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ELLIPTIC FREE BOUNDARY TRANSMISSION, THE BERNOULLI AND THE OBSTACLE PROBLEMS

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Abstract. An elliptic boundary problem with a solution-dependent measure is studied. Applications to elliptic transmission problems with a free interface and to a generalized elliptic obstacle problem with a nonzero Neumann term on the free boundary are given.

1. Introduction

The purpose of this paper is to study an elliptic boundary problem with a solution-dependent measure and to apply the abstract result to the obstacle problem with nonzero Neumann condition on the free boundary and to the Bernoulli problem.

Let Ω be a bounded open subset of R^3 with a smooth boundary and let $a_j(s)$ be a continuous function on R for $s \neq \alpha$ with a finite jump at $\alpha > 0$. Consider the transmission problem

$$\begin{aligned} & A(y_\pm)y_\pm = -\sum_{j=1}^{3} D_j\{a_j(y_\pm)D_j y_\pm\} = f \ \text{ in } \Omega_u^\pm(\alpha) , \\ & y_- = 0 \ \text{ on } \partial\Omega \ , \ \ y_+ = y_- = \alpha \ \text{ on } \partial\Omega_u^+(\alpha), \\ & \sum_{j=1}^{3} \nu_j\big\{a_j(\alpha^+)D_j y_+ - a_j(\alpha^-)D_j y_-\big\} = -g \ \text{ on } \partial\Omega_u^+(\alpha), \end{aligned} \tag{1.1}$$

where $\Omega_u^+(\alpha) = \big\{x : \ x \in \Omega, \, \alpha < u(x)\big\}, \ \Omega_u^-(\alpha) = \Omega/\overline{\Omega}_u^+(\alpha)$ and u is in the admissible set of controls $\mathcal{U}$

$$\mathcal{U} = \big\{u : \ u \geq 0, \ \|u\|_{C^{2,\lambda}(\Omega)\cap H_0^1(\Omega)} \leq 1\big\} \tag{1.2}$$

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with $\alpha \in (0, \inf_{u\in\mathcal{U}} \max_\Omega u)$. It is clear that $\partial\Omega_u^+(\alpha) = \big\{x : x \in \Omega,\, u(x) = \alpha\big\} \neq \emptyset,\ \partial\Omega \cap \partial\Omega_u^+(\alpha) = \emptyset$. The interface $\partial\Omega_u^+(\alpha)$ is of class C^1. We shall rewrite (1.1) as

$$A(y)y + \mu(u) = f \text{ in } \Omega \quad , \quad y = 0 \text{ on } \partial\Omega,\ \ y = \alpha \text{ on } \partial\Omega_u^+(\alpha),$$
$$\langle \mu(u), \varphi\rangle = \int_{\partial\Omega_u^+(\alpha)} g\varphi d\sigma \ \ \forall\varphi \in H_0^1(\Omega). \tag{1.3}$$

The existence of a solution y in $H_0^1(\Omega)$ of (1.3) with fixed interior interface $\partial\Omega_u^+(\alpha)$ will be established in Section 2. The interface in (1.1) plays the role of a control similar to that found in the theory of optimal shape domains and feedback controls and leads us to consider a new type of nonlinear problem, elliptic equations with a *solution-dependent measure.*

Let P be the projection of $L^2(\Omega)$ onto the compact convex subset $\mathcal{U}$ and consider the problem

$$A(y)y + \mu(Py) = f \text{ in } \Omega, y = 0 \text{ on } \partial\Omega\ , y = \alpha \text{ on } \partial\Omega_{Py}^+(\alpha),$$
$$\langle \mu(Py),\, \varphi\rangle = \int_{\partial\Omega_{Py}^+(\alpha)} g\varphi d\sigma \ \ \forall\varphi \in H_0^1(\Omega). \tag{1.4}$$

The existence of a solution y in $H^1(\Omega_{Py}^+(\alpha))$ of (1.4) is established in Section 3.

Elliptic free boundary transmission problems have been studied by Alt-Cafarelli-Friedman [1] and by many others. Our approach is different. By solving an elliptic problem with a solution-dependent measure we can treat the free boundary in a simple manner, using the “smooth” part Py of the solution instead of the “entire” solution y itself.

The interior Bernoulli free boundary problem will be shown to be a special case of (1.4). The nonstationary motion of an elastic membrane inside an incompressible viscous fluid has been recently the subject of extensive investigations by Beirao Da Viega [3], Coutland-Shkoller [8], Mouchair-Zolesio [11] and others.

We shall consider a stationary incompressible irrotational horizontal fluid flow in a planar domain Ω and let y be its stream function. Suppose the fluid moves in Ω around an elastic solid $\partial\Omega_u(\alpha)$ representing the membrane of a biological cell. Since both $\partial\Omega$ and $\partial\Omega_u(\alpha)$ are streamlines of the flow we have $y = 0$ on $\partial\Omega$ and $y = \alpha$ on $\partial\Omega_u(\alpha)$. The Bernoulli principle states that $|\nabla y| = \textit{constant}$ on $\partial\Omega_u(\alpha)$. The existence of a solution of the following problem will be shown:

$$-\Delta y = 0 \text{ in } \Omega_{Py}^-(\alpha), \qquad y = 0 \text{ on } \partial\Omega, \tag{1.5}$$

$$y = \alpha \text{ on } \partial\Omega^+_{Py}(\alpha), \quad \nabla y.\nu = g \text{ on } \partial\Omega^+_{Py}(\alpha)$$

where $\Omega^-_{Py}(\alpha) = \{x : x \in \Omega, 0 < Py(x) < \alpha\}$, $\Omega^+_{Py}(\alpha) = \Omega/\overline{\Omega}^-_{Py}(\alpha)$.

Pioneering work on nonlinear elliptic equations with a solution-dependent measure was done by Diaz-Padial-Rakotoson [9]. Using the Ambrosetti-Rabinowitz minimax theorem and a measure depending on the solution y, Bernoulli type problems were studied in [9]. In this paper we shall use Py instead of y; this allows us to treat the Laplacian operator as in general y may not be smooth enough.

In Section 4, we shall consider an obstacle problem. In the 60's and 70's the theory of variational inequalities was used by Browder [7], Brezis [6], Lions and Stampacchia [10], Stampacchia [14] and others to study elliptic obstacle problems. In spite of its elegance the theory does not allow us to treat the case of nonzero Neumann terms on the free boundary and the study of its smoothness presents tremendous difficulties. By using the projection P of $L^2(\Omega)$ onto the compact subset $\mathcal{U}$, by introducing the solution-dependent measure $\mu(y)$ and by requiring that $Py > 0$ instead of $y > 0$, the problem becomes easier. We shall consider the elliptic obstacle problem

$$\begin{aligned} A(y)y = f \text{ in } \Omega^+_{Py} \quad &, \quad y = 0 \text{ on } \partial\Omega^+_{Py}, \\ -\sum_{j=1}^{3} \{a_j(y)D_j y\}\nu_j \quad &= \quad g \text{ on } \partial\Omega^+_{Py}/\partial\Omega \end{aligned} \tag{1.6}$$

with $\Omega^+_{Py} = \{x : x \in \Omega,\, 0 < Py(x)\}, y = 0$ in $\{x : x \in \overline{\Omega}, Py(x) = 0\}$.

Elliptic boundary problems with Radon measure sources (independent of the solution) have been studied by Betta et al [3], by Benilan et al [2], by Boccardo-Gallouet [4] and others. Linear elliptic transmission problems with fixed interfaces have been the subject of extensive investigations by Schechter [13] and others.

2. Fixed interface

The main result of the section is the following theorem.

Theorem 2.1. *Let $\{f, g, u\}$ be in $L^2(\Omega) \times C^1(\Omega) \times \mathcal{U}$, and let $a_j(s)$ be continuous functions on $R/\{\alpha\}$ with a finite jump at $\alpha > 0$. Suppose that*

$$0 < a_j(s) \leq M \ \ \forall s, \quad 0 < a_0 \sum_{j=1}^{3} (D_j y)^2 \leq \sum_{j=1}^{3} a_j(y)(D_j y)^2.$$

Then there exists a solution $\{y,\, \mu(u)\}$ *of* (1.3) *with*

$$\{y,\, \mu(u)\} \in H_0^1(\Omega) \times H^{-1}(\Omega),\ y = \alpha \quad on\ \partial\Omega_u^+(\alpha).$$

Moreover,

$$\|y\|_{H_0^1(\Omega)} + \|\mu(u)\|_{H^{-1}(\Omega)} \le C\{1 + \|f\|_{L^2(\Omega)} + \|g\|_{C^1(\Omega)}\},$$

where C *is a constant independent of* u, α.

Remark. The key assertion of the theorem is the estimate on $\|\mu(u)\|_{H^{-1}(\Omega)}$. Note that an application of the trace theorem on the integral defining μ gives rise to an estimate depending on u and on α through the cone condition of the open set $\Omega_u^+(\alpha)$. The estimate of the theorem is needed when we consider the case of a free interface.

We now proceed to construct such a μ by considering functions approximating the delta function with mass at α. Similar F_k have been used by Diaz-Padial-Rakotoson in [9].

Let $F_k(s;\, \alpha)$ be positive Lipschitz continuous functions with

$$F_k(s;\alpha) = \begin{cases} k & \text{if } s = \alpha \\ 0 & \text{if } s \ge \alpha + k^{-1} \text{ or if } s \le \alpha - k^{-1} \end{cases}$$

and such that

$$\int_\infty^\alpha F_k(s;\, \alpha) ds = 1 = \int_\alpha^\infty F_k(s;\alpha) ds.$$

The functions F_k approximate the delta function with mass at $\alpha > 0$.

Consider the elliptic boundary problem

$$\begin{aligned} &-\Delta z_k + F_k((z_k + u)^+;\alpha) = 1 + \Delta u \text{ in } \Omega \\ &z_k = 0 \text{ on } \partial\Omega \quad,\ z_k = 0 \text{ on } \partial\Omega_u^+(\alpha) \\ &\nu.\{\nabla(z_k|_{\Omega_u^-(\alpha)}) - \nabla(z_k|_{\Omega_u^+(\alpha)})\} = 0 \text{ on } \partial\Omega_u^+(\alpha). \end{aligned} \tag{2.1}$$

Lemma 2.1. *Let* u *be in* $\mathcal{U}$; *then there exists a unique solution* z_k *of* (2.1). *Moreover,*

$$\|z_k + u\|_{H_0^1(\Omega)} + \|F_k((z_k + u)^+;\alpha)\|_{H^{-1}(\Omega)} \le C|\Omega|,$$

where C *is independent of* z_k, u, α.

Proof. 1) Consider the boundary problem

$$-\Delta \tilde{z}_k + F_k((\tilde{z}_k + u)^+;\alpha) = 1 + \Delta u \text{ in } \Omega_u^+(\alpha),\ \tilde{z}_k = 0 \text{ on } \partial\Omega_u^+(\alpha).$$

There exists a solution $\tilde{z}_k$ of the problem and $\tilde{z}_k$ in $H_0^1(\Omega_u^+(\alpha))$.

We now consider the elliptic boundary problem

$$-\Delta \hat{z}_k + F_k((\hat{z}_k+u)^+;\alpha) = 1 + \Delta u \text{ in } \Omega_u^-(\alpha), \;\; \hat{z}_k = 0 \text{ on } \partial\Omega_u^-(\alpha).$$

Again there exists a solution $\hat{z}_k$ of the boundary problem and $\hat{z}_k$ in $H_0^1(\Omega_u^-(\alpha))$.

Let $z_k = \tilde{z}_k$ in $\Omega_u^+(\alpha)$, $z_k = \hat{z}_k$ in $\Omega_u^-(\alpha)$; then $z_k \in H_0^1(\Omega)$ and $z_k = 0$ on $\partial\Omega_u^+(\alpha)$ as $\partial\Omega_u^-(\alpha) = \partial\Omega \cup \partial\Omega_u^+(\alpha)$. It is clear that z_k is the unique solution of (2.1).

2) We shall now establish the estimate. We have

$$c_0\|z_k+u\|^2_{H_0^1(\Omega)} + (F_k((z_k+u)^+;\alpha),\, z_k+u) \le C|\Omega|^2.$$

From the definition of F_k we get

$$\begin{aligned}(F_k((z_k+u)^+;\alpha),\, z_k+u) &= (F_k((z_k+u)^+;\alpha),\, (z_k+u)^+ - (z_k+u)^-)\\ &= (F_k((z_k+u)^+;\alpha), (z_k+u)^+) \ge 0,\end{aligned}$$

and thus,

$$\|z_k+u\|_{H_0^1(\Omega)} \le C|\Omega|.$$

From the equation we obtain

$$\|F_k((z_k+u)^+;\alpha)\|_{H^{-1}(\Omega)} \le \|\nabla(z_k+u)\|_{L^2(\Omega)} + c|\Omega| \le C|\Omega|.$$

The lemma is proved. □

Let j_ε be the usual mollifier, set $a_{j,\varepsilon} = j_\varepsilon * a_j$, and consider the elliptic problem

$$-\sum_{j=1}^{3} D_j\big\{a_{\varepsilon,j}(y_k+u)D_j(y_k+u)\big\} + gF_k((z_k+u)^+;\alpha) = f \;\text{ in } \Omega$$

$$\nu.\nabla\big\{y_k|_{\Omega_u^-(\alpha)} - y_k|_{\Omega_u^+(\alpha)}\big\} = 0\,,\; y_k = 0 \text{ on } \partial\Omega_u^+(\alpha)\cup\partial\Omega. \qquad (2.2)$$

Lemma 2.2. *Let F_k, z_k be as in Lemma* 2.1*. Then there exists a solution y_k of* (2.2)*. Moreover,*

$$\|y_k+u\|_{H_0^1(\Omega)} \le C\big\{\|f\|_{L^2(\Omega)} + \|g\|_{C^1(\Omega)} + |\Omega|\big\},$$

where C is independent of $\varepsilon, k, u, \alpha$.

Proof. 1) Consider the problem

$$-\sum_{j=1}^{3} D_j\{a_{\varepsilon,j}(\hat{y}_k+u)D_j(\hat{y}_k+u)\} \;\;=\;\; -gF_k((z_k+u)^+;\alpha) + f \;\text{ in }\; \Omega_u^-(\alpha),$$

$$\hat{y}_k = 0 \;\text{ on } \partial\Omega \quad , \quad \hat{y}_k = \alpha \;\text{ on } \partial\Omega_u^+(\alpha).$$

The existence of a solution $\hat{y}_k$ in $H^1(\Omega_u^-(\alpha))$ of the problem is known.

Next we consider the problem

$$-\sum_{j=1}^{3} D_j\{a_{\varepsilon,j}(\tilde{y}_k+u)D_j(\tilde{y}_k+u)\} = -gF_k((z_k+u)^+;\alpha)+f \text{ in } \Omega_u^+(\alpha),$$
$$\tilde{y}_k = \alpha \text{ on } \partial\Omega_u^+(\alpha).$$

Again there exists a solution $\tilde{y}_k$ in $H^1(\Omega_u^+(\alpha))$ of the problem. Set $y_k = \hat{y}_k$ in $\Omega_u^-(\alpha)$, $y_k = \tilde{y}_k$ in $\Omega_u^+(\alpha)$. Since $\tilde{y}_k = \hat{y}_k = \alpha$ on $\partial\Omega_u^+(\alpha) = \partial\Omega_u^-(\alpha)/\partial\Omega$ we get y_k in $H_0^1(\Omega)$. With f, F_k in $L^2(\Omega)$, it is trivial to check that y_k is a solution of (2.2).

2) We now establish the estimate of the lemma. With u being in $H_0^1(\Omega)$ we obtain

$$\begin{aligned} a_0\|y_k+u\|^2_{H_0^1(\Omega)} &\le C\{\|f\|^2_{L^2(\Omega)}+\|gF_k\|^2_{H^{-1}(\Omega)}\} \\ &\le C\{\|f\|^2_{L^2(\Omega)}+\|g\|^2_{C^1(\Omega)}\|F_k((z_k+u)^+;\alpha)\|^2_{H^{-1}(\Omega)}\} \\ &\le C\{1+\|g\|^2_{C^1(\Omega)}+\|f\|^2_{L^2(\Omega)}+|\Omega|^2\}, \end{aligned}$$

where C is a constant independent of $\varepsilon, k, u, \alpha$. □

Lemma 2.3. *Suppose all the hypotheses of Theorem* 2.1 *are satisfied and let* z_k, F_k *be as in Lemma* 2.1. *Then there exists* $\tilde{y}_k$ *in* $H_0^1(\Omega)$ *such that*

$$\begin{aligned} A(\tilde{y}_k)\tilde{y}_k+gF_k((z_k+u)^+;\alpha) &= f \ \text{ in } \Omega,\ \tilde{y}_k=0 \ \text{ on } \partial\Omega, \\ \nu.\nabla\{\tilde{y}_k|_{\Omega_u^-(\alpha)}-\tilde{y}_k|_{\Omega_u^+(\alpha)}\}=0 \quad &, \quad \tilde{y}_k=\alpha \ \text{ on } \partial\Omega_u^+(\alpha). \end{aligned} \tag{2.3}$$

Moreover,

$$\|\tilde{y}_k\|_{H_0^1(\Omega)}+\|gF_k((z_k+u)^+;\alpha)\|_{H^{-1}(\Omega)} \le C\{1+\|f\|_{L^2(\Omega)}+\|g\|_{C^1(\Omega)}+|\Omega|\},$$

where C is a constant independent of k, u, α.

Proof. Let $y_{\varepsilon,k}$ be as in Lemma 2.2 and set $\tilde{y}_{\varepsilon,k}=y_{\varepsilon,k}+u$; then $\|\tilde{y}_{\varepsilon,k}\|_{H_0^1(\Omega)} \le C$, $\tilde{y}_{\varepsilon,k}=\alpha$ on $\partial\Omega_u^+(\alpha)$. Clearly, $y_{\varepsilon,k}\to\tilde{y}_k$ in $(H_0^1(\Omega))_{weak}$, $\tilde{y}_k=\alpha$ on $\partial\Omega_u^+(\alpha)$ and an elementary argument gives

$$\int_{\Omega_u^\pm(\alpha)} a_{\varepsilon,j}(y_{\varepsilon,k})D_jy_{\varepsilon,k}D_j\varphi dx \to \int_{\Omega_u^\pm(\alpha)} a_j(\tilde{y}_k)D_j\tilde{y}_kD_j\varphi dx \ \ \forall\varphi\in H_0^1(\Omega).$$

The lemma is proved. □

Proof of Theorem 2.1. 1) Let $\{\tilde{y}_k,\, z_k\,, gF_k((z_k+u)^+;\alpha)\}$ be as in Lemmas 2.1, 2.3; then there exists a subsequence such that $\{\tilde{y}_k,\, z_k,\, gF_k((z_k+$

$u)^+;\alpha)\big\} \to \big\{\tilde{y},\, z,\, \mu\big\}$ in $\{(H_0^1(\Omega))_{weak} \cap L^2(\Omega)\} \times \{(H_0^1(\Omega))_{weak} \cap L^2(\Omega)\} \times (H^{-1}(\Omega))_{weak}$ and

$$\|\tilde{y}\|_{H^1(0,\Omega)} + \|\mu\|_{H^{-1}(\Omega)} \le C\big\{1 + |\Omega| + \|f\|_{L^2(\Omega)} + \|g\|_{C^1(\Omega)}\big\}.$$

2) It is clear that

$$\int_{\Omega_u^\pm(\alpha)} a_j(\tilde{y}_k) D_j \tilde{y}_k D_j \varphi dx \to \int_{\Omega_u^\pm(\alpha)} a_j(\tilde{y}) D_j \tilde{y} D_j \varphi dx$$

for all $\varphi \in H_0^1(\Omega)$. Since

$$\|\tilde{y}_k - \tilde{y}\|_{L^2(\partial\Omega_u^+(\alpha))} \le C(u,\alpha)\|\tilde{y}_k - \tilde{y}\|_{H^{1/2}(\Omega)}$$

we have $\tilde{y} = \alpha$ on $\partial\Omega_u^+(\alpha)$.

3) We now show that

$$\langle \mu,\, \varphi\rangle = \int_{\partial\Omega_u^+(\alpha)} g\varphi d\sigma \;\; \forall \varphi \in H_0^1(\Omega).$$

Since $z_k = 0$ on $\partial\Omega_u^+(\alpha)$ we have $\partial\Omega_u^+(\alpha) \subset \{x:\, z_k + u = \alpha\}$.

Let $\varphi \in C_0^1(\Omega)$ with $\mathrm{supp}(\varphi) \subset \{x:\, x \in \Omega, u < \alpha\}$. Let

$$\eta > \{\alpha - u_{\max}\}/2 > 0, \;\; u_{\max} = \max_{supp\varphi} u.$$

Since $z_k + u \to z + u$ in $L^2(\Omega)$ and almost everywhere, by Egorov's theorem there exists $\Omega_\varepsilon \subset \Omega$ such that $z_k \to z$ uniformly on Ω_ε and $\mathrm{meas}(\Omega/\Omega_\varepsilon) \le \varepsilon$. Since $(z_k + u)^+ = z_k^+ + u$ for almost all x we have

$$\begin{aligned}
&\{x:\, x \in \mathrm{supp}(\varphi) \cap \Omega_\varepsilon,\, 0 < z + u - \eta < \alpha + k^{-1} - \eta\} \\
&\subset \{x:\, x \in \mathrm{supp}(\varphi),\, 0 < z_k + u \le \alpha + k^{-1} - \{-u_{\max} + \alpha\}/2\} \\
&\subset \{x:\, x \in \mathrm{supp}(\varphi),\, 0 < z_k + u \le k^{-1} + \{u_{\max} + \alpha\}/2\} \\
&\subset \{x:\, x \in \Omega,\, 0 < z_k + u \le \alpha - k^{-1}\}
\end{aligned}$$

for $0 < 4k^{-1} < \alpha - u_{\max}$.

By construction $F_k(s;\, \alpha) = 0$ for $s \le \alpha - k^{-1}$ and therefore

$$\int_{\Omega_\varepsilon} gF_k((z_k + u)^+;\alpha)\varphi dx = 0 \;\forall \varphi \in C_0(\Omega),\, \mathrm{supp}\varphi \subset \{x:\, x \in \Omega,\, u(x) < \alpha\}.$$

We have

$$\begin{aligned}
&\Big|\int_\Omega gF_k((z_k + u)^+, a)\varphi dx\Big| \\
&= \Big|\int_{\Omega_\varepsilon} gF_k((z_k + u)^+, a)\varphi dx + \int_{\Omega/\Omega_\varepsilon} gF_k(z_k + u)^+, a)\varphi dx\Big|
\end{aligned}$$

$$= \Big| \int_{\Omega/\Omega_\varepsilon} gF_k((z_k+u)^+, a)\varphi dx \Big|$$
$$\le C\|F_k((z_k+u)^+, a)\|_{H^{-1}(\Omega)}\|g\|_{H_0^1(\Omega)}\|\varphi\|_{L^\infty(\Omega/\Omega_\varepsilon)}$$
$$\le C\|\varphi\|_{C(\Omega)}\{\varepsilon + [\text{meas}(\Omega/\Omega_\varepsilon)]^{1/s}\} \le C\varepsilon^{1/s}, \;\; s \ge s_0,$$

as

$$\|\varphi\|_{L^\infty(\Omega/\Omega_\varepsilon)} = \lim_{s\to\infty}\|\varphi\|_{L^s(\Omega/\Omega_\varepsilon)} \le \|\varphi\|_{L^s(\Omega/\Omega_\varepsilon)} + \varepsilon, \;\; s \ge s_0.$$

Hence,

$$\int_\Omega gF_k((z_k+u)^+, a)\varphi dx = 0 \;\; \forall\varphi \in C_0^1(\Omega), \;\text{supp}(\varphi) \subset \{x: \, x \in \Omega, \, u(x) < a\}.$$

Thus, $\text{supp}(\mu) \subset \big\{x: \, x \in \Omega, \, u(x) \ge \alpha\big\}$.

Let $\varphi \in C_0^1(\Omega)$ with $\text{supp}(\varphi) \subset \{x: \, x \in \Omega, \, u(x) > \alpha\}$. Set

$$u_{\min} = \min_{\text{supp}\varphi} u \ge \alpha - \varepsilon.$$

We have for almost all x

$$\{x: \, x \in \text{supp}\varphi, \; z + u \ge u_{\min} - k^{-1}\}$$
$$\subset \{x: \, x \in \text{supp}\varphi, \;\; z_k + u \ge (\alpha + u_{\min})/2 - k^{-1}\}$$
$$\subset \{x: \, x \in \Omega, \; z_k + u > \alpha + k^{-1}\}$$

for $2k^{-1} < u_{\min} - \alpha$. From the definition of F_k we deduce that

$$\int_\Omega gF_k((z_k+u)^+; \alpha)\varphi dx = 0 \;\; \forall\,\varphi \in C_0(\Omega), \;\text{supp}\varphi \subset \{x: \, x \in \Omega, u(x) > \alpha\}.$$

Therefore, $\text{supp}\mu \subset \big\{x: \, x \in \Omega, \, u(x) \le \alpha\big\}$ Combining the two cases we get $\text{supp}\mu \subset \big\{x: \, x \in \Omega, \, u(x) = \alpha\big\}$. Let $S = \big\{x: \, x \in \Omega, u(x) = \alpha\big\}/\text{supp}\mu$ and suppose that $S \ne \emptyset$. Let $\varphi \in C_0^1(\Omega)$ with $\text{supp}\,\varphi \subset S$, then $\langle \mu, \varphi\rangle = 0$. On the other hand we have

$$\langle \mu, \varphi\rangle = \lim_k \int_\Omega gF_k((z_k+u)^+; \alpha)\varphi dx$$
$$= \int_{\{x: x\in\Omega, u(x)=\alpha\}} \lim_k \Big\{ \int_{\alpha-k^{-1}}^{\alpha+k^{-1}} g((u+z_k)^{-1}(s))\varphi((u+z_k)^{-1}(s))F_k(s)ds\Big\} d\sigma$$
$$= \int_{\{x: x\in\Omega, \, u(x)=\alpha\}} \varphi(x)g(x)d\sigma \ne 0,$$

where $u^{-1}(s)$ denotes $\{x: \, x \in \Omega, \, u(x) = s\}$. We get a contradiction and therefore $\text{supp}\mu = \partial\Omega_u^+(\alpha)$. The theorem is proved. $\square$

3. Free interface

In this section we shall establish the existence of a solution for an elliptic free boundary transmission problem. The interface plays the role of a control as in the theory of optimal shape domains and feedback controls lead us to a new type of nonlinear problem, elliptic problems with *solution-dependent measure.* An application to the interior free boundary Bernoulli problem of fluid flows will be given.

Let P be the projection of $L^2(\Omega)$ onto the compact convex subset $\mathcal{U}$ defined by

$$\|y - Py\|_{L^2(\Omega)} = \inf\Big\{\|y-u\|_{L^2(\Omega)} : \ \ \forall u \in \mathcal{U}\Big\}.$$

It is well known that for a given $y \in L^2(\Omega)$ there exists a unique $Py \in \mathcal{U}$ and that P is a nonexpansive mapping in $L^2(\Omega)$

$$\|Py - Pz\|_{L^2(\Omega)} \le \|y - z\|_{L^2(\Omega)} \ \ \forall y, z \in L^2(\Omega).$$

The main result of the section is the following theorem.

Theorem 3.1. *Let $\{f,\, g,\, \alpha\}$ be as in Theorem 2.1. Then there exists a solution $\{y,\, \mu(y)\}$ in $H_0^1(\Omega)\times H^{-1}(\Omega)$ of the elliptic problem*

$$\begin{aligned} A(y)y + \mu(y) = f \ \text{ in } \Omega \quad , \quad y = 0 \ \text{ on } \partial\Omega,\ \ y = \alpha \ \text{ on } \partial\Omega^+_{Py}(\alpha), \\ \langle \mu(y), \varphi\rangle \ \ = \ \ \int_{\partial\Omega^+_{Py}(\alpha)} g\varphi d\sigma \ \forall \varphi \in H_0^1(\Omega). \end{aligned} \tag{3.1}$$

Moreover,

$$\|y\|_{H_0^1(\Omega)} + \|\mu(y)\|_{H^{-1}(\Omega)} \le C\big\{1 + |\Omega| + \|f\|_{L^2(\Omega)} + \|g\|_{C^1(\Omega)}\big\},$$

where C is a constant independent of α.

Remark. It seems that (3.1) represents a new type of nonlinear elliptic problems, an equation with a *solution-dependent measure.*

Let

$$\mathcal{B} = \Big\{z :\ \|z\|_{H_0^1(\Omega)} \le C\{1 + |\Omega| + \|f\|_{L^2(\Omega)} + \|g\|_{C^1(\Omega)}\}\Big\}$$

with C as in Theorem 2.1. Consider the elliptic transmission problem

$$\begin{aligned} -\sum_{k=1}^{3} D_k\{a_{\varepsilon,k}(z)D_k y\} \ \ &= \ \ -\mu(z) + f \ \text{ in } \Omega, \\ y = 0 \ \text{ on } \partial\Omega \quad &, \quad y = \alpha \ \text{ on } \partial\Omega^+_{Pz}(\alpha), \end{aligned} \tag{3.2}$$

$$\langle \mu(z), \varphi \rangle \;\; = \;\; \int_{\partial\Omega^+_{Pz}(\alpha)} g\varphi d\sigma \;\; \forall \varphi \in H^1_0(\Omega)$$

with $z \in \mathcal{B}$, $a_{\varepsilon,k} = a_k * j_\varepsilon$.

It follows from Theorem 2.1 that for each $z \in \mathcal{B}$ there exists a solution y of (3.2). Since the problem is linear the solution is unique. Let $\mathcal{A}$ be the mapping of $\mathcal{B}$, considered as a subset of $L^2(\Omega)$, into $L^2(\Omega)$ given by

$$\mathcal{A}z = y \tag{3.3}$$

where y is the unique solution of (3.2). We now show that $\mathcal{A}$ has a fixed point.

Lemma 3.1. *The mapping $\mathcal{A}$ takes $\mathcal{B}$ into $\mathcal{B}$ and is $L^2(\Omega)$-continuous.*

Proof. 1) It is clear from the estimate of Theorem 2.1 that $\mathcal{A}$ takes the compact convex subset $\mathcal{B}$ of $L^2(\Omega)$ into $\mathcal{B}$. We now show that it is $L^2(\Omega)$-continuous. Let $z_n \in \mathcal{B}$ with $z_n \to z$ in $L^2(\Omega)$. Set $u_n = Pz_n$, $y_n = \mathcal{A}z_n$. From the estimate of Theorem 2.1 we obtain (by taking subsequences) $\{y_n, z_n, Pz_n\} \to \{y, z, Pz\}$ in $\{(H^1_0(\Omega))_{weak} \cap L^2(\Omega)\} \times (H^1_0(\Omega))_{weak} \times \{C^{2,\lambda}(\Omega) \cap H^1_0(\Omega))\}$. We get

$$\int_\Omega a_{\varepsilon,j}(z_n) D_j y_n D_j \varphi dx \to \int_\Omega a_{\varepsilon,j}(z) D_j y D_j \varphi dx \;\; \forall \varphi \in H^1_0(\Omega).$$

2) Since $u_n = Pz_n \to u = Pz$ in $C^{2,\lambda}(\Omega)$, we have

$$u(x) - \varepsilon \le u_n(x) \le u(x) + \varepsilon \;\; \forall x \in \Omega, \forall n \ge n_0.$$

A simple argument gives $\Omega^-_{u+\varepsilon}(\alpha) \subset \Omega^-_{u_n}(\alpha) \subset \Omega^-_{u-\varepsilon}(\alpha)$. Thus,

$$\Omega^-_u(\alpha) = \bigcap_\varepsilon \Omega^-_{u+\varepsilon}(\alpha) \subset \bigcup_\varepsilon \Omega^-_{u+\varepsilon}(\alpha) \subset \bigcap_n \Omega^-_{u_n}(\alpha).$$

We have

$$\bigcap_n \Omega^-_{u_n}(\alpha) \subset \bigcap_\varepsilon \Omega^-_{u-\varepsilon}(\alpha) = \Omega^-_u(\alpha).$$

Combining the two inclusions we obtain $\Omega^-_u(\alpha) = \bigcap_n \Omega^-_{u_n}(\alpha)$.

4) We now show that $\mu(z_n) \to \mu(z)$ in $(H^{-1}(\Omega))_{weak}$. We have

$$\int_{\Omega^-_{u_n}(\alpha)/\Omega^-_u(\alpha)} \{gD_j\varphi - \varphi D_j g\} dx = \int_{\partial\Omega^-_{u_n}(\alpha)/\partial\Omega} g\varphi\nu_{j,n} d\sigma - \int_{\partial\Omega^-_u(\alpha)/\partial\Omega} g\varphi\nu_j d\sigma$$

for all $\varphi \in C_0^1(\Omega)$. Since $\partial\Omega^-_{u_n}(\alpha)/\partial\Omega = \partial\Omega^+_{u_n}(\alpha)$, it follows from the previous part that

$$\lim_n \int_{\partial\Omega^+_{u_n}(\alpha)} g\varphi\nu_{j,n} d\sigma = \int_{\partial\Omega^+_u(\alpha)} g\varphi\nu_j d\sigma.$$

With $u_n \to u$ in $C^{2,\lambda}(\Omega)$ we get $\nu_{j,n} \to \nu_j$ in $C^1(\Omega)$ and hence

$$\lim_n \int_{\partial\Omega^+_{u_n}(\alpha)} g\nu_j\varphi d\sigma = \int_{\partial\Omega^+_u(\alpha)} g\nu\varphi d\sigma \ \ \forall\varphi \in C_0^1(\Omega).$$

Take $\Phi = \varphi\nu_j / \sum_{k=1}^3 \nu_k^2$. It is clear that Φ is in $C_0^1(\Omega)$, and thus,

$$\lim_n \langle\mu(z_n),\,\varphi\rangle = \lim_n \int_{\partial\Omega^+_{u_n}(\alpha)} g\varphi d\sigma = \langle\mu(z),\varphi\rangle = \int_{\partial\Omega^+_u(\alpha)} g\varphi d\sigma \ \forall\varphi \in H_0^1(\Omega).$$

4) It remains to show that $y = \alpha$ on $\partial\Omega^+_{Pz}(\alpha)$. From the previous part, with $y_n - y$ instead of g,we obtain

$$\lim_n \int_{\partial\Omega^+_{u_n}(\alpha)} (y_n - y)\nu_{j,n}\varphi d\sigma = \lim_n \int_{\partial\Omega^+_u(\alpha)} (y_n - y)\nu_j\varphi d\sigma = 0$$

as $y_n \to y$ in $H^{1-\eta}(\Omega)$. Since $y_n = \alpha$ on $\partial\Omega^+_{u_n}(\alpha)$, we get

$$\begin{aligned} \lim_n \int_{\partial\Omega^+_{u_n}(\alpha)} (y_n - y)\nu_{j,n}\varphi d\sigma &= 0 = \lim_n \int_{\partial\Omega^+_{u_n}(\alpha)} (\alpha - y)\nu_{j,n}\varphi d\sigma \\ &= \int_{\partial\Omega^+_u(\alpha)} (\alpha - y)\nu_j\varphi d\sigma \ \ \forall\varphi \in H_0^1(\Omega). \end{aligned}$$

As before we deduce that $y = \alpha$ on $\partial\Omega^+_u(\alpha)$, where $u = Pz$. □

Lemma 3.2. *Suppose all the hypotheses of Theorem* 3.1 *are satisfied. Then there exists $y_\varepsilon \in H_0^1(\Omega)$, a solution of the elliptic problem*

$$\begin{aligned} -\sum_{k=1}^3 D_j\{a_{\varepsilon,j}(y_\varepsilon)D_j y_\varepsilon\} &= -\mu(y_\varepsilon) + f \ \textit{ in } \Omega, \\ y_\varepsilon = 0 \ \textit{on } \partial\Omega \quad , &\quad y_\varepsilon = \alpha \ \textit{on } \partial\Omega^+_{Py_\varepsilon(\alpha)}, \\ \langle\mu(y_\varepsilon),\varphi\rangle &= \int_{\partial\Omega^+_{Py_\varepsilon}(\alpha)} g\varphi d\sigma \ \ \forall\varphi \in H_0^1(\Omega). \end{aligned} \tag{3.4}$$

Moreover,

$$\|y_\varepsilon\|_{H_0^1(\Omega)} + \|\mu(y_\varepsilon)\|_{H^{-1}(\Omega)} \le C\{1 + |\Omega| + \|f\|_{L^2(\Omega)} + \|g\|_{C^1(\Omega)}\},$$

where C is a constant independent of ε.

Proof. The mapping $\mathcal{A}$ given by Lemma 3.2 satisfies all the hypotheses of the Schauder fixed-point theorem. Thus there exists $y_\varepsilon \in \mathcal{B}$ such that $\mathcal{A}y_\varepsilon = y_\varepsilon$.

The estimate of the theorem is an immediate consequence of that of Theorem 2.1. □

Proof of Theorem 3.1. Let y_ε , $\mu(y_\varepsilon)$ be as in Lemma 3.2; then there exists a subsequence such that $\{y_\varepsilon, Py_\varepsilon, \mu(y_\varepsilon)\} \to \{y, Py, \mu\}$ in $\{(H_0^1(\Omega))_{weak} \cap L^2(\Omega)\} \times C^{2,\lambda}(\Omega) \times (H^{-1}(\Omega))_{weak}$.

1) A proof as in that of Lemma 3.1, with $u_\varepsilon = Py_\varepsilon$ instead of $u_n = Pz_n$, gives

$$y = \alpha \text{ on } \partial\Omega^+_{Py}(\alpha),\ \langle \mu(y), \varphi\rangle = \int_{\partial\Omega^+_{Py}(\alpha)} g\varphi d\sigma\ \forall \varphi \in H_0^1(\Omega).$$

2) It remains to show that

$$\int_{\Omega^\pm_{Py_\varepsilon}(\alpha)} a_{\varepsilon,j}(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx \to \int_{\Omega^\pm_{Py}(\alpha)} a_j(y) D_j y D_j \varphi dx\ \ \forall\ \varphi \in H_0^1(\Omega).$$

Since a_j is continuous in $\Omega^\pm_{Py_\varepsilon}(\alpha)$ we have

$$\Big| \int_{\Omega^\pm_{Py_\varepsilon}(\alpha)} \{a_j(y_\varepsilon) - a_{\varepsilon,j}(y_\varepsilon)\} D_j y_\varepsilon D_j \varphi dx \Big| \le \eta \|y_\varepsilon\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)} \le \eta C \|\varphi\|_{H^1(\Omega)}.$$

As in Lemma 3.1 $\Omega^-_{Py}(\alpha) = \bigcap_\varepsilon \Omega^-_{Py_\varepsilon}(\alpha)$, and thus,

$$\begin{aligned}
&\lim_\varepsilon \int_{\Omega^-_{Py_\varepsilon}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx \\
&= \lim_\varepsilon \int_{\Omega^-_{Py_\varepsilon}(\alpha)/\Omega^-_{Py}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx + \lim_\varepsilon \int_{\Omega^-_{Py}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx \\
&= \lim_\varepsilon \int_{\Omega^-_{Py}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx = \int_{\Omega^-_{Py}(\alpha)} a_j(y) D_j y D_j \varphi dx\ \forall \varphi \in H_0^1(\Omega).
\end{aligned}$$

We have

$$\begin{aligned}
\Omega^+_{Py_\varepsilon}(\alpha)/\Omega^+_{Py}(\alpha) &= \{x : x \in \Omega,\ \alpha < Py_\varepsilon(x),\ Py(x) \le \alpha\} \\
&\subset \{x : \alpha - \eta < Py(x) \le \alpha\}
\end{aligned}$$

as $Py(x) - \eta \le Py_\varepsilon(x) \le Py(x) + \eta$ for all $x \in \Omega$. Thus,

$$\lim_\varepsilon \int_{\Omega^+_{Py_\varepsilon}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx$$

$$= \lim_{\varepsilon} \int_{\Omega^+_{Py_\varepsilon}(\alpha)/\Omega^+_{Py}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx + \lim_{\varepsilon} \int_{\Omega^+_{Py}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx$$
$$= \int_{\Omega^+_{Py}(\alpha)} a_j(y) D_j y D_j \varphi dx \ \forall \varphi \in H^1_0(\Omega)$$

since

$$\Big| \int_{\Omega^+_{Py_\varepsilon}(\alpha)/\Omega^+_{Py}(\alpha)} a_j(y_\varepsilon) D_j y_\varepsilon D_j \varphi dx \Big| \le C \|y_\varepsilon\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega^+_{Py_\varepsilon}(\alpha)/\Omega^+_{Py}(\alpha))}$$
$$\le C_1 \Big\{ \int_{\{x:\, \alpha-\eta < Py(x) \le \alpha\}} (|\varphi|^2 + |\nabla\varphi|^2) dx \Big\}^{1/2} \le C_2 \eta.$$

The theorem is proved. □

We shall give an application of Theorem 3.1 to a problem of fluid mechanics. The nonstationary motion of an elastic solid membrane inside an incompressible fluid has been the subject of many investigations. The interior free boundary for Bernoulli-type problems has been studied by Diaz, Padial and Rakotoson in [9].

In this paper we shall consider a stationary incompressible irrotational horizontal fluid flow in Ω. Let y be its stream function; then y is harmonic in Ω. Suppose that the fluid moves in Ω around an elastic membrane $\partial\Omega_u(\alpha)$, representing, e.g., the membrane of a biological cell. Since both $\partial\Omega$ and $\partial\Omega_u(\alpha)$ are streamlines, we have $y = 0$ on $\partial\Omega$ and $y = \alpha$ on $\partial\Omega_u(\alpha)$. The interacting force on the interface is g. The Bernoulli principle states that $|\nabla y| =$ constant on $\partial\Omega^+_u(\alpha)$ and thus we shall take g to be a constant.

We are led to the study of the boundary problem

$$\begin{aligned} -\Delta y = 0 \text{ in } \Omega^-_{Py}(\alpha) \quad &, \quad y = 0 \text{ on } \partial\Omega, \\ y = \alpha \text{ on } \partial\Omega^+_{Py} \quad &, \quad \nu.\nabla y = g \text{ on } \partial\Omega^+_{Py}(\alpha), \end{aligned} \tag{3.5}$$

where $\Omega^-_{Py}(\alpha) = \big\{x : x \in \Omega,\ 0 < (Py)(x) < \alpha\big\}$, $\Omega^+_{Py}(\alpha) = \Omega/\overline{\Omega}^-_{Py}$.

Theorem 3.2. *Let g, P be as in Theorem* 3.1. *Then there exists a solution y in $H^1_0(\Omega)$ of the problem* (3.5).

Proof. We apply Theorem 3.1 with $a_j(s) = 1$, $f = 0$. Then there exists $y = \{y_-, y_+\}$ in $H^1_0(\Omega)$ such that

$$\begin{aligned} -\Delta y_\pm = 0 \text{ in } \Omega^\pm_{Py}(\alpha) \quad &, \quad y_- = 0 \text{ on } \partial\Omega, \\ y_- = y_+ = \alpha \quad &, \quad \nu.\big\{\nabla y_+ - \nabla y_-\big\} = g \text{ on } \partial\Omega^+_{Py}(\alpha). \end{aligned}$$

Since α is a constant, the unique solution of the elliptic boundary problem

$$-\Delta z = 0 \text{ in } \Omega^+_{Py}(\alpha), \; z = \alpha \text{ on } \partial\Omega^+_{Py}(\alpha) \tag{3.6}$$

is $z = \alpha$. It is clear that y_+ is a solution of (3.6), and thus,

$$y_+ = \alpha, \; \nabla y_+ = 0.$$

Hence, $\{y_-, \Omega^-_{Py}(\alpha)\}$ is a solution of (3.5) and the theorem is proved. □

Thus the shape of the elastic membrane is determined and we have an elliptic equation with a solution-dependent measure.

4. Obstacle problem

The obstacle problem had been extensively studied in the 60's and 70's. The elegant approach of the theory of variational inequalities has two drawbacks: (i) the case of nonzero Neumann condition on the free boundary falls outside of the framework of the theory; and (ii) the study of the free boundary is difficult and sophisticated mathematical machinery has to be deployed to show that it is Hölder continuous.

The approach taken in this paper is different. The obstacle problem will be considered as a special case of a free boundary transmission problem with a solution-dependent measure; the free boundary depends not on the "entire" solution but only on its smooth part. Thus the nonzero Neumann condition can be treated and the difficult question on the regularity of the free boundary is no longer a problem.

Consider the elliptic boundary problem

$$\begin{aligned} A(y)y = f \text{ in } \Omega^+_{Py} \quad &, \quad y = 0 \text{ in } \Omega/\overline{\Omega}^+_{Py}, \\ -\sum_{j=1}^{3} \{a_j(y) D_j y\} \nu_j = g \quad &, \quad y = 0 \text{ on } \partial\Omega^+_{Py}/\partial\Omega, \end{aligned} \tag{4.1}$$

where

$$\Omega^+_{Py} = \{x : \; x \in \Omega, \; Py(x) > 0\} \tag{4.2}$$

and P is the projection of $L^2(\Omega)$ onto the convex compact subset $\mathcal{U}$ of $L^2(\Omega)$, defined in Section 3.

The main result of the section is the following theorem.

Theorem 4.1. *Let $\{f, g\}$ be as in Theorem* 2.1 *and let $a_j(s)$ be continuous functions on R. Suppose that*

$$|a_j| \leq M,\ 0 < a_0 \sum_{j=1}^{3} (D_j y)^2 \leq \sum_{j=1}^{3} a_j(y)(D_j y)^2$$

and

$$\sum_{j=1}^{3} \{a_j(y)D_j y - a_j(z)D_j z\} D_j(y-z) \geq c \sum_{j=1}^{3} |D_j(y-z)|^2.$$

Suppose further that

$$supp(f) \subset G \subset \bigcap_{u \in \mathcal{U}} \{x : x \in \Omega,\, u(x) > 0\}.$$

Then there exists a solution y in $H_0^1(\Omega)$ of (4.1)–(4.2).

Proof. 1) Let $\{y_\alpha, \mu(y_\alpha)\}$ be as in Theorem 3.1. From the estimate of the theorem we obtain by taking subsequences $\{y_\alpha, Py_\alpha, \mu(y_\alpha)\} \to \{\tilde{y}, P\tilde{y}, \mu\}$ in $\{(H_0^1(\Omega))_{weak} \cap L^2(\Omega)\} \times C^{2,\lambda}(\Omega) \times (H^{-1}(\Omega))_{weak}$. Let $\alpha \to 0^+$ and we obtain

$$\int_\Omega a_j(y_\alpha) D_j y_\alpha D_j \varphi dx \to \int_\Omega a_j(\tilde{y}) D_j \tilde{y} D_j \varphi dx \ \ \forall \varphi \in H_0^1(\Omega).$$

1) We now show that

$$\langle \mu, \varphi \rangle = \int_{\partial \Omega^+_{P\tilde{y}}} g \varphi d\sigma \ \forall \varphi \in H_0^1(\Omega).$$

We have

$$(P\tilde{y})(x) - \varepsilon < (Py_\alpha)(x) < (P\tilde{y})(x) + \varepsilon \ \ \forall x.$$

It is clear that $\Omega^+_{Py_\alpha} \subset \Omega^+_{P\tilde{y}}$. We get

$$\begin{aligned} \Omega^+_{P\tilde{y}} / \overline{\Omega}^+_{Py_\alpha(\alpha)} &= \Big\{x : x \in \Omega,\, 0 < (P\tilde{y})(x),\, 0 < (Py_\alpha)(x) \leq \alpha\Big\} \\ &\subset \Big\{x : x \in \Omega,\, 0 < (P\tilde{y})(x) \leq \alpha + \varepsilon\Big\} \subset \Omega^+_{P\tilde{y}}. \end{aligned}$$

Hence,

$$\bigcap_\alpha \{\Omega^+_{P\tilde{y}} / \overline{\Omega}^+_{Py_\alpha}(\alpha)\} \subset \bigcup_{\alpha,\varepsilon} \{x : x \in \Omega,\, 0 < (P\tilde{y})(x) \leq \varepsilon + \alpha\} \subset \Omega^+_{P\tilde{y}}$$

Since the sets $\Omega^-_{P\tilde y}(\alpha)$ are decreasing in α, we deduce that

$$\bigcap_{\alpha,\varepsilon}\Big\{x:\, x\in\Omega,\, 0<(P\tilde y)(x)\le \alpha+\varepsilon\Big\}=\Big\{x:\, x\in\Omega,(P\tilde y)(x)=0\Big\}\subset \Omega^+_{P\tilde y}.$$

We have a contradiction and thus $\big\{x:\, x\in\Omega,(P\tilde y)(x)=0\big\}$, when considered as a subset of $\Omega^+_{P\tilde y}$, must be empty. Therefore,

$$\bigcap_{\alpha>0}\{\Omega^+_{P\tilde y}/\overline{\Omega}^+_{Py_\alpha(\alpha)}\}=\emptyset.$$

2) We have

$$\int_{\Omega^+_{P\tilde y/\overline{\Omega}^+_{Py_\alpha(\alpha)}}}\{gD_j\varphi-\varphi D_jg\}dx=\int_{\partial\Omega^+_{P\tilde y}}g\nu_j\varphi d\sigma-\int_{\partial\Omega^+_{Py_\alpha(\alpha)}}g\nu_{j,\alpha}\varphi d\sigma$$

for all $\varphi\in H^1_0(\Omega)$. It follows from the previous part that

$$\lim_{\alpha\to 0}\int_{\Omega^+_{P\tilde y}/\overline{\Omega}^+_{Py_\alpha(\alpha)}}\{gD_j\varphi-\varphi D_jg\}dx=0.$$

Therefore,

$$\lim_{\alpha\to 0}\int_{\partial\Omega^+_{Py_\alpha(\alpha)}}g\nu_{j,\alpha}\varphi d\sigma=\int_{\partial\Omega^+_{P\tilde y}}g\nu_j\varphi d\sigma\ \ \forall\ \varphi\in H^1_0(\Omega).$$

Since $Py_\alpha\to P\tilde y$ in $C^2(\Omega)$ we obtain $\nu_{j,\alpha}\to\nu_j$ in $C^1(\Omega)$, and thus,

$$\lim_\alpha\int_{\partial\Omega^+_{Py_\alpha(\alpha)}}g\nu_{j,\alpha}\varphi d\sigma=\lim_\alpha\int_{\partial\Omega^+_{Py_\alpha(\alpha)}}g\nu_j\varphi d\sigma=\int_{\partial\Omega^+_{P\tilde y}}g\nu_j\varphi d\sigma\ \ \forall\varphi\in H^1_0(\Omega).$$

Take $\varphi=\nu_j\Phi/\sum_{k=1}^3\nu_k^2$ with $\Phi\in H^1_0(\Omega)$ and we get

$$\lim_\alpha\langle\mu(y_\alpha),\Phi\rangle=\lim_\alpha\int_{\partial\Omega^+_{Py_\alpha(\alpha)}}g\Phi d\sigma=\int_{\partial\Omega^+_{P\tilde y}}g\Phi d\sigma=\langle\mu,\Phi\rangle\ \forall\Phi\in H^1_0(\Omega).$$

3) We now show that

$$\tilde y=0\quad\text{on } \partial\Omega^+_{P\tilde y}=\{x:\, x\in\Omega,\,(P\tilde y)(x)=0\}.$$

With $y_\alpha-\tilde y$ instead of g, we obtain

$$\begin{aligned}|\lim_\alpha\int_{\partial\Omega^+_{Py_\alpha(\alpha)}}\{y_\alpha-\tilde y\}\varphi d\sigma| &= |\lim_\alpha\int_{\partial\Omega^+_{P\tilde y}}\{y_\alpha-\tilde y\}\varphi d\sigma|\\ &\le C(\Omega^+_{P\tilde y})\|\varphi\|_{H^1(\Omega)}\|y_\alpha-\tilde y\|_{H^{1/2}(\Omega)}.\end{aligned}$$

Therefore,

$$\lim_{\alpha} \int_{\partial\Omega^+_{Py_\alpha}(\alpha)} y_\alpha \varphi d\sigma = 0 = \lim_{\alpha} \int_{\partial\Omega^+_{Py_\alpha}(\alpha)} \tilde{y}\varphi d\sigma = \int_{\partial\Omega^+_{P\tilde{y}}} \tilde{y}\varphi d\sigma \ \forall\varphi \in H^1_0(\Omega).$$

Hence, $\tilde{y} = 0$ on $\partial\Omega^+_{P\tilde{y}}$.

4) We now have

$$\sum_{j=1}^{3} \{ \int_{\Omega^+_{P\tilde{y}}} a_j(\tilde{y}) D_j\tilde{y} D_j\varphi dx + \int_{\Omega/\overline{\Omega}^+_{P\tilde{y}}} a_j(\tilde{y}) D_j\tilde{y} D_j\varphi dx \} + \int_{\partial\Omega^+_{P\tilde{y}}} g\varphi d\sigma = \int_{\Omega^+_{P\tilde{y}}} f\varphi dx \ \ \forall\varphi \in H^1_0(\Omega).$$

Suppose that $\Omega/\overline{\Omega}^+_{P\tilde{y}} \neq \emptyset$. Let $\varphi \in H^1_0(\Omega/\overline{\Omega}^+_{P\tilde{y}})$; then we get

$$-\sum_{j=1}^{3} D_j\{a_j(\tilde{y}) D_j\tilde{y}\} = 0 \text{ in } \Omega/\overline{\Omega}^+_{P\tilde{y}},\ \tilde{y} = 0 \ \text{ on } \partial\Omega^+_{P\tilde{y}}.$$

On the other hand we know that the elliptic problem

$$-\sum_{j=1}^{3} D_j\{a_j(z) D_j z\} = 0 \text{ in } \Omega/\Omega^+_{P\tilde{y}},\ z = 0 \ \text{ on } \partial\Omega \cup \partial\Omega^+_{P\tilde{y}}$$

has a unique solution, namely the trivial one. Therefore, $\tilde{y} = 0$ in $\Omega/\Omega^+_{P\tilde{y}}$.

We consider the case when

$$\Omega/\overline{\Omega}^+_{P\tilde{y}} = \{x : x \in \Omega, (P\tilde{y})(x) = 0\} = \emptyset.$$

Then we have $\Omega = \Omega^+_{P\tilde{y}}$ and $\partial\Omega^+_{P\tilde{y}}/\partial\Omega = \emptyset$. This is the trivial case as $\langle\mu,\varphi\rangle = 0$ for all $\varphi \in H^1_0(\Omega)$. The theorem is proved. □

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