi_{1,2}(z,\xi)$ and $\Theta_{1,2}(z,\xi)$

$$\Psi_1^+(p,\xi) = \Phi_1^+(p,t) - \xi\Theta_1^+(p,\xi), \tag{2.22}$$
$$\Psi_2^-(p,\xi) = \Phi_2^-(p,t) - \xi\Theta_2^-(p,\xi).$$

Also observe that from (2.19) and (2.21) by formula (2.3)

$$K(p)\left(\Theta_j^+(p,\xi) - \Theta_j^-(p,\xi)\right) = \Psi_j^+(p,\xi) - \Psi_j^-(p,\xi),\ j = 1,2.$$

Substituting (2.18) and (2.22) into this system we obtain two nonhomogeneous Riemann problems

$$\begin{cases} \frac{K(p)+\xi}{\xi}(\Psi_1^+(p,\xi) - \Phi_1^+(p,t)) = \Psi_1^-(p,\xi) + K(p)\Lambda_1^-(p,\xi) - \Phi_1^+(p,t), \\ \Psi_2^+(p,\xi) = \frac{K(p)+\xi}{\xi}\left(\Psi_2^-(p,\xi) - \Phi_2^-(p,t)\right) - K(p)\Lambda_2^+(p,\xi) + \Phi_2^-(p,t). \end{cases} \tag{2.23}$$

It is required to find functions for some fixed point $\xi$, $\mathrm{Re}(\xi) > 0$, $\Psi_j^+(z,\xi)$, $j = 1,2$ , analytic in $\mathrm{Re}(z) < 0$ and $\Psi_j^-(z,\xi)$, analytic in $\mathrm{Re}(z) > 0$, which satisfy on the contour $\mathrm{Re}(p) = 0$ the system of relations (2.23). Here, for some fixed point $\xi$, $\mathrm{Re}(\xi) > 0$.

The method for solving the Riemann problem

$$A^+(p) = \varphi(p)A^-(p) + \phi(p)$$

is based on Theorems 2.2 and 2.3.

First, let us introduce some notation and let us establish certain auxiliary relationships. Setting $K_1(p) = p^{\frac{1}{2}}$ we introduce the function

$$\widetilde{W}(p,\xi) = \Big(\frac{K(p)+\xi}{K_1(p)+\xi}\Big)\frac{w^-(p)}{w^+(p)},$$

where for some fixed point $z_0$ ($\mathrm{Re}(z_0) > 0$)

$$w^-(z) = \Big(\frac{z}{z+z_0}\Big)^{\frac{1}{4}}, \quad w^+(z) = \Big(\frac{z}{z-z_0}\Big)^{\frac{1}{4}}.$$

We make a cut in the plane $z$ from point $z_0$ to point $-\infty$ through 0. Owing to the manner of performing the cut the functions $w^-(z)$, $K_1(z)$ are analytic for $\mathrm{Re}(z) > 0$ and the function $w^+(z)$ is analytic for $\mathrm{Re}(z) < 0$.

We observe that the function $\widetilde{W}(p,\xi)$ given on the contour $\mathrm{Re}(p) = 0$, satisfies the Hölder condition and under the assumption $\mathrm{Re}(K_1(p)) > 0$ does not vanish for any $\mathrm{Re}(\xi) > 0$. Also we have

$$\mathrm{Ind}\widetilde{W}(p,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\ln \widetilde{W}(p,\xi) = 0.$$

Therefore in accordance with Theorem 2.3 the function $\widetilde{W}(p,\xi)$ can be represented in the form of the ratio

$$\widetilde{W}(p,\xi) = \frac{X^+(p,\xi)}{X^-(p,\xi)}, \tag{2.24}$$

where

$$X^{\pm}(p,\xi) = e^{\Gamma^{\pm}(p,\xi)}, \quad \Gamma(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\widetilde{W}(q,\xi)dq.$$

Now we return to the system (2.23). Multiplying and dividing the expression $\frac{K(p)+\xi}{\xi}$ by $\frac{1}{K_1(p)+\xi}\frac{w^-(p)}{w^+(p)}$ and making use of the formula (2.24) we get

$$W(p,\xi) = \frac{K(p)+\xi}{\xi} = \frac{Y^+(p,\xi)}{Y^-(p,\xi)}\Big(\frac{K_1(p)+\xi}{\xi}\Big), \tag{2.25}$$

where

$$Y^{\pm}(p,\xi) = X^{\pm}(p,\xi)w^{\pm}(p).$$

Replacing $W(p,\xi)$ in system (2.23) by (2.25) we reduce (2.23) to the form

$$\begin{aligned}
&(\Psi_1^+(p,\xi) - \Phi_1^+)Y^+(p,\xi) = \frac{\xi}{K_1(p)+\xi}(\Psi_1^-(p,\xi) - \Lambda_1^-\xi)Y^-(p,\xi) + \xi g_1,\\
&\frac{\Psi_2^+(p,\xi)}{Y^+(p,\xi)} = \Big(\frac{K_1(p)+\xi}{\xi}\Big)\frac{\Psi_2^-(p,\xi) - \Phi_2^-}{Y^-(p,\xi)} + g_2,
\end{aligned} \tag{2.26}$$

where the functions $g_1(p,\xi)$ and $g_2(p,\xi)$ are defined as

$$\begin{aligned} g_1(p,\xi) &= \Big(\Lambda_1^-(p,\xi) - \frac{1}{K(p)+\xi}\Phi_1^+\Big)Y^+(p,\xi), \\ g_2(p,\xi) &= \frac{1}{Y^+(p,\xi)}\left(\Phi_2^- - K(p)\Lambda_2^+(p,\xi)\right). \end{aligned} \tag{2.27}$$

In subsequent consideration we shall have to use the following property of the limiting values of a Cauchy type integral, the statement of which we now quote. The proofs may be found in [1].

**Lemma 2.5.** *If $L$ is a smooth closed contour and $\phi(q)$ a function that satisfies the Hölder condition on $L$, then the limiting values of the Cauchy type integral*

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{1}{q-z}\phi(q)dq$$

*also satisfy this condition.*

On basis of this lemma the functions $g_j(p,\xi), j = 1,2$ satisfy a Hölder condition. Therefore in accordance with Theorem 2.2 it can be uniquely represented in the form of the difference of the functions $U_j^+(p,\xi)$ and $U_j^-(p,\xi)$, constituting the boundary values of the analytic function $U_j(z,\xi)$, given by the formula

$$U_j(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z} g_j(q,\xi)dq. \tag{2.28}$$

Therefore, the system (2.26) takes the form

$$\begin{aligned} &(\Psi_1^+(p,\xi) - \Phi_1^+)Y^+(p,\xi) - \xi U_1^+(p,\xi) \\ &\quad = \frac{\xi}{K_1(p)+\xi}(\Psi_1^-(p,\xi) - \Lambda_1^-\xi)Y^-(p,\xi) - \xi U_1^-(p,\xi), \\ &\frac{\Psi_2^+(p,\xi)}{Y^+(p,\xi)} - U_2^+(p,\xi) = \frac{K_1(p)+\xi}{\xi}\frac{\Psi_2^-(p,\xi) - \Phi_2^-}{Y^-(p,\xi)} - U_2^-(p,\xi). \end{aligned}$$

Every equation of the last system indicates that the function, analytic in $\mathrm{Re}(z) < 0$, and the function, analytic in $\mathrm{Re}(z) > 0$, constitute the analytic continuation of each other through the contour $\mathrm{Re}(z) = 0$. Consequently, they are branches of a unique analytic function in the complex plane. Moreover, this function has a zero at infinity. According to Liouville's theorem this function is identically zero. Thus, bearing in mind the relation (2.25) by formula (2.5) we get

$$
\begin{aligned}
e^p\frac{\Phi}{K(p)+\xi} &= \frac{1}{K(p)}(\Psi_1^+(p,\xi)-\Psi_1^-(p,\xi))\\
&= \frac{\Phi_1^+}{K(p)+\xi}+\frac{1}{Y^+}\big(\widetilde{\Lambda}_1^-(q,\xi)+V_1^-\big),\\
\frac{\Phi}{K(p)+\xi} &= \frac{1}{K(p)}(\Psi_2^+(p,\xi)-\Psi_2^-(p,\xi))\\
&= -\frac{\Phi_2^-}{K(p)+\xi}+\frac{Y^+}{K(p)+\xi}(V_2^+-\widetilde{\Lambda}_2^+(q,\xi)),
\end{aligned}\tag{2.29}
$$

where $V_1^-$, $V_2^+$, $\widetilde{\Lambda}_1^-(p,\xi)$ and $\widetilde{\Lambda}_2^+(p,\xi)$ are left and right limiting values of the sectionally analytic functions $V_1(z,t)$, $V_2(z,t)$, $\widetilde{\Lambda}_1(z,t)$ and $\widetilde{\Lambda}_2(z,t)$ given by the formulas

$$
\begin{aligned}
V_1(z,\xi) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{Y^+}{K(q)+\xi}\Phi_1^+dq, \qquad (2.30)\\
V_2(z,\xi) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{\Phi_2^-}{Y^+}dq,\\
\widetilde{\Lambda}_1(z,\xi) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^q}{q-z}\frac{Y^+}{K(q)+\xi}u_0(q)dq,\\
\widetilde{\Lambda}_2(z,\xi) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{u_0(q)}{Y^+}dq.
\end{aligned}
$$

From (2.29) taking into account (2.17) we get

$$
\begin{aligned}
e^p\frac{\Phi_2^-}{K(p)+\xi} &= -\frac{1}{Y^+}\left(\widetilde{\Lambda}_1^-(p,\xi)+V_1^-\right),\\
\frac{e^{-p}\Phi_1^+}{K(p)+\xi} &= \frac{Y^+}{K(p)+\xi}(V_2^+-\widetilde{\Lambda}_2^+(q,\xi)).
\end{aligned}\tag{2.31}
$$

We now proceed to find the unknown functions $\widehat{\Phi}_1^+(p,\xi)$ and $\widehat{\Phi}_2^-(p,\xi)$ involved in the formula (2.15) for the solution $\widehat{\widehat{u}}(p,\xi)$ of the problem (2.4). With the help of the integral representations (2.30) for the sectionally analytic functions $V_j(z,\xi)$ and $\widetilde{\Lambda}_j(z,\xi)$, making use of the Sokhotski-Plemelj formula (2.1) we get

$$
\Phi_2^- = Y^+(V_2^+-V_2^-), \qquad \frac{\Phi_1^+}{K(p)+\xi}=\frac{1}{Y^+}(V_1^+-V_1^-). \tag{2.32}
$$

Substituting (2.32) into the system (2.31) we obtain

$$\begin{aligned} e^p \frac{Y^+}{K(p)+\xi}(V_2^+(p,\xi) - V_2^-(p,\xi)) &= -\frac{1}{Y^+}\big(\widetilde{\Lambda}_1^-(p,\xi) + V_1^-\big), \\ \frac{e^{-p}}{Y^+}(V_1^+(p,\xi) - V_1^-(p,\xi)) &= \frac{Y^+}{K(p)+\xi}(V_2^+ - \widetilde{\Lambda}_2^+(,\xi)). \end{aligned} \tag{2.33}$$

Expressing from the first equation of the system (2.33) the term $\frac{e^{-p}V_1^-}{Y^+}$

$$\frac{e^{-p}V_1^-}{Y^+} = -\frac{Y^+}{K(p)+\xi}(V_2^+(p,\xi) - V_2^-(p,\xi)) - \frac{e^{-p}}{Y^+}\widetilde{\Lambda}_1^-(q,\xi)$$

and substituting it in the second equation via (2.25) we get the following relation:

$$\frac{e^{-p}}{Y^+}V_1^+(p,\xi) - \Omega^+ = \frac{Y^-}{K_1(p)+\xi}V_2^-(p,\xi) - \Omega^-, \tag{2.34}$$

where

$$\Omega(z) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\Big(\frac{e^{-q}}{Y^+}\widetilde{\Lambda}_1^-(q,\xi) + \frac{Y^+}{K(p)+\xi}\widetilde{\Lambda}_2^+(q,\xi)\Big).$$

From (2.34) by Liouville's theorem

$$V_1^+(p,\xi) = e^p Y^+ \Omega^+ \tag{2.35}$$

and

$$V_2^-(p,\xi) = \frac{K_1(p)+\xi}{Y^-}\Omega^-. \tag{2.36}$$

Now substituting (2.35) into the second equation of system (2.33) we get

$$Y^+V_2^+ = -e^{-p}\frac{K_1(p)+\xi}{Y^-}V_1^-(p,\xi) + Y^+\widetilde{\Lambda}_2^+(p,\xi) + \frac{K_1(p)+\xi}{Y^-}Y^+\Omega^+.$$

Replacing by the difference $\Theta^+ - \Theta^-$ the term $\frac{K_1(p)+\xi}{Y^-}Y^+\Omega^+ = (K(p)+\xi)\Omega^+$, where

$$\Theta(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}(K(q)+\xi)\Omega^+ dq,$$

according to Liouville's theorem we obtain

$$V_2^+ = \widetilde{\Lambda}_2^+(p,\xi) + \frac{1}{Y^+}\Theta^+, \qquad e^{-p}V_1^-(p,\xi) = -\frac{Y^-}{K_1(p)+\xi}\Theta^-. \tag{2.37}$$

Having determined the functions $V_1^{\pm}, V_2^{\pm}$ by (2.35)-(2.37) and bearing in mind formulas (2.32) and (2.17) we determine the required function $\frac{\Phi}{K(p)+\xi}$ :

$$\frac{\Phi}{K(p)+\xi} = \frac{e^{-p}\Phi_1^+}{K(p)+\xi} - \frac{\Phi_2^-}{K(p)+\xi} \tag{2.38}$$
$$= \Omega^+ + \Omega^- - \frac{1}{K(p)+\xi}\left(\Theta^+ - \Theta^-\right) - \frac{Y^+}{K(p)+\xi}\widetilde{\Lambda}_2^+(q,\xi) = -\Omega_1^- - \Omega_2^+,$$

where

$$\Omega_1(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-q}}{q-z}\frac{1}{Y^+}\widetilde{\Lambda}_1^-(q,\xi)dq, \tag{2.39}$$
$$\Omega_2(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{q-z}\frac{Y^+}{K(p)+\xi}\widetilde{\Lambda}_2^+(q,\xi)dq.$$

Here $\widetilde{\Lambda}_1^-(q,\xi)$ and $\widetilde{\Lambda}_2^+(q,\xi)$ was defined by (2.30). We now return to the Laplace transformation $\widehat{\widehat{u}}(p,\xi)$ of the solution $u(x,t)$ of the problem (2.4). Substituting (2.38) into formula (2.15) we get

$$\widehat{\widehat{u}}(p,\xi) = \frac{1}{K(p)+\xi}\widehat{u}_0(p) - \Omega_1^- - \Omega_2^+. \tag{2.40}$$

Now we prove that, in accordance with the last relation, the function $\widehat{\widehat{u}}(p,\xi)$ is an analytic function in $\mathbb{C}$.

With the help of the integral representations (2.39), (2.30) after making use of the Sokhotski-Plemelj formula (2.1) we have

$$\Omega_1^- = \lim_{z\to p,\mathrm{Re}z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-q}}{q-z}\frac{1}{Y^+}\widetilde{\Lambda}_1^-(q,\xi)dq$$
$$= \lim_{z\to p,\mathrm{Re}z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-q}}{q-z}\frac{1}{Y^+}\widetilde{\Lambda}_1^+(q,\xi)dq$$
$$- \lim_{z\to p,\mathrm{Re}z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{u_0(q)}{K(q)+\xi}dq$$

and after making use of (2.25) by Cauchy's theorem

$$\Omega_2^+ = \lim_{z\to p,\mathrm{Re}z<0}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{Y^+}{K(p)+\xi}\widetilde{\Lambda}_2^+(q,\xi)dq$$
$$= \lim_{z\to p,\mathrm{Re}z<0}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{Y^-}{K_1(p)+\xi}\widetilde{\Lambda}_2^-(q,\xi)dq$$

$$+\lim_{z\to p,\mathrm{Re}z<0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{u_0(q)}{K(q)+\xi}dq$$
$$=\lim_{z\to p,\mathrm{Re}z<0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{u_0(q)}{K(q)+\xi}dq.$$

Substituting this relation into (2.40) we get

$$\widehat{\widehat{u}}(p,\xi)=-\lim_{z\to p,\mathrm{Re}z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-q}}{q-z}\frac{1}{Y^+}\widetilde{\Lambda}_1^+(q,\xi)dq \tag{2.41}$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{q-p}-1}{q-p}\frac{e^{-q}}{Y^+}\widetilde{\Lambda}_1^+(q,\xi)dq=\mathbb{P}\Big\{\frac{e^{-q}}{Y^+}\widetilde{\Lambda}_1^+(q,\xi)\Big\}.$$

Here we are using the fact that, by Cauchy's theorem,

$$\lim_{z\to p,\mathrm{Re}z>0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+}\widetilde{\Lambda}_1^+(q,\xi)dq=0.$$

Thus the function $\widehat{\widehat{u}}\,(p,\xi)$ is an analytic function in the complex plane for $p\in\mathbb{C}$. Moreover, $\widehat{\widehat{u}}=\mathbb{P}\{\widehat{\widehat{u}}\}$, and, as a consequence, its inverse Laplace transform vanishes for all $x\notin[0,1]$. We now return to the solution $u(x,t)$ of the problem (2.4).

Via (2.11) taking the inverse Laplace transform with respect to time and the inverse Fourier transform with respect to space variables we obtain

$$u(x,t)=\mathcal{G}(t)u_0=\theta(x)\int_0^1 G(x,y,t)u_0(y)dy,$$

where the function $G(x,y,t)$ was defined by formula (2.7). Proposition 2.4 is proved. $\square$

Now we collect some preliminary estimates of the Green's operator $\mathcal{G}\,(t)$ (see 2.6). Let the contours $\mathcal{C}_i$ be defined as

$$\mathcal{C}_1=\Big\{p\in\Big(\infty e^{-i(\frac{\pi}{2}+\varepsilon)},0\Big)\bigcup\Big(0,\infty e^{i(\frac{\pi}{2}+\varepsilon)}\Big)\Big\}, \tag{2.42}$$

$$\mathcal{C}_2=\Big\{q\in\Big(\infty e^{-i(\frac{\pi}{2}+2\varepsilon)},0\Big)\bigcup\Big(0,\infty e^{i(\frac{\pi}{2}+2\varepsilon)}\Big)\Big\}, \tag{2.43}$$

$$\mathcal{C}_3=\Big\{q\in\Big(\infty e^{-i(\frac{\pi+\varepsilon}{2})},0\Big)\bigcup\Big(0,\infty e^{i(\frac{\pi+\varepsilon}{2})}\Big)\Big\}, \tag{2.44}$$

where $\varepsilon>0$ can be chosen such that all functions under integration are analytic.

**Lemma 2.6.** *The estimates are true, provided that the right-hand sides are finite,*

$$\left\|\mathcal{G}\left(t\right)\phi\right\|_{\mathbf{L}^{s}} \leq Ct^{-2\left(\frac{1}{r}-\frac{1}{s}\right)}\left\|\phi(\cdot)\right\|_{\mathbf{L}^{r}}, \tag{2.45}$$

$$\left\|\mathcal{G}\phi\right\|_{\mathbf{L}^{\infty}} \leq Ct^{-2}\left\|\phi\right\|_{\mathbf{L}^{1}}, \qquad \left\|\mathcal{G}\phi\right\|_{\mathbf{L}^{\infty}} \leq Ct^{-\gamma}\left\|\phi\right\|_{\mathbf{L}^{\infty}},$$

$$\mathcal{G}\left(t\right)\phi - t^{-2}\Lambda f(\phi) = t^{-2-a}\left\|\phi\right\|_{\mathbf{L}^{1,\,\frac{a}{2}}}, \tag{2.46}$$

*where* $a \in (0,1)$, $\gamma > 0$ *is small,* $s$, $r \geq 1$, *and* $f(\phi)$ *is given by* (1.4). *The constant* $\Lambda$ *is defined below* (*see* (2.53)).

**Proof.** For subsequent considerations it is required to investigate the behavior of the function $\Gamma(z,\xi)$,which is defined in (2.9). Set

$$\phi(p,\xi) = \ln\Big\{\Big(\frac{K(p)+\xi}{K_1(p)+\xi}\Big)\frac{w^-(p)}{w^+(p)}\Big\} \neq 0,\ \operatorname{Re}(p)=0,\quad \operatorname{Re}(\xi)<0.$$

Observe that the function $\phi(p,\xi)$ obeys the Hölder condition for all finite $p$ and tends to a definite limit $\phi(\infty,\xi)$ as $p \to \pm i\infty$, $\phi(\infty,\xi) = -i\frac{\pi}{4}$. Also it can be easily obtained that for large $p$ and some fixed $\xi$ the following inequality holds:

$$|\phi(p,\xi) - \phi(\infty,\xi)| \leq C\Big(\frac{|\xi|^{\mu}}{|p|^{2\mu}}\Big), \quad \mu > 0. \tag{2.47}$$

Therefore, $\left|e^{\Gamma^{\pm}(z,\xi)}\right| \leq C$ for all $\xi \in \mathcal{C}_1$ and $\operatorname{Re}(z) \neq 0$. Moreover, we get the following asymptotics:

$$\begin{aligned}\Gamma(z,\xi) &= \frac{1}{8} + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\ln\Big\{\frac{(1+\frac{\xi}{K(q)})}{(1+\frac{\xi}{K_1(q)})}(1+O(q))\Big\}dq \\ &= \frac{1}{8} + A(\xi,z) + O\Big\{\Big[\frac{\xi}{K(q)}\Big]^{1+\gamma}\Big\},\end{aligned} \tag{2.48}$$

where

$$A(\xi,z) = \frac{\xi}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{K_1(q)-K(q)}{K(q)K_1(q)}dq.$$

Now making the changes of variables $\xi t = \xi_1$, $p = p_1 t^{-2}$, $q = q_1 t^{-2}$ in the formula (2.7) we get

$$\begin{aligned}G(x,y,t) &= \frac{1}{2\pi i}\frac{1}{2\pi i}\theta(x)\theta(y)t^{-2}\int_{C_1} d\xi_1 e^{\xi_1} \\ &\times \int_{-i\infty}^{i\infty} dp_1 e^{p_1\widetilde{x}}\frac{1}{Y^+(p_1,\xi_1)}I^+(p_1,\xi_1,\widetilde{y})dp_1,\end{aligned} \tag{2.49}$$

where $\widetilde{x} = \frac{x-1}{t^2}$ and $\widetilde{y} = \frac{1-y}{t^2}$ and

$$I(z, \xi, y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{q(1-y)}}{q-z} \frac{Y^+(q,\xi)}{K(q)+\xi} dq. \tag{2.50}$$

Replacing by the difference $I^- - e^{p\widetilde{y}} \frac{Y^+(p_1,\xi_1)}{K(p_1)+\xi_1}$ the term $I^+(p_1, \xi_1, \widetilde{y})$ in (2.49) and taking into account the asymptotic formula (2.48), by Cauchy's theorem we have $G(x, y, t) = I_1 + I_2$, where

$$\begin{aligned} I_1(x, y, t) = \frac{1}{2\pi i} \frac{1}{2\pi i} \theta(x)\theta(y) t^{-2} \int_{C_1} d\xi e^{\xi_1} \int_{-i\infty}^{i\infty} dp_1 (e^{p_1 \widetilde{x}} - 1) \\ \times \Big( \frac{1}{Y^+(p_1, \xi_1)} - 1 - A^+(\xi_1, p_1) \Big) I^+(p_1, \xi_1, \widetilde{y}) dp_1 \end{aligned} \tag{2.51}$$

and

$$\begin{aligned} I_2(x, y, t) = -\frac{\theta(x)\theta(y)}{4\pi^2} t^{-2} \int_{C_1} d\xi e^{\xi_1} \xi_1 \\ \times \int_{-i\infty}^{i\infty} dp_1 e^{p_1 \widetilde{x}} \left(1 + A^+(\xi_1, p_1)\right) I^+(p_1, \xi_1, \widetilde{y}) dp_1. \end{aligned} \tag{2.52}$$

Via Lemma 2.5 the integral we are interested in, in (2.50), takes the form

$$I^+(p_1, \xi_1, \widetilde{y}) = O(p_1^{\frac{1}{2}}).$$

It follows from the last relation and (2.48) that an estimate for the integral $I_1$ (see (2.51)) is

$$|I_1| \le C t^{-2} |\widetilde{x}|^{\gamma} \int_{C_1} d\xi_1 e^{\mathrm{Re}\xi_1} |\xi_1|^{1+\gamma} \int_{-i\infty}^{i\infty} dp_1 |p_1|^{\gamma - \frac{1}{2}} O(\xi^{\gamma} p_1^{-\frac{\gamma+1}{2}}) dp_1 = O(t^{-2-\gamma}).$$

Bearing in mind the formula (2.48) we get the following estimate for the integral $I_2$ (see (2.52)):

$$\begin{aligned} & I_2(x, y, t) = \theta(x)\theta(y) \Big( t^{-2} \Lambda \\ & + t^{-2-\gamma} O\Big( \int_{C_1} d\xi e^{\mathrm{Re}\xi_1} \int_{-i\infty}^{i\infty} dp_1 \frac{|p_1|^{\gamma}}{|K(p_1)+\xi|} O(\frac{\xi^{1+\gamma}}{K(p_1)^{1+\gamma}}) dp_1 \Big) \Big) \\ = \ & \theta(x)\theta(y) \left( t^{-2} \Lambda + O(t^{-2-\gamma}(1-y)^{\gamma}) \right), \end{aligned}$$

where $\gamma > 0$ is small and the constant $\Lambda$ is defined as

$$\Lambda = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-K(p)} \Big( e^{\frac{1}{8}} + \frac{K(p)}{2\pi i} \lim_{z \to p, \mathrm{Re} z < 0} \int_{-i\infty}^{i\infty} \frac{1}{q-z} \Big( \frac{1}{K(q)} - \frac{1}{K_1(q)} \Big) dq \Big) dp. \tag{2.53}$$

Collecting the derived estimates for $I_1$ and $I_2$ we verify the validity of the asymptotic

$$G(x,y,t) = \theta(x)\theta(y)\big(\frac{1}{t^2}\Lambda + O(t^{-2-\gamma}(1-y)^{\gamma})\big).$$

Moreover, obviously,

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^\infty} \le Ct^{-2}\|\phi\|_{\mathbf{L}^1}.$$

Now we get an estimate for small time $t<1$ in the $L^\infty$ norm. After integrating by parts with respect to $q$ the variable formula (2.50) takes the form

$$\begin{aligned} I(z,\xi,y) = &-\frac{1}{2\pi i}\frac{1}{1-y}\int_{-i\infty}^{i\infty}\frac{(e^{q(1-y)}-1)Y^-(q,\xi)}{q-z} \\ &\times\Big(\frac{1}{q-z} - \frac{1}{(K_1(q)+\xi)^2} + \Gamma_q^-(q,\xi)\Big)dq, \end{aligned}$$

where

$$\Gamma_z(z,\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dq}{q-z}\Big(\frac{K'(q)}{K(q)+\xi} - \frac{K_1'(q)}{K_1(q)+\xi}\Big) = O(z^{-1}).$$

Thus the left limiting value can be estimated as

$$\begin{aligned} I^+(p,\xi,y) &= -\frac{1}{2\pi i}\frac{1}{1-y}\lim_{z\to p,\mathrm{Re}z<0}\int_{-i\infty}^{i\infty}\frac{(e^{q(1-y)}-1)Y^-(q,\xi)}{q-z}\phi(q,z,\xi)dq \\ &= (1-y)^{-1+\gamma}O(p^{-1+\gamma}), \end{aligned}$$

where

$$\phi(q,z,\xi) = \frac{1}{q-z} - \frac{1}{(K_1(q)+\xi)^2} + \Gamma_q^-(q,\xi).$$

Substituting the derived estimate for $I^+$ into (2.51) we get, for $x\in[0,1]$, $y\in[0,1]$,

$$|I_1(x,y,t)| \le C(1-y)^{-1+\gamma}t^{-\gamma}\int_{C_1}d\xi e^{\xi_1}\int_{-i\infty}^{i\infty}O(\langle p\rangle^{-1-\gamma})dp.$$

In an entirely similar way we obtain the estimate for (2.52) for $x\in[0,1]$, $y\in[0,1]$,

$$\begin{aligned} &|I_2(x,y,t)| \\ &= -\frac{1}{4\pi^2}t^{-2}\Big|\int_{C_1}d\xi e^{\xi_1}\xi_1\int_{-i\infty}^{i\infty}dp_1 e^{p_1\widetilde{x}}\left(1+A^+(\xi_1,p_1)\right)I^+(p_1,\xi_1,\widetilde{y})dp_1\Big| \end{aligned}$$

$$\leq C\Big|\int_{-i\infty}^{i\infty} dp_1 e^{p_1(x-y)-K(p)t}dp_1\Big|$$
$$+ (1-y)^{-1+\gamma}t^{-\gamma}O\Big(\int_{C_1} d\xi e^{\xi_1}\int_{-i\infty}^{i\infty} O(\langle p\rangle^{1+\gamma})dp\Big)$$
$$\leq C(1-y)^{-1+\gamma}t^{-\gamma}y^{-1+\gamma}.$$

Collecting the derived estimates for $I_1$ and $I_2$ we verify the validity of the following estimate:

$$\|\mathcal{G}\phi\|_{\mathbf{L}^\infty} \leq Ct^{-\gamma}\left\|(1-y)^{-1+\gamma}y^{-1+\gamma}\phi\right\|_{\mathbf{L}^1} \leq Ct^{-\gamma}\|\phi\|_{\mathbf{L}^\infty}.$$

Now we estimate $\|\mathcal{G}(t)\phi\|_{\mathbf{L}^s}$ for $t<1$. We have for $l>0$

$$\left\|e^{-C|q|\cdot}\right\|_{L^l} = \Big(\int_0^1 e^{-C|q|ly}dy\Big)^{\frac{1}{l}} \leq C\left|e^{-C|q|}-1\right|^{\frac{1}{l}}|q|^{-\frac{1}{l}}.$$

Therefore, in accordance with (2.7) by Hölder's inequality we have arrived at the following estimate for $s\geq r>1$, $l^{-1}=1-r^{-1}$ :

$$\left\|\theta(x)\int_0^1 G(\cdot,y,t)\phi(y)dy\right\|_{\mathbf{L}^s}$$
$$\leq C\int_{\mathcal{C}_3} e^{-C|\xi|t}\int_{\mathcal{C}_1} dp\left|e^{-C|p|}-1\right|^{\frac{1}{s}}|p|^{-\frac{1}{s}}\int_{\mathcal{C}_2}\frac{dq}{|q-p|}\left\|e^{-C|q|\cdot}\right\|_{\mathbf{L}^l}\|\phi\|_{\mathbf{L}^r}$$
$$\leq C\int_{\mathcal{C}_3} d\xi e^{-C|\xi|t}\int_{\mathcal{C}_1} dp\left|e^{-C|p|}-1\right|^{\frac{1}{s}}|p|^{-\frac{1}{s}}\int_{\mathcal{C}_2} dq\frac{\left|e^{-C|q|}-1\right|^{1-\frac{1}{r}}}{|q-p|}\frac{|q|^{-1+\frac{1}{r}}}{|K(q)+\xi|}$$
$$\leq Ct^{-2(\frac{1}{r}-\frac{1}{s})}\|\phi(\cdot)\|_{\mathbf{L}^r}.$$

The lemma is proved. □

## 3. Proof of Theorem 1.1

By Proposition 2.4 we rewrite the initial-boundary-value problem (1.1) as the following integral equation:

$$\begin{aligned} u(t) &= \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau, \\ \mathcal{N}(u) &= |u|^\sigma u, \end{aligned} \tag{3.1}$$

where $\mathcal{G}$ is the Green's operator of the linear problem (2.4). We choose the space $\mathbf{Z}=\{\phi\in\mathbf{L}^\infty(\mathbf{0},\mathbf{1})\}$ and the space

$$\mathbf{X} = \left\{\phi\in\mathbf{C}\left([0,\infty);\mathbf{L}^1(\mathbf{0},\mathbf{1})\cap(0,\infty);\mathbf{L}^\infty(\mathbf{0},\mathbf{1})\right):\ \|\phi\|_{\mathbf{X}}<\infty\right\},$$

where now the norm for small $\gamma > 0$,

$$\|\phi\|_{\mathbf{X}} = \sup_{t\geq 0}\left(\left(\{t\}^{\gamma}\langle t\rangle^{2}\|\phi(t)\|_{\mathbf{L}^{\infty}} + \langle t\rangle^{2}\|\phi(t)\|_{\mathbf{L}^{1}}\right)\right),$$

reflects the optimal time decay properties of the solution. We apply the contraction mapping principle in a ball

$$\mathbf{X}_{\rho} = \{\phi \in \mathbf{X} : \ \|\phi\|_{\mathbf{X}} \leq \rho\}$$

in the space $\mathbf{X}$ of a radius

$$\rho = \frac{1}{2C}\|u_0\|_{\mathbf{Z}} > 0.$$

For $v \in \mathbf{X}_{\rho}$ we define the mapping $\mathcal{M}(u)$ by the formula

$$\mathcal{M}(u) = \mathcal{G}(t)\,u_0 - \int_0^t \mathcal{G}(t-\tau)\,\mathcal{N}(u(\tau))\,d\tau. \tag{3.2}$$

We first prove that $\|\mathcal{M}(u)\|_{\mathbf{X}} \leq \rho$, where $\rho > 0$ is sufficiently small. By virtue of Lemma 2.6 we have

$$\|\mathcal{G}(t)\,\phi\|_{\mathbf{L}^{1}} \leq \|\phi\|_{\mathbf{L}^{1}}, \qquad \|\mathcal{G}\phi\|_{\mathbf{L}^{\infty}} \leq Ct^{-2}\|\phi\|_{\mathbf{L}^{1}}.$$

Therefore, for $t > 0$,

$$\|\mathcal{G}u_0\|_{\mathbf{X}} \leq C\|u_0\|_{\mathbf{Z}}. \tag{3.3}$$

Also since $v \in \mathbf{X}_{\rho}$ we get for all $\tau > 0$

$$\|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1}} \leq C\|u\|_{\mathbf{L}^{\infty}}^{\sigma}\|u\|_{\mathbf{L}^{1}} \leq C\langle\tau\rangle^{-2(\sigma+1)}\{\tau\}^{-\gamma\sigma}\|u\|_{\mathbf{X}}^{\sigma+1}.$$

Therefore by Lemma 2.6 we get for $p = 1, \infty$

$$\begin{aligned}
&\left\|\int_0^t \mathcal{G}(t-\tau)\,\mathcal{N}(u(\tau))\,d\tau\right\|_{\mathbf{L}^{p}} \\
&\leq \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-2}\|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{1}}\,d\tau + \int_{\frac{t}{2}}^{t}(t-\tau)^{-\gamma}\|\mathcal{N}(u(\tau))\|_{\mathbf{L}^{\infty}}\,d\tau \\
&\leq C\|u\|_{\mathbf{X}}^{\sigma+1}\langle t\rangle^{-2}\{t\}^{-\gamma(1-\frac{1}{p})},
\end{aligned}$$

for all $t \geq 0$. Thus we get

$$\left\|\int_0^t \mathcal{G}(t-\tau)\,\mathcal{N}(u(\tau))\,d\tau\right\|_{\mathbf{X}} \leq C\|u\|_{\mathbf{X}}^{\sigma+1}, \tag{3.4}$$

hence in view of (3.2) , (3.3) and (3.4),

$$\|\mathcal{M}(u)\|_{\mathbf{X}} \leq \|\mathcal{G}u_0\|_{\mathbf{X}} + \left\|\int_0^t \mathcal{G}(t-\tau)\,\mathcal{N}(u(\tau))\,d\tau\right\|_{\mathbf{X}}$$

$$\leq C \left\| u_0 \right\|_{\mathbf{Z}} + C \left\| u \right\|_{\mathbf{X}}^{\sigma+1} \leq \tfrac{\rho}{2} + C\rho^{\sigma+1} < \rho,$$

since $\rho > 0$ is sufficiently small. Hence the mapping $\mathcal{M}$ transforms a ball $\mathbf{X}_\rho$ into itself. In the same manner we estimate the difference

$$\left\| \mathcal{M}(w) - \mathcal{M}(v) \right\|_{\mathbf{X}} \leq \tfrac{1}{2} \left\| w - v \right\|_{\mathbf{X}},$$

which shows that $\mathcal{M}$ is a contraction mapping. Therefore we see that there exists a unique solution $u \in \mathbf{X}$ to the initial-boundary-value problem (1.1). Now we can prove the asymptotic formula

$$u(x,t) = A_1 \Lambda t^{-2} + O\left(t^{-2-\gamma}\right), \tag{3.5}$$

where $\gamma > 0$ is small , the constant $\Lambda$ was defined by (2.30) and the constant $A_1$ is

$$A_1 = f(u_0) - \int_0^{+\infty} d\tau \int_0^{+\infty} |u|^\sigma u dy.$$

Denote $G_0(t) = t^{-2}\ \Lambda$. From Lemma 2.6 we have

$$t^2 \left\| \mathcal{G}(t)\phi - G_0(t) f(\phi) \right\|_{\mathbf{L}^\infty} \leq C \left\| \phi \right\|_{\mathbf{Z}} \tag{3.6}$$

for all $t > 1$. Also in view of the definition of the norm $\mathbf{X}$ we have

$$\left| f\left(\mathcal{N}\left(u(\tau)\right)\right) \right| \leq \left\| \mathcal{N}\left(u(\tau)\right) \right\|_{\mathbf{L}^1} \leq C \left\{\tau\right\}^{-\gamma} \langle \tau \rangle^{-2(\sigma+1)} \left\| u \right\|_{\mathbf{X}}^{\sigma+1}.$$

By a direct calculation we have for some small $\gamma_1 > 0$, $\gamma > 0$,

$$\begin{aligned} &\left\| \int_0^{\frac{t}{2}} e^{-\tau} \left| G_0(t-\tau) - G_0(t) \right| f\left(\mathcal{N}\left(u(\tau)\right)\right) d\tau \right\|_{\mathbf{L}^\infty} \\ &\leq \langle t \rangle^{-2} C \left\| u \right\|_{\mathbf{X}}^{\sigma+1} \int_0^{\frac{t}{2}} \left\| \left(G_0(t-\tau) + G_0(t)\right) \right\|_{\mathbf{L}^\infty} \left\{\tau\right\}^{-\gamma_1} \langle \tau \rangle^{-\gamma_2} d\tau \\ &\leq C \langle t \rangle^{-2} \int_0^{\frac{t}{2}} \left\{\tau\right\}^{-\gamma_1} \langle \tau \rangle^{-\gamma_2} d\tau \leq C \langle t \rangle^{-\gamma-2} \end{aligned} \tag{3.7}$$

and in the same way

$$\left\| \langle t \rangle^\gamma G_0(t) \int_{\frac{t}{2}}^\infty f\left(\mathcal{N}\left(u(\tau)\right)\right) d\tau \right\|_{\mathbf{L}^\infty} \leq C \left\| u \right\|_{\mathbf{X}}^{\sigma+1}. \tag{3.8}$$

Also we have

$$\begin{aligned} &\left\| \int_0^{\frac{t}{2}} \left( \mathcal{G}(t-\tau) \mathcal{N}\left(u(\tau)\right) - G_0(t-\tau) f\left(\mathcal{N}\left(u(\tau)\right)\right) \right) d\tau \right\|_{\mathbf{L}^\infty} \\ &+ \left\| \int_{\frac{t}{2}}^t \mathcal{G}(t-\tau) \mathcal{N}\left(u(\tau)\right) d\tau \right\|_{\mathbf{L}^\infty} \end{aligned} \tag{3.9}$$

$$\leq C\int_0^{\frac{t}{2}} (t-\tau)^{-2} \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^1} d\tau + C\int_{\frac{t}{2}}^t (t-\tau)^{-\gamma} \|\mathcal{N}(u(\tau))\|_{\mathbf{L}^\infty} d\tau$$
$$\leq Ct^{-2-\gamma} \|u\|_{\mathbf{X}}^{\sigma+1},$$

for all $t > 1$. By virtue of the integral equation (3.1) we get

$$\begin{aligned}
&\langle t\rangle^{\gamma+2} \|(u(t) - AG_0(t))\|_{\mathbf{X}} \leq \|(\mathcal{G}(t) u_0 - G_0(t) f(u_0)\|_{\mathbf{L}^\infty} \\
&+ \langle t\rangle^{\gamma+2} \left\| \int_0^{\frac{t}{2}} (\mathcal{G}(t-\tau)\mathcal{N}(u(\tau)) - G_0(t-\tau) f(\mathcal{N}(u(\tau))))\, d\tau \right\|_{\mathbf{L}^\infty} \\
&+ \langle t\rangle^{\gamma+2} \left\| \int_{\frac{t}{2}}^t \mathcal{G}(t-\tau)\mathcal{N}(u(\tau))\, d\tau \right\|_{\mathbf{L}^\infty} \\
&+ \langle t\rangle^{\gamma+2} \left\| G_0(t) \int_{\frac{t}{2}}^\infty f(\mathcal{N}(u(\tau)))\, d\tau \right\|_{\mathbf{L}^\infty} \\
&+ \langle t\rangle^{\gamma+2} \left\| \int_0^{\frac{t}{2}} (G_0(t-\tau) - G_0(t)) f(\mathcal{N}(u(\tau)))\, d\tau \right\|_{\mathbf{L}^\infty}. \qquad (3.10)
\end{aligned}$$

All summands in the right-hand side of (3.10) are estimated by $C\|u_0\|_{\mathbf{Z}} + C\|u\|_{\mathbf{X}}^{\sigma+1}$ via estimates (3.7) - (3.9). Thus by (3.10) the asymptotic (3.5) is valid. The theorem is proved.

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