MULTIPLE SOLUTIONS FOR CRITICAL ELLIPTIC SYSTEMS VIA PENALIZATION METHOD

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Abstract. We consider the system

$$\begin{cases} -\varepsilon^2\Delta u + W(x)u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V(x)v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $\varepsilon > 0$ is a parameter, W and V are positive potentials, and Q and K are homogeneous function with K having critical growth. We relate the number of solutions to the topology of the set where W and V attain their minimum values. In the proof, we apply Ljusternik-Schnirelmann theory.

1. Introduction

In this paper, we are concerned with the existence of multiple solutions for the following class of elliptic systems:

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + W(x)u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V(x)v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v \in H^1(\mathbb{R}^N),\ u(x),\ v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

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where $\varepsilon > 0$ and $N \geq 3$. The potentials W, V are Hölder continuous functions and there exist an open bounded set $\Lambda \subset \mathbb{R}^N$, $x_0 \in \Lambda$ and $\rho_0 > 0$ such that

(WV_0) $W(x), V(x) \geq \rho_0$ for each $x \in \partial\Lambda$;
(WV_1) $W(x_0), V(x_0) < \rho_0$;
(WV_2) $W(x) \geq W(x_0) > 0,\ V(x) \geq V(x_0) > 0$ for each $x \in \mathbb{R}^N$.

Let $\mathbb{R}^2_+ := [0, \infty) \times [0, \infty)$. For any given any $q \geq 1$ we denote by $\mathcal{H}^q$ the collection of all functions $F \in C^2(\mathbb{R}^2_+, \mathbb{R})$ satisfying the following properties.

$(\mathcal{H}^q_0)$ F is q-homogeneous; that is,

$$F(\lambda s, \lambda t) = \lambda^q F(s,t) \text{ for each } \lambda > 0,\ (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{H}^q_1)$ There exists $c_1 > 0$ such that

$$|F_s(s,t)| + |F_t(s,t)| \leq c_1 \left(s^{q-1} + t^{q-1}\right) \text{ for each } (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{H}_2)$ $F(s,t) > 0$ for each $s, t > 0$.
$(\mathcal{H}_3)$ $\nabla F(0,1) = \nabla F(1,0) = (0,0)$.
$(\mathcal{H}_4)$ $F_s(s,t), F_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

We set $2^* := 2N/(N-2)$ and state our main hypothesis on the nonlinear terms of the system as follows.

(A_1) $K \in \mathcal{H}^{2^*}$ and $Q \in \mathcal{H}^p$ for some $2 < p < 2^*$.
(A_2) The 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave.
(A_3) $Q(s,t) \geq \lambda s^\alpha t^\beta$ for all $(s,t) \in \mathbb{R}^2_+$, with $\alpha, \beta > 1$, $\alpha + \beta =: p_1 \in (2, 2^*)$ and λ satisfies
  (i) $\lambda > 0$ if either $N \geq 4$, or $N = 3$ and $2^* - 2 < p_1 < 2^*$;
  (ii) λ is sufficiently large if $N = 3$ and $2 < p_1 \leq 2^* - 2$.

Since we are interested in positive solutions, we extend the functions Q and K to all of $\mathbb{R}^2$ by setting $Q(u,v) = K(u,v) = 0$ if $u \leq 0$ or $v \leq 0$.

In order to get precise statements about our result, we need to introduce some notation. Thus, we fix $\xi \in \mathbb{R}^N$ and consider the autonomous system associated to (S_ε), namely

$$(AP)_\xi \qquad \begin{cases} -\Delta u + W(\xi)u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\Delta v + V(\xi)v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N),\ u(x),\ v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In view of conditions (WV_2) and (A_1), the above problem has a variational structure. The associated functional

$$I_\xi(u,v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 + W(\xi)|u|^2 + V(\xi)|v|^2\right) \mathrm{d}x$$

$$- \int_{\mathbb{R}^N} Q(u,v)\mathrm{d}x - \frac{1}{2^*}\int_{\mathbb{R}^N} K(u,v)\mathrm{d}x,$$

is well defined for $(u,v) \in X := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Since I_ξ has the mountain pass geometry we can set the the minimax level $C(\xi)$ in the following way:

$$C(\xi) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\xi(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1],X) : \gamma(0) = 0,\ I_\xi(\gamma(1)) \leq 0\}$. Moreover, the map $\xi \mapsto C(\xi)$ is continuous and $C(\xi)$ can be further characterized as

$$C(\xi) = \inf_{(u,v)\in\mathcal{M}_\xi} I_\xi(u,v),$$

where $\mathcal{M}_\xi$ is the Nehari manifold of I_ξ; that is,

$$\mathcal{M}_\xi := \{(u,v) \in X \setminus \{(0,0)\} : I'_\xi(u,v)(u,v) = 0\}.$$

By adapting some well-known arguments (see Section 2) we shall prove that, for each ξ fixed, the minimax level $C(\xi)$ is achieved and therefore

$$M := \left\{x \in \mathbb{R}^N : C(x) = \inf_{\xi\in\mathbb{R}^N} C(\xi)\right\} \neq \varnothing.$$

Moreover, arguing as in [1], we can prove that

$$(C_0) \qquad C^* := C(x_0) = \inf_{\xi\in\Lambda} C(\xi) < \min_{\xi\in\partial\Lambda} C(\xi).$$

If Y is a closed subset of a topological space Z, we denote by $\mathrm{cat}_Z(Y)$ the Ljusternik-Schnirelmann category of Y in Z, namely the least number of closed and contractible sets in Z which cover Y. We are now ready to state the main result of this paper.

Theorem 1.1. *Suppose* $(WV_0)-(WV_2)$ *and* $(A_1)-(A_3)$. *Then, for any* $\delta > 0$ *satisfying* $M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x,M) < \delta\} \subset \Lambda$, *there exists* $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0,\varepsilon_\delta)$, *the system* (S_ε) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *solutions. Moreover, if* $(u_\varepsilon, v_\varepsilon)$ *is a solution for* (S_ε) *and if* P_ε *and* Q_ε *are maximum points of* u_ε *and* v_ε *respectively, then* $P_\varepsilon, Q_\varepsilon \in \Lambda$ *and* $C(P_\varepsilon), C(Q_\varepsilon) \to C(x_0)$ *as* $\varepsilon \to 0^+$.

The proof of Theorem 1.1 follows the same lines as the subcritical case found in [2] and it will be done in three main steps. First, we use a version of the penalization method of del Pino and Felmer [10] which was recently introduced by the first author in [1]. It consists in modifying the nonlinearity outside the set Λ in such a way that the modified functional satisfies the Palais-Smale condition. In the second step, by using a technique due to

Benci and Cerami [8], we relate the category of the set M to the number of positive solutions for the modified problem. Finally, we prove that, for $\varepsilon > 0$ small, the solutions of the modified problem are in fact solutions of (S_ε).

It is worthwhile to emphasize that, since we deal with the critical functional restricted to an appropriated manifold, the calculations performed to get compactness are much more involved than those of [10, 1, 2]. We apply the ideas of Brezis and Nirenberg [7], besides the paper [11], where it is proved that the number

$$\widetilde{S}_K := \inf\Big\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\mathrm{d}x \ :\ u, v \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} K(u^+, v^+)\mathrm{d}x = 1 \Big\}$$

plays an important role when dealing with critical systems. Actually, we use the above constant, (A_3) and some calculations performed in [15] to localize the energy levels where the Palais-Smale condition fails.

Concerning the class of nonlinearities we are dealing with, we have the following examples from [11]. Let $q \geq 1$ and

$$P_q(s,t) = \sum_{\alpha_i + \beta_i = q} a_i s^{\alpha_i} t^{\beta_i},$$

where $i \in \{1, \ldots, k\}$, $\alpha_i, \beta_i \geq 1$ and $a_i \in \mathbb{R}$. The following functions and their possible combinations, with appropriate choices of the coefficients a_i, satisfy our hypothesis on Q

$$Q_1(s,t) = P_p(s,t),\ Q_2(s,t) = \sqrt[r]{P_l(s,t)} \text{ and } Q_3(s,t) = \frac{P_{l_1}(s,t)}{P_{l_2}(s,t)},$$

with $r = pl$ and $l_1 - l_2 = p$. Condition (A_2) restricts the expression of the critical function H. However, it can have the polynomial form $H(s,t) = P_{2^*}(s,t)$.

Our theorem extends or complements the results found in [12, 1, 3, 6] (see also [4, 5] for some related results for Hamiltonian systems). As far as we know, this is the first time that penalization methods jointly with Ljusternik-Schnirelmann theory are used to get multiple solutions for gradient systems with critical growth.

The paper is organized as follows: In Section 2, we present the abstract framework. Section 3 is devoted to the proof of the local Palais-Smale condition for the modified functional. Finally, we present the proof of Theorem 1.1 in Section 4.

2. The variational framework

In this section, we present the variational framework to deal with the system and also study the autonomous problem related to it. We write only $\int u$ instead of $\int_{\mathbb{R}^N} u(x)\mathrm{d}x$. Moreover, we notice that, for any function $F \in \mathcal{H}^q$, we can use the homogeneity condition $(\mathcal{H}_0^q)$ to conclude that

$$qF(s,t) = sF_s(s,t) + tF_t(s,t), \tag{2.1}$$

$$q(q-1)F(s,t) = s^2F_{ss}(s,t) + 2stF_{st}(s,t) + t^2F_{tt}(s,t), \tag{2.2}$$

for any $(s,t) \in \mathbb{R}^2$.

Our first aim is to consider, for any fixed $\xi \in \mathbb{R}^N$, the autonomous problem

$$(AP)_\xi \qquad \begin{cases} -\Delta u + W(\xi)|u|^{p-2}u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V(\xi)|v|^{p-2}v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) & \text{in } \mathbb{R}^N, \\ u(x), v(x) > 0 \text{ for all } x \in \mathbb{R}^N, \end{cases}$$

whose solutions are the critical points of $I_\xi : X \to \mathbb{R}$ defined as

$$I_\xi(u,v) := \|(u,v)\|_\xi^2 - \int Q(u,v) - \frac{1}{2^*}\int K(u,v),$$

where

$$\|(u,v)\|_\xi^2 := \int |\nabla u|^2 + |\nabla v|^2 + W(\xi)u^2 + V(\xi)v^2.$$

We denote by m_ξ the ground state level of I_ξ; that is,

$$m_\xi := \inf_{(u,v)\in X\setminus\{(0,0)\}} \max_{t\geq 0} I_\xi(tu,tv) > 0.$$

As usual, we denote by S the best constant of the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. As stated in the introduction we shall consider the following number, which was introduced in [11].

$$\widetilde{S}_K := \inf\left\{\int (|\nabla u|^2 + |\nabla v|^2) \ : \ u,v \in H^1(\mathbb{R}^N) \text{ and } \int K(u^+,v^+) = 1\right\}.$$

Proposition 2.1. *For any $\xi \in \mathbb{R}^N$, $m_\xi < \frac{1}{N}\widetilde{S}_K^{N/2}$.*

Proof. In view of the definition of m_ξ, it suffices to obtain $(u,v) \in X$ such that

$$\max_{t\geq 0} I_\xi(tu,tv) < \frac{1}{N}\widetilde{S}_K^{N/2}.$$

We first recall (see [16]) that, for any $\delta > 0$, the instanton

$$w_\delta(x) := [\delta N(N-2)]^{(N-2)/4}\left(\delta + |x|^2\right)^{(2-N)/2}$$

satisfies

$$\int |\nabla w_\delta|^2 = \int |w_\delta|^{2^*} = S^{N/2}.$$

By [11, Lemma 1], there exist $A, B \in \mathbb{R}$ such that $\widetilde{S}_K$ is attained by $(A\omega_\delta, B\omega_\delta)$. So,

$$\widetilde{S}_K = \frac{\int |\nabla(Aw_\delta)|^2 + |\nabla(Bw_\delta)|^2}{\left(\int K(Aw_\delta, Bw_\delta)\right)^{2/2^*}} = \frac{S^{N/2}(A^2+B^2)}{\left(\int K(Aw_\delta, Bw_\delta)\right)^{2/2^*}}. \tag{2.3}$$

Let $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$. Setting

$$\psi_\delta(x) := \frac{\eta(x) w_\delta(x)}{|\eta w_\delta|_{p^*}},$$

we can use the definition of ψ_δ and (A_3) to get

$$\begin{aligned} I_\xi(tA\psi_\delta, tB\psi_\delta) \le \frac{t^2}{2} D_\delta(A^2+B^2) - \frac{t^{2^*}}{2^*} \int K(Aw_\delta, Bw_\delta) \\ - \lambda t^{p_1} A^{p_1} B^{p_1} \int_{B_2(0)} |\psi_\delta|^{p_1}, \end{aligned}$$

where $p_1 \in (2, 2^*)$ is given by condition (A_3) and

$$D_\delta = \int |\nabla \psi_\delta|^2 + \max\{W(\xi), V(\xi)\} |\psi_\delta|^2.$$

Let $h_\delta(t)$ be the the right-hand side of the above expression and denote by t_δ the maximum point of h_δ on $(0, \infty)$. Since $h'_\delta(t_\delta) = 0$ we have that

$$\bar{t}_\delta := \left(\frac{D_\delta(A^2+B^2)}{\int K(Aw_\delta, Bw_\delta)}\right)^{1/(2^*-2)} \ge t_\delta > 0.$$

Since the function $t \mapsto t^2 D_\delta(A^2+B^2)/2 - t^{2^*} \int K(Aw_\delta, Bw_\delta)/2^*$ is increasing in $(0, \bar{t}_\delta)$, we can use the definition of h_δ to get

$$h_\delta(t_\delta) \le \frac{1}{N}\left(\frac{D_\delta(A^2+B^2)}{\left(\int K(Aw_\delta, Bw_\delta)\right)^{2/2^*}}\right)^{N/2} - \lambda t^{p_1} A^{p_1} B^{p_1} \int_{B_2(0)} |\psi_\delta|^{p_1}. \tag{2.4}$$

If $a, b \ge 0$ and $s \ge 1$, then $(a+b)^s \le a^s + s(a+b)^{s-1} b$. Therefore, there exists $C_1 > 0$ such that

$$D_\delta^{N/2} \le S^{N/2} + O(\delta^{(N-2)/2}) + C_1 \int_{B_2(0)} |\psi_\delta|^2.$$

Moreover, we can obtain $\rho > 0$ such that $t_\delta > \rho$ for any δ small. Hence, it follows from the above inequality, (2.4) and (2.3) that

$$h_\delta(t_\delta) \le \frac{1}{N}\widetilde{S}_K^{N/2} + \delta^{(N-2)/2}\Big[C_2 + \frac{C_3}{\delta^{(N-2)/2}}\Big(\int_{B_2(0)} |\psi_\delta|^2 - \lambda C_4 |\psi_\delta|^{p_1}\Big)\Big],$$

for positive constants C_2, C_3, and C_4. In view of the hypotheses on $\lambda > 0$ given in (A_3), we can argue as in the proof of [15, Claim 2] to check that, if δ is sufficiently small, the second term in the right-hand side above is negative. Thus,

$$\max_{t\ge 0} I_\xi(tA\psi_\delta, tB\psi_\delta) \le \max_{t\ge 0} h_\delta(t) = h_\delta(t_\delta) < \frac{1}{N}\widetilde{S}_K^{N/2},$$

and the proposition is proved. □

In order to prove that m_ξ is achieved we need the following lemma.

Lemma 2.2. *Let $((u_n, v_n)) \subset X$ be a $(\mathrm{PS})_d$ sequence for the functional I_ξ with $d < \frac{1}{N}\widetilde{S}_K^{N/2}$. Then we have either*

(i) $\|(u_n, v_n)\|_\xi \to 0$, *or*

(ii) *there exists a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R,\ \gamma > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) \ge \gamma > 0.$$

Proof. Suppose that (ii) does not hold. Then we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 = 0 \text{ and } \lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2 = 0,$$

for any $R > 0$. Thus, the fact that $2 < p < 2^*$ and [14, Lemma I.1] imply that

$$\lim_{n\to\infty} \int |u_n|^p = 0 = \lim_{n\to\infty} \int |v_n|^p.$$

Thus, we can use the growth condition $(\mathcal{H}_1^p)$ to conclude that $\int Q(u_n, v_n) \to 0$. Since $((u_n, v_n))$ is bounded, $I'_\xi(u_n, v_n)(u_n, v_n) \to 0$. Taking a subsequence, we obtain $l \ge 0$ such that

$$\|(u_n, v_n)\|_\xi^2 \to l \text{ and } \int K(u_n, v_n) \to l. \tag{2.5}$$

Since $I_\xi(u_n, v_n) \to d$, we can use (2.5) to conclude that $l = Nd$. Recalling the definition of $\widetilde{S}_K$ we get

$$\|(u_n, v_n)\|_\xi^2 \ge \widetilde{S}_K \Big(\int K(u_n, v_n)\Big)^{2/2^*}.$$

Taking the limit we conclude that $l \geq \widetilde{S}_K l^{2/2^*}$. If $l > 0$ we obtain $Nd = l \geq \widetilde{S}_K^{N/2}$, which does not make sense. Hence $l = 0$ and therefore (i) holds. □

Proposition 2.3. *For any $\xi \in \mathbb{R}^N$ the problem $(AP)_\xi$ has a weak solution.*

Proof. Standard calculations show that I_ξ has the mountain pass geometry. This and Proposition 2.1 provide a sequence $((u_n, v_n)) \subset X$ such that

$$I_\xi(u_n, v_n) \to m_\xi < \frac{1}{N}\widetilde{S}_K^{N/2} \quad \text{and} \quad I'_\xi(u_n, v_n) \to 0.$$

It suffices now to use Lemma 2.2 and argue as in [3, Proposition 2.1] to prove that the sequence $(\widetilde{u}_n, \widetilde{v}_n) := (u_n(\cdot + y_n), v_n(\cdot + y_n)$ weakly converges to a solution of $(AP)_\xi$. We omit the details. □

2.1. **The penalization scheme.** We present now the penalization scheme which will avoid the lack of compactness originated by the unboundedness of $\mathbb{R}^N$. This kind of idea has first appeared in the paper of del Pino and Felmer [10]. Here, we use an adaptation for systems introduced by Alves in [1].

We start by choosing $a > 0$ and considering $\eta \in C^2(\mathbb{R}, \mathbb{R})$ a non-increasing function such that

$$\eta \equiv 1 \text{ on } (-\infty, a], \;\; \eta \equiv 0 \text{ on } [5a, +\infty), \;\; |\eta'(t)| \leq \frac{C}{a} \text{ and } |\eta''(t)| \leq \frac{C}{a^2}, \quad (2.6)$$

for each $t \in \mathbb{R}$ and for some positive constant $C > 0$. By using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\widehat{Q}(s,t) = \eta(|(s,t)|)\Big(Q(s,t) + \frac{1}{2^*}K(s,t)\Big) + A(1 - \eta(|(s,t)|)(s^2 + t^2),$$

where

$$A := \max\Big\{\frac{Q(s,t) + \frac{1}{2^*}K(s,t)}{s^2 + t^2} : (s,t) \in \mathbb{R}^2, \; a \leq |(s,t)| \leq 5a\Big\}.$$

Notice that, since A tends to zero as $a \to 0^+$, we may suppose that $A \in (0, \gamma/4)$ where

$$\gamma := \min\{V(x_0), W(x_0)\}. \quad (2.7)$$

Finally, denoting by I_Λ the characteristic function of the set Λ, we define $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$H(x,s,t) := I_\Lambda(x)\Big(Q(s,t) + \frac{1}{2^*}K(s,t)\Big) + (1 - I_\Lambda(x))\widehat{Q}(s,t).$$

By using the penalized function H, in the sequel we are going to study the number of solutions of the following modified problem:

$$(S_{\varepsilon,a}) \qquad \begin{cases} -\Delta u + W(\varepsilon x)u = H_u(\varepsilon x, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V(\varepsilon x)v = H_v(\varepsilon x, u, v) & \text{in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{cases}$$

Our interest in this problem can be justified in the following way. Suppose that $u_\varepsilon, v_\varepsilon \in H^1(\mathbb{R}^N)$ are positive functions such that $(u_\varepsilon, v_\varepsilon)$ solves $(S_{\varepsilon,a})$ and moreover $|(u_\varepsilon(x), v_\varepsilon(x))| \le a$ for each $x \in \mathbb{R}^N \setminus \Lambda$. It follows from the definition of H and $\widehat{Q}$ that $H(\cdot, u_\varepsilon, v_\varepsilon) \equiv Q(u_\varepsilon, v_\varepsilon) + \frac{1}{2^*}K(u_\varepsilon, v_\varepsilon)$. Hence, the function $(u(x), v(x)) := (u_\varepsilon(x/\varepsilon), v_\varepsilon(x/\varepsilon))$ is a solution of the problem (S_ε). Thus, we shall look for solutions $(u_\varepsilon, v_\varepsilon)$ of $(S_{\varepsilon,a})$ satisfying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \le a \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$.

For each $\varepsilon > 0$ we denote by X_ε the Hilbert space

$$X_\varepsilon := \Big\{(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int \big(W(\varepsilon x)|u|^2 + V(\varepsilon x)|v|^2\big) < \infty\Big\},$$

endowed with the norm

$$\|(u, v)\|_\varepsilon^2 := \int (|\nabla u|^2 + |\nabla v|^2) + \int \big(W(\varepsilon x)|u|^2 + V(\varepsilon x)|v|^2\big).$$

The growth conditions on Q and K, and (WV_2) imply that the critical points of the C^1-functional $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$J_\varepsilon(u, v) := \frac{1}{2}\int \big(|\nabla u|^2 + |\nabla v|^2 + W(\varepsilon x)|u|^2 + V(\varepsilon x)|v|^2\big) - \int H(\varepsilon x, u, v)$$

are weak solutions of $(S_{\varepsilon,a})$. We recall that these critical points belong to the Nehari manifold of J_ε, namely on the set

$$\mathcal{N}_\varepsilon := \{(u, v) \in X_\varepsilon \setminus \{(0, 0)\} : J_\varepsilon'(u, v)(u, v) = 0\}.$$

It is well known that, for any nontrivial element $(u, v) \in X_\varepsilon$, the function $t \mapsto J_\varepsilon(tu, tv)$, for $t \ge 0$, achieves its maximum value at a unique point $\bar{t} > 0$ such that $(\bar{t}u, \bar{t}v) \in \mathcal{N}_\varepsilon$.

We end this section by presenting some useful properties of the penalized function H.

Lemma 2.4. *For all $(s, t) \in \mathbb{R}^2$, the function H satisfies the following:*

(i) *$pH(x, s, t) \le sH_s(x, s, t) + tH_t(x, s, t)$, for all $x \in \Lambda$;*
(ii) *$2H(x, s, t) \le sH_s(x, s, t) + tH_t(x, s, t)$, for all $x \in \mathbb{R}^N \setminus \Lambda$;*

(iii) $sH_s(x,s,t) + tH_t(x,s,t) \leq \frac{1}{2}\big(W(x)s^2 + V(x)t^2\big)$, *for all* $x \in \mathbb{R}^N \setminus \Lambda$.

Proof. Since $H(x,s,t) = Q(s,t) + \frac{1}{2^*}K(s,t)$ on the set Λ, we can use the fact that $p < 2^*$ and (2.1), to get

$$\begin{aligned} pH(x,s,t) &\leq sQ_s + tQ_t + K(s,t) \\ &= sQ_s + tQ_t + \frac{1}{2^*}(sK_s + tK_t) = sH_s + tH_t, \end{aligned}$$

for all $x \in \Lambda$. This proves (i).

In what follows we denote $|z| := \sqrt{s^2+t^2}$. Notice that $H(x,s,t) = \widehat{Q}(s,t)$ in the set $\mathbb{R}^N \backslash \Lambda$. Thus, on this set,

$$H_s = \widehat{Q}_s = \frac{\eta' s}{|z|}\Big(Q + \frac{1}{2^*}K\Big) + \eta Q_s + \eta\frac{1}{2^*}K_s - A\eta' s|z| + 2A(1-\eta)s,$$

and

$$H_t = \widehat{Q}_t = \frac{\eta' t}{|z|}\Big(Q + \frac{1}{2^*}K\Big) + \eta Q_t + \eta\frac{1}{2^*}K_t - A\eta' t|z| + 2A(1-\eta)t.$$

So,

$$sH_s + tH_t = \eta'|z|\big(Q + \frac{1}{2^*}K - A|z|^2\big) + p\eta Q + \eta K + 2A(1-\eta)|z|^2. \quad (2.8)$$

Notice that, in view of the definition of A, we have that

$$Q(s,t) + \frac{1}{2^*}K(s,t) - A|z|^2 \leq 0,$$

for all x belonging in the support of η'. Hence, recalling that $\eta' \leq 0$, we can use the above estimate, (2.8) and the fact that $2 < p < 2^*$, to obtain

$$\begin{aligned} sH_s + tH_t &\geq p\eta Q + \eta K + 2A(1-\eta)|z|^2 \\ &\geq 2\Big(\eta\Big(Q + \frac{1}{2^*}K\Big) + A(1-\eta)|z|^2\Big) = 2H, \end{aligned}$$

for all $x \in \mathbb{R}^n \setminus \Lambda$. Thus, (ii) holds.

In order to verify (iii) we note that, by (2.8),

$$\frac{sH_s + tH_t}{s^2+t^2} = \eta'|z|\Big(\frac{Q + \frac{1}{2^*}K}{s^2+t^2} - A\Big) + \eta\frac{pQ}{s^2+t^2} + \eta\frac{K}{s^2+t^2} + 2A(1-\eta).$$

Since η is smooth, $\operatorname{supp}\eta' \subset [a, 5a]$ and $\lim_{a\to 0^+} A = 0$, we have that

$$\frac{sH_s + tH_t}{s^2+t^2} = \eta\frac{pQ}{s^2+t^2} + \eta\frac{K}{s^2+t^2} + o(1), \text{ as } a \to 0^+. \quad (2.9)$$

Now, we can use (2.1), $(\mathcal{H}_1^p)$ and $(\mathcal{H}^{2^*})$ to conclude that

$$\lim_{(s,t)\to(0,0)} \frac{pQ(s,t)+K(s,t)}{s^2+t^2} = 0.$$

Hence, the function on the right-hand side of (2.9) is continuous on the compact set $B_{5a}(0)$. It follows that, for $a > 0$ sufficiently small, we have

$$\frac{sH_s + tH_t}{s^2+t^2} \leq \frac{\gamma}{2}.$$

Thus, for this choice of a, we can use the above estimate, (2.7) and (WV_2) to obtain, for each $x \in \mathbb{R}^N \backslash \Lambda$,

$$sH_s + tH_t \leq \tfrac{1}{2}(W(x)s^2 + V(x)t^2).$$

This establishes (iii) and concludes the proof of the lemma. □

3. A compactness result

Since we are intending to apply critical point theory we need to prove some compactness properties for the functional J_ε. So, let E be a Banach space, $\mathcal{V}$ be a C^1-manifold of E and $I : E \to \mathbb{R}$ a C^1-functional. We say that $I|_{\mathcal{V}}$ satisfies the Palais-Smale condition at level d ($(\mathrm{PS})_d$ for short) if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to d$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here, we are denoting by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u. The main result of this section can be stated as follows.

Proposition 3.1. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies the Palais-Smale condition at any level $c < \frac{1}{N}\widetilde{S}_K^{N/2}$.*

In order to prove the above result, we first obtain a compactness property of the unconstrained functional.

Lemma 3.2. *Any sequence $((u_n, v_n)) \subset X_\varepsilon$ such that*

$$J_\varepsilon(u_n, v_n) \to c < \frac{1}{N}\widetilde{S}_K^{N/2} \quad \text{and} \quad \|J'_\varepsilon(u_n, v_n)\| \to 0$$

possesses a convergent subsequence.

Proof. Standard calculations show that $((u_n, v_n))$ is bounded in X_ε. Hence, $J'_\varepsilon(u_n, v_n)(u_n, v_n) \to 0$ and we have that

$$\|(u_n, v_n)\|_\varepsilon^2 = \int u_n H_u(\varepsilon x, u_n, v_n) + v_n H_u(\varepsilon x, u_n, v_n) + o_n(1), \tag{3.1}$$

where $o_n(1)$ denotes a quantity approaching zero as $n \to \infty$. Up to a subsequence, we may suppose that

$$\begin{array}{lll} (u_n, v_n) \rightharpoonup (u, v) & \text{weakly in } X_\varepsilon, & \\ u_n \to u,\ v_n \to v & \text{in } L^r_{loc}(\mathbb{R}^N) \text{ for any } 2 \le r < 2^*, & \quad (3.2) \\ (u_n(x), v_n(x)) \to (u(x), v(x)) & \text{for a.e. } x \in \mathbb{R}^N. & \end{array}$$

This, the Lebesgue theorem and standard calculations show that (u, v) is a critical point of J_ε. Hence

$$\|(u, v)\|_\varepsilon^2 = \int uH_u(\varepsilon x, u, v) + \int vH_v(\varepsilon x, u, v). \tag{3.3}$$

Claim 1. The following limit holds as $n \to \infty$:

$$\int u_n H_u(\varepsilon x, u_n, v_n) + v_n H_u(\varepsilon x, u_n, v_n) \to \int (uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)).$$

This claim, (3.1) and (3.3) imply that $\|(u_n, v_n)\|_\varepsilon^2 \to \|(u, v)\|_\varepsilon^2$, from which it follows that $(u_n, v_n) \to (u, v)$ in X_ε.

We proceed now with the proof of Claim 1. For any given $\zeta > 0$, we can argue as in the proof of [1, Lemma 3.2, Claim 1] to obtain $R > 0$ such that $\Lambda_\varepsilon \subset B_R(0)$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} \left(|\nabla u_n|^2 + |\nabla v_n|^2 + W(\varepsilon x)|u_n|^2 + V(\varepsilon x)|v_n|^2\right) \le \zeta.$$

It follows from the above expression and Lemma 2.4 (iii) that

$$\int_{\mathbb{R}^N \setminus B_{R(0)}} u_n H_u(x, u_n, v_n) + v_n H_v(x, u_n, v_n) \le \frac{\zeta}{4}, \tag{3.4}$$

for all n large. On the other hand, taking R larger if necessary, we can suppose that

$$\int_{\mathbb{R}^N \setminus B_R(0)} (uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)) \le \frac{\zeta}{4}.$$

Since $\zeta > 0$ is arbitrary, the above expression and (3.4) imply that

$$\begin{aligned} & \lim_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} \left(u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)\right) \\ = & \int_{\mathbb{R}^N \setminus B_R(0)} \left(uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)\right). \end{aligned} \tag{3.5}$$

Since the set $B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)$ is bounded, we can use Lemma 2.4 (iii), (3.2) and Lebesgue's theorem to conclude that

$$\begin{aligned} &\lim_{n\to\infty}\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)}(u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)) \\ &= \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)}(uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)). \end{aligned}\tag{3.6}$$

Claim 2. $\displaystyle\lim_{n\to\infty}\int_{\Lambda_\varepsilon}K(u_n,v_n)=\int_{\Lambda_\varepsilon}K(u,v).$

By using the above claim, $(\mathcal{H}_1^p)$, (3.2) and Lebesgue's theorem again, we conclude that

$$\begin{aligned} &\lim_{n\to\infty}\int_{\Lambda_\varepsilon}(u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)) \\ &= \int_{\Lambda_\varepsilon}(uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)). \end{aligned}\tag{3.7}$$

This convergence, (3.5) and (3.6)) conclude the proof of Claim 1.

We now prove Claim 2. First notice that, since $((u_n,v_n))$ is bounded, we may suppose that

$$|\nabla u_n|^2\rightharpoonup\mu,\ |\nabla v_n|^2\rightharpoonup\sigma \ \text{ and } \ K(u_n,v_n)\rightharpoonup\nu \text{ (weak}^*\text{-sense of measures).}$$

By adapting the concentration compactness principle [13, Lemma 1.2] (see also [11, Lemma 3]), we obtain an at most countable index set Γ, sequences $(x_i)\subset\mathbb{R}^N$, $(\mu_i),(\sigma_i),(\nu_i)\subset(0,\infty)$, such that

$$\begin{aligned} &\mu\geq|\nabla u|^2+\sum_{i\in\Gamma}\mu_i\delta_{x_i},\ \sigma\geq|\nabla v|^2+\sum_{i\in\Gamma}\sigma_i\delta_{x_i}, \\ &\nu=K(u,v)+\sum_{i\in\Gamma}\nu_i\delta_{x_i} \ \text{ and } \ \widetilde{S}_K\nu_i^{2/2^*}\leq\mu_i+\sigma_i, \end{aligned}\tag{3.8}$$

for all $i\in\Gamma$, where δ_{x_i} is the Dirac mass at the point $x_i\in\mathbb{R}^N$.

It suffices to show that $\{x_i\}_{i\in\Gamma}\cap\Lambda_\varepsilon=\varnothing$. Suppose, by contradiction, that $x_i\in\Lambda_\varepsilon$ for some $i\in\Gamma$. Define, for $\varrho>0$, the function $\psi_\varrho(x):=\psi((x-x_i)/\varrho)$ where $\psi\in C_0^\infty(\mathbb{R}^N,[0,1])$ is such that $\psi\equiv1$ on $B_1(0)$, $\psi\equiv0$ on $\mathbb{R}^N\setminus B_2(0)$ and $|\nabla\psi|_\infty\leq2$. We suppose that ϱ is chosen in such a way that the support of ψ_ϱ is contained in Λ_ε. Since $(\psi_\varrho u_n,\psi_\varrho v_n)$ is bounded, $J'_\varepsilon(u_n,v_n)(\psi_\varrho u_n,\psi_\varrho v_n)\to0$. This and (2.1) imply that

$$\int\big(\psi_\varrho|\nabla u_n|^2+\psi_\varrho|\nabla v_n|^2\big)\leq-\int\big(u_n(\nabla u_n\cdot\nabla\psi_\varrho)+v_n(\nabla v_n\cdot\nabla\psi_\varrho)\big)$$

$$+\int (u_n\psi_\varrho, v_n\psi_\varrho)\cdot\nabla Q(u_n,v_n) + \int \psi_\varrho K(u_n,v_n) + o_n(1).$$

Since Q has subcritical growth and ψ_ϱ has compact support, we can let $n\to\infty$, $\varrho\to 0$ and use (3.8) to conclude that $\nu_i \geq \mu_i + \sigma_i$. It follows from the last statement in (3.8) that $\nu_i \geq \widetilde{S}_K^{N/2}$, and therefore we can use Lemma 2.4, $p>2$ and (2.1) to compute

$$\begin{aligned} c &= J_\varepsilon(u_n,v_n) - \tfrac{1}{2}J'_\varepsilon(u_n,v_n)(u_n,v_n) + o_n(1)\\ &= \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \big(\tfrac{1}{2}\big(u_nH_u(\varepsilon x,u_n,v_n) + v_nH_v(\varepsilon x,u_n,v_n)\big) - H(\varepsilon x,u_n,v_n)\big)\\ &+ \int_{\Lambda_\varepsilon} \big(\tfrac{1}{2}\big(u_nQ_u(u_n,v_n) + v_nQ_v(u_n,v_n)\big) - Q(u_n,v_n)\big)\\ &+ \frac{1}{N}\int_{\Lambda_\varepsilon} K(u_n,v_n) + o_n(1)\\ &\geq \frac{1}{N}\int_{\Lambda_\varepsilon} K(u_n,v_n) + o_n(1) \geq \frac{1}{N}\int_{\Lambda_\varepsilon} \psi_\varrho K(u_n,v_n) + o_n(1). \end{aligned}$$

By taking the limit and using (3.8) we get

$$c \geq \frac{1}{N}\sum_{\{i\in\Gamma: x_i\in\Lambda_\varepsilon\}} \psi_\varrho(x_i)\nu_i = \frac{1}{N}\sum_{\{i\in\Gamma: x_i\in\Lambda_\varepsilon\}} \nu_i \geq \frac{1}{N}\widetilde{S}_K^{N/2},$$

which does not make sense. This concludes the proof of Claim 2 and therefore the lemma is proved. □

We shall need the following technical results.

Lemma 3.3. *There exist constants $a_1, c>0$ such that, for each $a\in(0,a_1)$, $(u,v)\in\mathcal{N}_\varepsilon$,*

$$\int_{\Lambda_\varepsilon} (pQ(u,v) + K(u,v)) \geq c$$

and

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (W(\varepsilon x)u^2 + V(\varepsilon x)v^2) \leq 2\int_{\Lambda_\varepsilon} (pQ(u,v) + K(u,v)).$$

Proof. The growth conditions on Q and K, and Sobolev's embeddings, provide $c_2>0$, independent of (u,v), such that $\|(u,v)\|_\varepsilon \geq c_2$. By setting

$B(u,v) := pQ(u,v) + K(u,v)$ we can use (2.1) and Lemma 2.4(iii) to get

$$\begin{aligned}\|(u,v)\|_\varepsilon^2 &= \int_{\Lambda_\varepsilon}\big(uQ_u + vQ_v + \frac{1}{2^*}(uK_u + vK_v)\big) + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(uH_u + vH_v)\\ &\le \int_{\Lambda_\varepsilon} B(u,v) + \frac{1}{2}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(W(\varepsilon x)u^2 + V(\varepsilon x)v^2).\end{aligned}$$

Hence,

$$\frac{c_2^2}{2} \le \frac{1}{2}\|(u,v)\|_\varepsilon^2 \le \int_{\Lambda_\varepsilon} B(u,v),$$

which proves the first statement of the lemma. For the second one we use $(u,v) \in \mathcal{N}_\varepsilon$ and (2.1) to obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}(W(\varepsilon x)u^2 + V(\varepsilon x)v^2) \le \|(u,v)\|_\varepsilon^2 = \int uH_u + vH_v\\ &= \int_{\Lambda_\varepsilon} B(u,v) + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(uH_u + vH_v)\\ &\le \int_{\Lambda_\varepsilon} B(u,v) + \frac{1}{2}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(W(\varepsilon x)u^2 + V(\varepsilon x)v^2).\end{aligned}$$

The proof is finished. □

Lemma 3.4. *Let $\phi_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by*

$$\phi_\varepsilon(u,v) := \|(u,v)\|_\varepsilon^2 - \int\left(uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)\right).$$

Then there exist constants $a_2, b > 0$ such that, for each $a \in (0, a_2)$,

$$\phi_\varepsilon'(u,v)(u,v) \le -b < 0 \textit{ for each } (u,v) \in \mathcal{N}_\varepsilon. \tag{3.9}$$

Proof. In what follows we suppose that a is small in such a way that Lemma 3.3 holds. Given $(u,v) \in \mathcal{N}_\varepsilon$, we can use the definition of H, (2.1) and (2.2) to get

$$\begin{aligned}&\phi_\varepsilon'(u,v)(u,v) = \int_{\Lambda_\varepsilon}(uQ_u + vQ_v + \frac{1}{2^*}(uK_u + vK_v))\\ &- \int_{\Lambda_\varepsilon}(u^2Q_{uu} + v^2Q_{vv} + 2uvQ_{uv})\\ &- \int_{\Lambda_\varepsilon}\frac{1}{2^*}(u^2K_{uu} + v^2K_{vv} + 2uvK_{uv}) + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(D_1 - D_2)\\ &= -(p-2)\int_{\Lambda_\varepsilon} pQ(u,v) - (2^*-2)\int_{\Lambda_\varepsilon} K(u,v) + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(D_1 - D_2),\end{aligned}$$

with $D_1 := (uH_u + vH_v)$ and $D_2 := \left(u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}\right)$. Now, we can use $p < 2^*$ to get

$$\phi_\varepsilon'(u,v)(u,v) \leq -(p-2)\int_{\Lambda_\varepsilon} B(u,v) + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(D_1 - D_2). \tag{3.10}$$

Arguing as in the proof of [2, Lemma 2.3], we can prove that, for $a > 0$ small enough, we have

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} |D_1| + |D_2| \leq \frac{p-2}{4}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(W(\varepsilon x)u^2 + V(\varepsilon x)v^2\right).$$

This inequality, (3.10) and Lemma 3.3 imply that

$$\phi_\varepsilon'(u,v)(u,v) \leq -\frac{p-2}{2}\int_{\Lambda_\varepsilon} B(u,v) \leq -\frac{p-2}{2}c = -b < 0.$$

The lemma is proved. □

We are now ready to prove our compactness result.

Proof of Proposition 3.1. Let $((u_n,v_n)) \subset \mathcal{N}_\varepsilon$ be such that

$$J_\varepsilon(u_n,v_n) \to c \text{ and } \|J_\varepsilon'(u_n,v_n)\|_* = o_n(1).$$

Then there exists $(\lambda_n) \subset \mathbb{R}$ satisfying

$$J_\varepsilon'(u_n,v_n) = \lambda_n\phi_\varepsilon'(u_n,v_n) + o_n(1), \tag{3.11}$$

where ϕ_ε was defined in Lemma 3.4. Since $(u_n,v_n) \in \mathcal{N}_\varepsilon$ we have that

$$0 = J_\varepsilon'(u_n,v_n)(u_n,v_n) = \lambda_n\phi_\varepsilon'(u_n,v_n)(u_n,v_n) + o_n(1)\|(u_n,v_n)\|_\varepsilon.$$

Straightforward calculations show that $((u_n,v_n))$ is bounded. Moreover, in view of Lemma 3.4, we may suppose that $\phi_\varepsilon'(u_n,v_n)(u_n,v_n) \to l < 0$. Hence, the above expression shows that $\lambda_n \to 0$ and, therefore, we conclude that $J_\varepsilon'(u_n,v_n) \to 0$ in the dual space of X_ε. It follows from Lemma 3.2 that $((u_n,v_n))$ has a convergent subsequence. □

As a byproduct of the above proof we have the following.

Corollary 3.5. *The critical points of the functional J_ε constrained to $\mathcal{N}_\varepsilon$ are critical points of J_ε in X_ε.*

4. Proof of the main theorem

In this section, we prove our multiplicity result. Firstly, we shall obtain multiple solutions of the modified problem. Later, we show that these solutions also solve the original problem if $\varepsilon > 0$ is sufficiently small.

We start by noticing that, by Proposition 2.3, there exist $(w_1, w_2) \in X$ such that $w_1, w_2 > 0$ in $\mathbb{R}^N$ and

$$I'_{x_0}(w_1, w_2) = 0 \text{ and } I_{x_0}(w_1, w_2) = C(x_0) = C^* < \frac{1}{N}\widetilde{S}_K^{N/2}.$$

Let us consider $\delta > 0$ such that $M_\delta \subset \Lambda$ and $\psi \in C_0^\infty(\mathbb{R}^+, [0,1])$ a non-increasing function such that $\psi \equiv 1$ on $[0, \delta/2]$ and $\psi \equiv 0$ on $[\delta, \infty)$. For any $y \in M$, we define $\Psi_{i,\varepsilon,y} \in X_\varepsilon$ by setting

$$\Psi_{i,\varepsilon,y}(x) = \psi(|\varepsilon x - y|) w_i\Big(\frac{\varepsilon x - y}{\varepsilon}\Big), \ i = 1, 2,$$

and denote by $t_\varepsilon > 0$ the unique positive number satisfying

$$\max_{t \geq 0} J_\varepsilon(t(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})) = J_\varepsilon(t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})).$$

Let $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ be given by

$$\Phi_\varepsilon(y) = t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y}).$$

Since $I_{x_0}(w_1, w_2) = C^*$ and M is compact, it follows from the above definitions and Lebesgue's theorem that

$$\lim_{\varepsilon \to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = C^*, \text{ uniformly for } y \in M. \tag{4.1}$$

By following some known arguments and using the definition of H we can establish the following technical result.

Lemma 4.1. *Let $\varepsilon_n \to 0^+$ and $((u_n, v_n)) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}((u_n, v_n)) \to C^*$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(\widetilde{u}_n, \widetilde{v}_n) := (u_n(\cdot + \widetilde{y}_n), v_n(\cdot + \widetilde{y}_n))$ has a convergent subsequence in X. Moreover, up to a subsequence, $(y_n) := (\varepsilon_n \widetilde{y}_n)$ is such that $y_n \to y \in M$.*

Proof. The proof is an adaptation of the subcritical case proved in [2, Lemma 3.6] and therefore we only sketch the main arguments. Since the sequence $((u_n, v_n))$ is bounded we can argue as in Lemma 2.2 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(\widetilde{y}_n)} \big(|u_n|^2 + |v_n|^2\big) \geq \gamma > 0.$$

Thus, by setting $(\widetilde{u}_n(x), \widetilde{v}_n(x)) := (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n))$ and going to a subsequence if necessary, we can assume that

$$(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}, \widetilde{v}) \neq (0,0) \text{ weakly in } X. \tag{4.2}$$

Let $(t_n) \subset \mathbb{R}^+$ be such that $(\widehat{u}_n, \widehat{v}_n) := t_n(\widetilde{u}_n, \widetilde{v}_n) \in \mathcal{M}_{x_0}$. The definition of I_{x_0} and H, along with condition (WV_2), imply that $I_{x_0}(\widehat{u}_n, \widehat{v}_n) \to C^*$ and therefore $(\widehat{u}_n, \widehat{v}_n) \not\to (0,0)$ in X. It follows that (t_n) is bounded and, up to a subsequence, $t_n \to t_0 > 0$. Arguing as in [3, Proposition 2.1] we conclude that $(\widehat{u}_n, \widehat{v}_n) \to t_0(\widetilde{u}, \widetilde{v}) = (\widehat{u}, \widehat{v})$ strongly in X. This is the first part of the lemma.

If we now define $y_n := \varepsilon_n \widetilde{y}_n$, it can be proved that, up to a subsequence, $y_n \to y \in \overline{\Lambda}$. We shall prove that $C(y) = C^*$. If this is true, we can use the property (C_0) stated in the introduction to conclude that $y \in M$. Arguing by contradiction, we suppose that $C^* < C(y)$. So, recalling that $(\widehat{u}_n, \widehat{v}_n) \to (\widehat{u}, \widehat{v})$, we can use (WV_2) and Fatou's lemma to get

$$\begin{aligned} C^* &< C(y) = I_y(\widehat{u}, \widehat{v}) \\ &= \liminf_{n\to\infty} \Big[\frac{1}{2} \int \Big(|\nabla \widehat{u}_n|^2 + |\nabla \widehat{v}_n|^2 + W(\varepsilon_n x + y_n)\widehat{u}_n^2 + V(\varepsilon_n x + y_n)\widehat{v}_n^2 \Big) \\ &\quad - \int_{\mathbb{R}^N} \big(Q(\widehat{u}_n, \widehat{v}_n) + \frac{1}{2^*} K(\widehat{u}_n, \widehat{v}_n) \big) \Big] \\ &\leq \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n, t_n v_n) \leq \liminf_{n\to\infty} J_{\varepsilon_n}(u_n, v_n) = C^*, \end{aligned}$$

which does not make sense. Thus, $C(y) = C^*$ and the proof is concluded. □

We now choose $\rho = \rho(\delta) > 0$ in such a way that $M_\delta \subset B_\rho(0)$ and consider $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ defined as $\Upsilon(x) := x$ for $|x| \leq \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. Let $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ be given by

$$\beta_\varepsilon(u,v) := \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)|u(x)|^2}{\int_{\mathbb{R}^N} |u(x)|^2} + \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)|v(x)|^2}{\int_{\mathbb{R}^N} |v(x)|^2}.$$

Since $M \subset B_\rho(0)$, we can use the definition of Υ and Lebegue's theorem to conclude that

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly for } y \in M. \tag{4.3}$$

Consider the set $\Sigma_\varepsilon := \{(u,v) \in \mathcal{N}_\varepsilon : J_\varepsilon(u,v) \leq C^* + h(\varepsilon)\}$, where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use (4.1) to conclude that $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - C^*|$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \varnothing$ for any $\varepsilon > 0$. Moreover, the following holds.

Lemma 4.2. *For any $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0^+} \sup_{u \in \Sigma_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u,v), M_\delta) = 0. \tag{4.4}$$

Proof. Let $(\varepsilon_n) \subset \mathbb{R}$ be such that $\varepsilon_n \to 0^+$. By definition, there exists $((u_n, v_n)) \subset \Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(u_n, v_n), M_\delta) = \sup_{(u,v) \in \Sigma_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(u,v), M_\delta) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n, v_n) - y_n| = o_n(1). \tag{4.5}$$

Since $((u_n, v_n)) \subset \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we can use (WV_2) to obtain

$$C^* \le \inf_{(u,v) \in \mathcal{N}_{\varepsilon_n}} J_{\varepsilon_n}(u,v) \le J_{\varepsilon_n}(u_n, v_n) \le C^* + h(\varepsilon_n),$$

and, therefore, $J_{\varepsilon_n}(u_n, v_n) \to C^*$. We may now invoke Lemma 4.1 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\varepsilon_n \widetilde{y}_n) \subset M_\delta$. We set

$$(\widetilde{u}_n(x), \widetilde{v}_n(x)) := (u_n(\varepsilon_n x + y_n), v_n(\varepsilon_n x + y_n)),$$

and observe that, since $(\widetilde{u}_n, \widetilde{v}_n) \to (u, v)$ in X and $\varepsilon_n x + y_n \to y \in M$, a direct calculation shows that $\beta_{\varepsilon_n}(u_n, v_n) = y_n + o_n(1)$. The lemma is proved. □

Theorem 4.3. *For any $\delta > 0$ satisfying $M_\delta \subset \Lambda$, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system $(S_{\varepsilon,a})$ has at least $\operatorname{cat}_{M_\delta}(M)$ solutions.*

Proof. Given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can use (4.1), (4.3), (4.4), and argue as in [9, Section 6] to obtain $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram $M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$ is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Since $C^* = C(x_0) < \frac{1}{N} \widetilde{S}_K^{N/2}$, we can use the definition of Σ_ε and Proposition 3.1 to guarantee that J_ε satisfies the Palais-Smale condition in Σ_ε (taking ε_δ smaller if necessary). Standard Ljusternik-Schnirelmann theory provides at least $\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon)$ non-zero critical points (u_i, v_i) of J_ε restricted to $\mathcal{N}_\varepsilon$. The same ideas contained in the proof of [8, Lemma 4.3] show that $\operatorname{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \ge \operatorname{cat}_{M_\delta}(M)$. Moreover, by Corollary 3.5, each (u_i, v_i) is a critical point of J_ε. The theorem is proved. □

Once we have obtained multiple solutions for the modified problem $(S_{\varepsilon,a})$ the proof of Theorem 1.1 follows by some standard arguments (see [10, 1]). For the sake of completeness, we give a sketch below.

Proof of Theorem 1.1. For any $\varepsilon > 0$, we define

$$m_\varepsilon := \sup\left\{ \max_{\partial\Lambda_\varepsilon} |(u_\varepsilon, v_\varepsilon)| : (u_\varepsilon, v_\varepsilon) \in \mathcal{N}_\varepsilon \text{ is a solution of } (S_{\varepsilon,a}) \right\}.$$

We claim that, for $\varepsilon > 0$ small, the number m_ε is finite. Indeed, suppose by contradiction that for some sequence $\varepsilon_n \to 0^+$ we have $m_{\varepsilon_n} = \infty$. Then, there exists $b > 0$ and a sequence $(x_n) \subset \partial\Lambda_{\varepsilon_n}$ such that

$$\min\{u_{\varepsilon_n}(x_n), v_{\varepsilon_n}(x_n)\} \geq b > 0.$$

It follows from [1, Proposition 3.1] that $\lim_{n\to\infty} C(x_n) = C^*$, which contradicts (C_0). The same kind of argument shows that $\lim_{\varepsilon\to 0^+} m_\varepsilon = 0$.

Let $\delta > 0$ such that $M_\delta \subset \Lambda$. In view of Theorem 4.3 we can choose $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the problem $(S_{\varepsilon,a})$ has at least $\operatorname{cat}_{M_\delta}(M)$ solutions and moreover $m_\varepsilon < \frac{a}{2}$. If we denote by $(u_\varepsilon, v_\varepsilon)$ one of these solutions, this last inequality and the calculations performed in [1, Section 4] show that

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq a \text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

Hence, $H(\cdot, u_\varepsilon, v_\varepsilon) \equiv Q(u_\varepsilon, v_\varepsilon) + \frac{1}{2^*}K(u_\varepsilon, v_\varepsilon)$, and therefore, $(u(x), v(x)) := (u_\varepsilon(x/\varepsilon), v_\varepsilon(x/\varepsilon))$ is a solution of the original system (S_ε).

By denoting $u_\varepsilon^- = \max\{-u_\varepsilon, 0\}$ and $v_\varepsilon^- = \max\{-v_\varepsilon, 0\}$ the negative part of u_ε and v_ε, respectively, we can use $(u_\varepsilon^-, v_\varepsilon^-)$ as a test function in the weak formulation of the system (S_ε) to conclude that u_ε and v_ε are nonnegative functions. By combining $(\mathcal{H}_4)$ with the maximum principle, it follows that both u_ε and v_ε are positive in $\mathbb{R}^N$.

The asymptotic behavior of the maximum points of the solutions can be proved by using Lemma 2.4 and following the same lines as in [1, Section 5]. We omit the details. □

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