Differential and Integral Equations Volume 23, Numbers 7-8 (2010), 671–684

EXISTENCE OF GLOBAL SOLUTIONS TO THE CAUCHY PROBLEM FOR SOME REACTION-DIFFUSION SYSTEM

Munemitsu Hirose
Department of Mathematics, School of Science and Technology
Meiji University, Higashimita Tama-ku Kawasaki-shi
Kanagawa 214-8571, Japan

(Submitted by: Tohru Ozawa)

Abstract. We consider the Cauchy problem for the following reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} = \Delta u_i + g_i(x,t)\displaystyle\prod_{j=1}^{m} {u_j}^{p_{ij}}, & x \in \mathbf{R}^n,\ t>0,\ i=1,2,\cdots,m, \\ u_i(x,0) = f_i(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n,\ i=1,2,\cdots,m, \end{cases}$$

where $n \geq 3$, $m \geq 2$, $p_{ij} \geq 0$ $(1 \leq i,j \leq m)$, $\prod_{j=1}^{m} {u_j}^{p_{ij}} = {u_1}^{p_{i1}}{u_2}^{p_{i2}}\cdots {u_m}^{p_{im}}$, $(i=1,2,\cdots,m)$ and $f_i(x)$ $(i=1,2,\cdots,m)$ is a non-negative, bounded and continuous function in $\mathbf{R}^n$. In this paper, we show the existence of non-negative and global solutions $u_i(x,t)$ $(i=1,2,\cdots,m)$ for the above Cauchy problem when $g_i(x,t)$ $(i=1,2,\cdots,m)$ and $p_{ij} \geq 0$ $(1 \leq i,j \leq m)$ satisfy some conditions.

1. Introduction

In this paper, we study the following Cauchy problem:

$$\begin{cases} \frac{\partial u_1}{\partial t} = \Delta u_1 + g_1(x,t){u_1}^{p_{11}}{u_2}^{p_{12}}\cdots{u_m}^{p_{1m}}, & x \in \mathbf{R}^n,\ t>0, \\ \frac{\partial u_2}{\partial t} = \Delta u_2 + g_2(x,t){u_1}^{p_{21}}{u_2}^{p_{22}}\cdots{u_m}^{p_{2m}}, & x \in \mathbf{R}^n,\ t>0, \\ \qquad\vdots & \\ \frac{\partial u_m}{\partial t} = \Delta u_m + g_m(x,t){u_1}^{p_{m1}}{u_2}^{p_{m2}}\cdots{u_m}^{p_{mm}}, & x \in \mathbf{R}^n,\ t>0, \\ u_1(x,0) = f_1(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \\ u_2(x,0) = f_2(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \\ \qquad\vdots & \\ u_m(x,0) = f_m(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

Accepted for publication: November 2009.
AMS Subject Classifications: 35K45, 35K57.

where $n \geq 3$, $m \geq 2$, $p_{ij} \geq 0$ $(i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, m)$ and $f_i(x)$ $(i = 1, 2, \cdots, m)$ is a non-negative, bounded and continuous function in $\mathbf{R}^n$. Our aim is to show the existence of non-negative and global solutions $u_i(x,t)$ $(i = 1, 2, \cdots, m)$ of (1.1) when we give $g_i(x,t)$ $(i = 1, 2, \cdots, m)$ the following assumptions:

(G_1) For any $i = 1, 2, \cdots, m$, $g_i(x,t)$ is a continuous function in $\mathbf{R}^n \times [0, \infty)$ and a locally Hölder continuous function in $\mathbf{R}^n \times (0, \infty)$.

(G_2) There exist some $\theta \geq 0$, $l > -2$ and $C > 0$ such that $g_i(x,t)$ satisfies

$$0 \leq g_i(x,t) \leq C(t+1)^\theta |x|^l \tag{1.2}$$

for any $i = 1, 2, \cdots, m$ and $(x,t) \in \mathbf{R}^n \times [0, \infty)$.

On the Cauchy problem for the single equation

$$\begin{cases} \dfrac{\partial w}{\partial t} = \Delta w + H(x,t,w), & x \in \mathbf{R}^n,\ t > 0, \\ w(x,0) = f(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.3}$$

many results have been shown. In 1966, Fujita [3] has shown that, if $p > 1 + 2/n$, then the solution of (1.3) with $H(x,t,w) = w^p$ has a global classical solution when f satisfies $0 < f(x) < \delta \exp(-|x|^2)$ where δ is a sufficiently small positive number. In addition, Lee and Ni [6] have shown that, if $p > 1 + 2/n$ and f satisfies $f(x) \sim (1 + |x|^2)^{-1/(p-1)}$ as $x \to \infty$, then (1.3) with $H(x,t,w) = w^p$ has a global classical solution and the solution $w(x,t)$ satisfies $||w(\cdot,t)||_{L^\infty(\mathbf{R}^n)} \sim t^{-1/(p-1)}$ as $t \to \infty$.

In [8], Wang has proved that, if $H(x,t,w) = |x|^l w^p$, $n \geq 3$, $l > -2$ and $p \geq (n + 2 + 2l)/(n-2)$, then there exists a small $\varepsilon > 0$ such that, if $0 \leq f(x) \leq \varepsilon(1 + |x|)^{-(2+l)/(p-1)}$ in $\mathbf{R}^n$, then (1.3) has a global solution $w(x,t)$ which satisfies $w(x,t) \geq 0$ for any $(x,t) \in \mathbf{R}^n \times [0, \infty)$ and $||w(\cdot,t)||_{L^\infty(\mathbf{R}^n)} \leq Mt^{-(2+l)/\{2(p-1)\}}$ for any $t > 0$. Moreover, in case $n \geq 3$, $-2 < l \leq 0$ and $1 + (2+l)/n < p < (n + 2 + 2l)/(n-2)$, we have proved that the same result holds in [4]. In addition, Pinsky has stated the following result in [7]: Let $n \geq 2$ and $H(x,t,w) = g(x)w^p$. Assume that $g(x)$ is a positive and Hölder continuous function in $\mathbf{R}^n$ and satisfies $c_1|x|^l \leq g(x) \leq c_2|x|^l$ $(l > -2)$ for large $|x|$ and constants c_1, $c_2 > 0$. If $p > 1 + (2+l)/n$, then (1.3) has a global solution. (Note that these results yield the existence of global solutions for (1.3) with $H(x,t,w) = |x|^l w^p$ if $n \geq 3$, $l > -2$ and $p > 1 + (2+l)/n$.)

Now, let $H(x,t,w) = h(x,t)w^p$ and set $\mu := (2 + 2\theta + l)/\{2\,(p-1)\}$. Then we have the following result from Theorem 1.4 and its proof in [5].

Proposition 1.1. *Let $n \geq 3$. Assume that $h(x,t)$ is a continuous function in $\mathbf{R}^n \times [0, \infty)$ and a locally Hölder continuous function in $\mathbf{R}^n \times (0, \infty)$, and* *there exist some* $\theta \geq 0$, $l > -2$ *and* $C_0 > 0$ *such that* $h(x,t)$ *satisfies* $0 \leq h(x,t) \leq C_0(t+1)^\theta |x|^l$ *for any* $(x,t) \in \mathbf{R}^n \times [0,\infty)$. *If* $p > 1 + (2+2\theta+l)/n$, *then there exists a small* $\varepsilon_0 > 0$ *such that, if* $0 \leq f(x) \leq \varepsilon_0(1+|x|)^{-2\mu}$ *in* $\mathbf{R}^n$, *then* (1.3) *with* $H(x,t,w) = h(x,t)w^p$ *has a global solution* $w(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ *which satisfies* $0 \leq w(x,t) \leq (t+1)^{-\mu}$ *for any* $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

Remark 1.1. In order to prove Theorem 1.4 in [5], we have shown the existence of an entirely positive solution with $\lim_{r\to\infty} r^{2\mu}u(r) \in (0,\infty)$ for

$$\begin{cases} u'' + \Big(\dfrac{n-1}{r} + \dfrac{r}{2}\Big)u' + \mu u + r^l u^p = 0, \quad r > 0, \\ u(0) = \alpha > 0 \end{cases} \tag{1.4}$$

by using Theorem 2 in [9]. The assumption $n \geq 3$ has been needed to apply the theorem for (1.4), so that we also need it in Theorem 1.1 and Proposition 1.2 below since these results follow from Proposition 1.1.

Next, we consider the existence of global solutions for the system (1.1) in case $m = 2$, namely,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 + g_1(x,t){u_1}^{p_{11}}{u_2}^{p_{12}}, & x \in \mathbf{R}^n,\ t > 0, \\ \dfrac{\partial u_2}{\partial t} = \Delta u_2 + g_2(x,t){u_1}^{p_{21}}{u_2}^{p_{22}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_1(x,0) = f_1(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \\ u_2(x,0) = f_2(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n. \end{cases} \tag{1.5}$$

Escobedo and Levine [2] have studied (1.5) with $g_1(x,t) = g_2(x,t) = 1$, $p_{ij} \geq 0$ $(1 \leq i,j \leq 2)$ and $0 < p_{11} + p_{12} \leq p_{21} + p_{22}$. By setting $\alpha_0 = (p_{12} - p_{22} + 1)/\{p_{21}p_{12} - (1-p_{11})(1-p_{22})\}$ and $\beta_0 = (p_{21} - p_{11} + 1)/\{p_{21}p_{12} - (1-p_{11})(1-p_{22})\}$, they have shown the following results.

(a) If $p_{11} \leq 1$ and $0 \leq \max\{\alpha_0, \beta_0\} < n/2$, then there exist global solutions of (1.5) for small initial data.

(b) If $p_{11} \leq 1$ and $\max\{\alpha_0, \beta_0\} < 0$, then there exist global solutions of (1.5) for every initial data.

(c) If $p_{11} > 1$ and $p_{11} + p_{12} > 1 + 2/n$, then there exist global solutions of (1.5) for small initial data.

Moreover, Aoyagi, Tsutaya, and Yamauchi [1] have studied (1.5) with $g_1(x,t) = |x|^{l_1}$, $g_2(x,t) = |x|^{l_2}$, $p_{ij} \geq 0$ $(1 \leq i,j \leq 2)$, $l_i > \max\{-2, -n\}$ $(i = 1,2)$ and $0 < (p_{11} + p_{12} - 1)/(2 + l_1) \leq (p_{21} + p_{22} - 1)/(2 + l_2)$. Let

$$\alpha = \frac{p_{12}(l_2+2) + (1-p_{22})(l_1+2)}{2\{p_{21}p_{12} - (1-p_{11})(1-p_{22})\}}, \quad \beta = \frac{p_{21}(l_1+2) + (1-p_{11})(l_2+2)}{2\{p_{21}p_{12} - (1-p_{11})(1-p_{22})\}},$$

$$\delta_1 = \frac{p_{12}l_2 + (1-p_{22})l_1}{p_{21}p_{12} - (1-p_{11})(1-p_{22})}, \quad \delta_2 = \frac{p_{21}l_1 + (1-p_{11})l_2}{p_{21}p_{12} - (1-p_{11})(1-p_{22})}$$

and

$$I^a = \Big\{ \ w \in C(\mathbf{R}^n) : w(x) \geq 0, \ \limsup_{|x|\to\infty} |x|^a w(x) < \infty \Big\}.$$

Assuming $(f_1, f_2) \in I^{\delta_1} \times I^{\delta_2}$, they have shown the following results.

(i) If $p_{11} < 1$, $p_{22} < 1$ and $0 < \max\{\alpha, \beta\} < n/2$, then there exist global solutions of (1.5) for small initial data.

(ii) If $p_{11} < 1$, $p_{22} < 1$ and $\max\{\alpha, \beta\} < 0$, then there exist global solutions of (1.5) for every initial data.

(iii) If $p_{11} > 1$, $p_{22} < 1$, $\alpha < n/2$ and $p_{11} + p_{12} > 1 + (2 + l_1)/n$, then there exist global solutions of (1.5) for small initial data.

(iv) If $p_{11} < 1$, $p_{22} > 1$, $\beta < n/2$ and $p_{21} + p_{22} > 1 + (2 + l_2)/n$, then there exist global solutions of (1.5) for small initial data.

(v) If $p_{11} > 1$, $p_{22} > 1$, $p_{11} + p_{12} > 1 + (2 + l_1)/n$ and $p_{21} + p_{22} > 1 + (2 + l_2)/n$, then there exist global solutions of (1.5) for small initial data.

Now, by setting

$$q := \min\Big\{ \sum_{j=1}^m p_{1j}, \ \sum_{j=1}^m p_{2j}, \ \cdots \ , \ \sum_{j=1}^m p_{mj} \Big\} \quad \text{and} \quad \lambda := \frac{2 + 2\theta + l}{2(q-1)}, \tag{1.6}$$

we state our main result as follows.

Theorem 1.1. *Assume that* $n \geq 3$, $p_{ij} \geq 0$ $(1 \leq i, j \leq m)$, (G_1), (G_2),

$$q > 1 + \frac{2 + 2\theta + l}{n} \tag{1.7}$$

and $f_i(x)$ $(i = 1, 2, \cdots, m)$ *is a non-negative, bounded and continuous function in* $\mathbf{R}^n$. *Then there exists a small* $\varepsilon > 0$ *such that, if* $0 \leq f_i(x) \leq \varepsilon(1 + |x|)^{-2\lambda}/m$ *in* $\mathbf{R}^n$ *for each* $i = 1, 2, \cdots, m$, *then there exist some non-negative and global solutions* $u_i(x,t)$ $(i = 1, 2, \cdots, m)$ *of* (1.1).

Remark 1.2. (i) In order to prove Theorem 1.1, we consider the following system of integral equations associated with (1.1):

$$u_i(x,t) = \Big[e^{t\Delta} f_i + \int_0^t e^{(t-s)\Delta} g_i(\cdot, s) \prod_{j=1}^m \Big(u_j(\cdot, s) \Big)^{p_{ij}} ds \Big](x), \qquad (x,t) \in \mathbf{R}^n \times [0, \infty), \ i = 1, 2, \cdots, m, \tag{1.8}$$

where

$$e^{t\Delta} f_i(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} \exp\Big(-\frac{|x-y|^2}{4t}\Big) f_i(y)dy,$$

and show the existence of non-negative functions $u_i(x,t)$ $(i = 1, 2, \cdots, m)$ satisfying (1.8).

(ii) Considering the case $m = 2$ and $g_1(x,t) = g_2(x,t) = 1$, we have

$$0 \le g_i(x,t) \le 1 \cdot (t+1)^0 |x|^0 \quad \text{for any} \quad (x,t) \in \mathbf{R}^n \times [0,\infty) \quad \text{and} \quad i = 1, 2,$$

so that, if $p_{ij} \ge 0$ $(1 \le i, j \le 2)$ and $\min\{p_{11}+p_{12}, p_{21}+p_{22}\} > 1+2/n$, then there exist some non-negative and global solutions $u_1(x,t)$ and $u_2(x,t)$ of (1.1). Thus Theorem 1.1 guarantees the result (c) by Escobedo and Levine [2]. Moreover, consider the case $m = 2$, $g_1(x,t) = g_2(x,t) = |x|^l$ and $l \ge 0$. Then $g_1(x,t)$ and $g_2(x,t)$ are continuous functions in $\mathbf{R}^n \times [0,\infty)$ and locally Hölder continuous functions in $\mathbf{R}^n \times (0,\infty)$, and

$$0 \le g_i(x,t) \le 1 \cdot (t+1)^0 |x|^l \quad \text{for any} \quad (x,t) \in \mathbf{R}^n \times [0,\infty) \quad \text{and} \quad i = 1, 2.$$

Therefore, if $p_{ij} \ge 0$ $(1 \le i, j \le 2)$ and $\min\{p_{11}+p_{12}, p_{21}+p_{22}\} > 1+(2+l)/n$, then there exist some non-negative and global solutions $u_1(x,t)$ and $u_2(x,t)$ of (1.1). Thus Theorem 1.1 also guarantees the result (v) with $l_1 = l_2 \ge 0$ by Aoyagi, Tsutaya and Yamauchi [1].

(iii) For example, we can apply Theorem 1.1 for (1.1) with

- $g_i(x,t) = C_i > 0$ $(i = 1, 2, \cdots, m)$ and $q > 1 + 2/n$,
- $g_i(x,t) = C_i\, t^{\theta_i} (1+|x|)^{-l_i}$, where $C_i > 0$, $\theta_i \ge 0$ and $l_i \ge 0$ $(i = 1, 2, \cdots, m)$, and $q > 1 + (2 + 2\max\{\theta_1, \theta_2, \cdots, \theta_m\})/n$

or

- $g_i(x,t) = C_i\, t^{\theta_i} |x|^l$, where $C_i > 0$, $\theta_i \ge 0$ $(i = 1, 2, \cdots, m)$ and $l \ge 0$, and $q > 1 + (2 + 2\max\{\theta_1, \theta_2, \cdots, \theta_m\} + l)/n$.

In order to prove Theorem 1.1, we first note the following result which is a consequence of Proposition 1.1 by noting that $\sum_{i=1}^m g_i(x,t)$ is a continuous function in $\mathbf{R}^n \times [0,\infty)$ and a locally Hölder continuous function in $\mathbf{R}^n \times (0,\infty)$ from (G_1) and satisfies

$$0 \le \sum_{i=1}^m g_i(x,t) \le mC(t+1)^\theta |x|^l$$

for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$ from (G_2).

Proposition 1.2. *Under the same assumptions as in Theorem* 1.1, *there exists a small* $\varepsilon > 0$ *such that, if* $0 \le f_i(x) \le \varepsilon(1+|x|)^{-2\lambda}/m$ *in* $\mathbf{R}^n$ *for each*

$i = 1, 2, \cdots, m$, then

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} = \Delta\varphi + \Big(\displaystyle\sum_{i=1}^{m} g_i(x,t)\Big)\varphi^q, & x \in \mathbf{R}^n,\ t > 0, \\ \varphi(x,0) = \displaystyle\sum_{i=1}^{m} f_i(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n \end{cases} \tag{1.9}$$

has a global solution $\varphi(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $0 \leq \varphi(x,t) \leq (t+1)^{-\lambda}$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

Remark 1.3. Since λ is a positive number from (1.6), $l > -2$ and $\theta \geq 0$, we also have $0 \leq \varphi(x,t) \leq 1$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

Next, we will show the following result in Section 2 below.

Lemma 1.1. *Suppose the same assumptions as in Theorem* 1.1. *Let $\varphi(x,t)$ be a solution of* (1.9).

(i) *For each $i = 1, 2, \cdots m$, there exists a sequence of functions $\{u_i^{(k)}(x,t)\}_{k\in\mathbf{N}}$ which satisfies the following conditions*:

(a) *For each $k \in \mathbf{N}$, $u_i^{(k)}$ and $u_j^{(k-1)}$ $(j = 1, 2, \cdots, m)$ satisfy*

$$\begin{cases} \dfrac{\partial u_i^{(k)}}{\partial t} = \Delta u_i^{(k)} + g_i(x,t) \displaystyle\prod_{j=1}^{m} \left(u_j^{(k-1)}\right)^{p_{ij}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_i^{(k)}(x,0) = f_i(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{1.10}$$

where we define $u_j^{(0)} \equiv 0$ $(j = 1, 2, \cdots, m)$.

(b) *For each $k \in \mathbf{N}$, $u_i^{(k)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$.*

(c) *For any $(x,t) \in \mathbf{R}^n \times [0,\infty)$,*

$$0 \leq u_i^{(1)}(x,t) \leq u_i^{(2)}(x,t) \leq \cdots \leq u_i^{(k)}(x,t) \leq \cdots \leq \varphi(x,t) \leq (t+1)^{-\lambda}.$$

(ii) *For each $i = 1, 2, \cdots, m$ and $(x,t) \in \mathbf{R}^n \times [0,\infty)$, there exists a non-negative number $\overline{u}_i(x,t)$ such that*

$$\lim_{k\to\infty} u_i^{(k)}(x,t) = \overline{u}_i(x,t).$$

Thus we obtain non-negative and bounded functions $\overline{u}_i(x,t)$ $(i = 1, 2, \ldots, m)$ in $\mathbf{R}^n \times [0,\infty)$. Moreover, using the Lebesgue convergence theorem, we obtain the following result which is proved in Section 3 below.

Lemma 1.2. *Suppose the same assumptions as in Theorem* 1.1. *For each* $i = 1, 2, \cdots, m$ *and* $(x, t) \in \mathbf{R}^n \times [0, \infty)$, $\overline{u}_i(x, t)$ *and* $\overline{u}_j(x, t)$ $(j = 1, 2, \cdots, m)$ *satisfy the following integral equation:*

$$\overline{u}_i(x,t) = \Big[e^{t\Delta} f_i + \int_0^t e^{(t-s)\Delta} g_i(\cdot, s) \prod_{j=1}^m \Big(\overline{u}_j(\cdot, s)\Big)^{p_{ij}} ds\Big](x). \tag{1.11}$$

Therefore, it follows from (1.11) that there exist non-negative and global solutions of (1.1) $\overline{u}_1(x, t)$, $\overline{u}_2(x, t)$, $\cdots$, $\overline{u}_m(x, t)$, so that we conclude Theorem 1.1.

2. Proof of Lemma 1.1

Note that we call a function w an upper (lower) solution of (1.3) if w satisfies $w \in C^{2,1}(\mathbf{R}^n \times (0, \infty)) \cap C(\mathbf{R}^n \times [0, \infty))$, $w(x, 0) \geq (\leq) f(x)$ and

$$\frac{\partial w}{\partial t} \geq (\leq) \Delta w + H(x, t, w), \quad x \in \mathbf{R}^n, \ t > 0.$$

Then the following result follows from Definition 1.1 and Lemma 1.2 in [8].

Lemma 2.1. *Suppose that* $F(x, t)$ *is a continuous function in* $\mathbf{R}^n \times [0, \infty)$ *and a locally Hölder continuous function in* $\mathbf{R}^n \times (0, \infty)$. *If* $\overline{u}$ *and* $\underline{u}$ *are upper and lower solutions of*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + F(x, t), & x \in \mathbf{R}^n, \ t > 0, \\ u(x, 0) = f(x) \geq 0, \ \not\equiv 0, & x \in \mathbf{R}^n \end{cases} \tag{2.1}$$

with $\overline{u} \geq \underline{u}$ *in* $\mathbf{R}^n \times [0, \infty)$, *then* (2.1) *has a solution* $u \in C^{2,1}(\mathbf{R}^n \times (0, \infty)) \cap C(\mathbf{R}^n \times [0, \infty))$ *satisfying* $\underline{u} \leq u \leq \overline{u}$ *in* $\mathbf{R}^n \times [0, \infty)$.

Now, we will show Lemma 1.1.

Proof of Lemma 1.1. Let $\varphi(x, t)$ be a global solution of (1.9) in Proposition 1.2. Using Lemma 2.1, we have the following results.

Step 1. (1) Observe that φ is an upper solution and the trivial solution is a lower solution of

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1, & x \in \mathbf{R}^n, \ t > 0, \\ u_1(x, 0) = f_1(x) \geq 0, \ \not\equiv 0, & x \in \mathbf{R}^n. \end{cases} \tag{2.2}$$

(Here, note that

$$\Delta\varphi - \frac{\partial \varphi}{\partial t} \leq \Delta\varphi + \Big(\sum_{i=1}^m g_i(x, t)\Big)\varphi^q - \frac{\partial \varphi}{\partial t} = 0 \quad \text{and} \quad \varphi(x, 0) \geq f_1(x)$$

hold.) Thus, (2.2) has a non-negative and global solution $u_1^{(1)}(x,t) \in C^{2,1}(\mathbf{R}^n \times(0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $0 \le u_1^{(1)} \le \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

(2) Since φ is an upper solution and the trivial solution is a lower solution of

$$\begin{cases} \dfrac{\partial u_2}{\partial t} = \Delta u_2, & x \in \mathbf{R}^n,\ t > 0, \\ u_2(x,0) = f_2(x) \ge 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{2.3}$$

(2.3) has a non-negative and global solution $u_2^{(1)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $0 \le u_2^{(1)} \le \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

$$\vdots$$

(m) Since φ is an upper solution and the trivial solution is a lower solution of

$$\begin{cases} \dfrac{\partial u_m}{\partial t} = \Delta u_m, & x \in \mathbf{R}^n,\ t > 0, \\ u_m(x,0) = f_m(x) \ge 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{2.4}$$

(2.4) has a non-negative and global solution $u_m^{(1)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $0 \le u_m^{(1)} \le \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

Step 2. (1) Observe that φ is an upper solution and $u_1^{(1)}$ is a lower solution of

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 + g_1(x,t) \displaystyle\prod_{j=1}^m \left(u_j^{(1)}\right)^{p_{1j}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_1(x,0) = f_1(x) \ge 0,\ \not\equiv 0, & x \in \mathbf{R}^n \end{cases} \tag{2.5}$$

since we obtain $\varphi(x,0) \ge f_1(x)$, $u_1^{(1)}(x,0) = f_1(x)$,

$$\begin{aligned} &\Delta\varphi + g_1(x,t) \prod_{j=1}^m \left(u_j^{(1)}\right)^{p_{1j}} - \frac{\partial\varphi}{\partial t} \le \Delta\varphi + g_1(x,t) \prod_{j=1}^m \varphi^{p_{1j}} - \frac{\partial\varphi}{\partial t} \\ &= \Delta\varphi + g_1(x,t)\varphi^{p_{11}+p_{12}+\cdots+p_{1m}} - \frac{\partial\varphi}{\partial t} \le \Delta\varphi + g_1(x,t)\varphi^q - \frac{\partial\varphi}{\partial t} \\ &\le \Delta\varphi + \Big(\sum_{i=1}^m g_i(x,t)\Big)\varphi^q - \frac{\partial\varphi}{\partial t} = 0 \end{aligned}$$

and

$$\Delta u_1^{(1)} + g_1(x,t) \prod_{j=1}^m \left(u_j^{(1)}\right)^{p_{1j}} - \frac{\partial u_1^{(1)}}{\partial t} \ge \Delta u_1^{(1)} - \frac{\partial u_1^{(1)}}{\partial t} = 0$$

by noting $0 \le u_j^{(1)} \le \varphi$ $(j = 1, 2, \cdots, m)$, $0 \le \varphi \le 1$ and $1 < q \le p_{11} + p_{12} + \cdots + p_{1m}$. Thus (2.5) has a non-negative and global solution $u_1^{(2)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $u_1^{(1)} \le u_1^{(2)} \le \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$. (Here, in order to apply Lemma 2.1, we must note that $g_1(x,t) \prod_{j=1}^m \left(u_j^{(1)}(x,t)\right)^{p_{1j}}$ is continuous in $\mathbf{R}^n \times [0,\infty)$ and locally Hölder continuous in $\mathbf{R}^n \times (0,\infty)$ since $p_{1j} \ge 0$ $(j = 1, 2, \cdots, m)$, $u_j^{(1)}(x,t)$ $(j = 1, 2, \ldots, m)$ are smooth functions in $\mathbf{R}^n \times [0,\infty)$ and $g_1(x,t)$ satisfies (G_1).)

(2) Since φ is an upper solution and $u_2^{(1)}$ is a lower solution of

$$\begin{cases} \dfrac{\partial u_2}{\partial t} = \Delta u_2 + g_2(x,t) \displaystyle\prod_{j=1}^m \left(u_j^{(1)}\right)^{p_{2j}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_2(x,0) = f_2(x) \ge 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{2.6}$$

(2.6) has a non-negative and global solution $u_2^{(2)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $u_2^{(1)} \le u_2^{(2)} \le \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

$$\vdots$$

(m) Since φ is an upper solution and $u_m^{(1)}$ is a lower solution of

$$\begin{cases} \dfrac{\partial u_m}{\partial t} = \Delta u_m + g_m(x,t) \displaystyle\prod_{j=1}^m \left(u_j^{(1)}\right)^{p_{mj}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_m(x,0) = f_m(x) \ge 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{2.7}$$

(2.7) has a non-negative and global solution $u_m^{(2)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $u_m^{(1)} \le u_m^{(2)} \le \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

Step 3. (1) Observe that φ is an upper solution and $u_1^{(2)}$ is a lower solution of

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 + g_1(x,t) \displaystyle\prod_{j=1}^m \left(u_j^{(2)}\right)^{p_{1j}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_1(x,0) = f_1(x) \ge 0,\ \not\equiv 0, & x \in \mathbf{R}^n \end{cases} \tag{2.8}$$

since we obtain $\varphi(x,0) \ge f_1(x)$, $u_1^{(2)}(x,0) = f_1(x)$,

$$\Delta\varphi + g_1(x,t) \prod_{j=1}^m \left(u_j^{(2)}\right)^{p_{1j}} - \frac{\partial \varphi}{\partial t} \le \Delta\varphi + g_1(x,t) \prod_{j=1}^m \varphi^{p_{1j}} - \frac{\partial \varphi}{\partial t}$$

$$\leq \Delta\varphi + g_1(x,t)\varphi^q - \frac{\partial \varphi}{\partial t} \leq \Delta\varphi + \Big(\sum_{i=1}^{m} g_i(x,t)\Big)\varphi^q - \frac{\partial \varphi}{\partial t} = 0$$

and

$$\Delta u_1^{(2)} + g_1(x,t)\prod_{j=1}^{m}(u_j^{(2)})^{p_{1j}} - \frac{\partial u_1^{(2)}}{\partial t} \geq \Delta u_1^{(2)} + g_1(x,t)\prod_{j=1}^{m}(u_j^{(1)})^{p_{1j}} - \frac{\partial u_1^{(2)}}{\partial t} = 0$$

by noting $u_j^{(1)} \leq u_j^{(2)} \leq \varphi$ $(j = 1, 2, \cdots, m)$, $0 \leq \varphi \leq 1$ and $1 < q \leq p_{11} + p_{12} + \cdots + p_{1m}$. Thus (2.8) has a non-negative and global solution $u_1^{(3)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $u_1^{(2)} \leq u_1^{(3)} \leq \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

(2) Since φ is an upper solution and $u_2^{(2)}$ is a lower solution of

$$\begin{cases} \dfrac{\partial u_2}{\partial t} = \Delta u_2 + g_2(x,t)\displaystyle\prod_{j=1}^{m}\Big(u_j^{(2)}\Big)^{p_{2j}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_2(x,0) = f_2(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{2.9}$$

(2.9) has a non-negative and global solution $u_2^{(3)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $u_2^{(2)} \leq u_2^{(3)} \leq \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

$$\vdots$$

(m) Since φ is an upper solution and $u_m^{(2)}$ is a lower solution of

$$\begin{cases} \dfrac{\partial u_m}{\partial t} = \Delta u_m + g_m(x,t)\displaystyle\prod_{j=1}^{m}\Big(u_j^{(2)}\Big)^{p_{mj}}, & x \in \mathbf{R}^n,\ t > 0, \\ u_m(x,0) = f_m(x) \geq 0,\ \not\equiv 0, & x \in \mathbf{R}^n, \end{cases} \tag{2.10}$$

(2.10) has a non-negative and global solution $u_m^{(3)}(x,t) \in C^{2,1}(\mathbf{R}^n \times (0,\infty)) \cap C(\mathbf{R}^n \times [0,\infty))$ which satisfies $u_m^{(2)} \leq u_m^{(3)} \leq \varphi$ for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

$$\vdots$$

$$\vdots$$

Repeating this argument, we see that there exist m sequences of smooth functions $\{u_1^{(k)}(x,t)\}_{k\in\mathbf{N}}$, $\{u_2^{(k)}(x,t)\}_{k\in\mathbf{N}}, \dots, \{u_m^{(k)}(x,t)\}_{k\in\mathbf{N}}$ such that

$$\begin{aligned} 0 &\le u_1^{(1)}(x,t) \le u_1^{(2)}(x,t) \le \cdots \le u_1^{(k)}(x,t) \le \cdots \le \varphi(x,t) \le (t+1)^{-\lambda}, \\ 0 &\le u_2^{(1)}(x,t) \le u_2^{(2)}(x,t) \le \cdots \le u_2^{(k)}(x,t) \le \cdots \le \varphi(x,t) \le (t+1)^{-\lambda}, \\ &\vdots \\ 0 &\le u_m^{(1)}(x,t) \le u_m^{(2)}(x,t) \le \cdots \le u_m^{(k)}(x,t) \le \cdots \le \varphi(x,t) \le (t+1)^{-\lambda} \end{aligned} \tag{2.11}$$

for any $(x,t) \in \mathbf{R}^n \times [0,\infty)$, and $u_i^{(k)}(x,t)$ and $u_j^{(k-1)}(x,t)$ $(j = 1,2,\cdots,m)$ satisfy (1.10) for each $i = 1,2,\cdots,m$ and $k \in \mathbf{N}$. Thus we conclude (i). Statement (ii) is an easy consequence of (2.11) by noting Remark 1.3. □

3. Proof of Lemma 1.2

In this section, we prove Lemma 1.2.

Proof of Lemma 1.2. From (1.10), for each $i = 1,2,\cdots,m$, we have

$$u_i^{(k)}(x,t) = \Big[e^{t\Delta} f_i + \int_0^t e^{(t-s)\Delta} g_i(\cdot,s) \prod_{j=1}^m \Big(u_j^{(k-1)}(\cdot,s)\Big)^{p_{ij}} ds\Big](x), \tag{3.1}$$

$(x,t) \in \mathbf{R}^n \times [0,\infty)$, where $u_j^{(0)} \equiv 0$ $(j = 1,2,\cdots,m)$. Here, note that

$$\begin{aligned} &\Big[\int_0^t e^{(t-s)\Delta} g_i(\cdot,s) \prod_{j=1}^m \Big(u_j^{(k-1)}(\cdot,s)\Big)^{p_{ij}} ds\Big](x) \\ &= \int_0^t \int_{\mathbf{R}^n} \frac{1}{\{4\pi(t-s)\}^{n/2}} \exp\left(-\frac{|y-x|^2}{4(t-s)}\right) g_i(y,s) \prod_{j=1}^m \big(u_j^{(k-1)}(y,s)\big)^{p_{ij}} dyds. \end{aligned} \tag{3.2}$$

Now, for any fixed point $(x,t) \in \mathbf{R}^n \times [0,\infty)$, it follows from (ii) of Lemma 1.1 that

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{\{4\pi(t-s)\}^{n/2}} \exp\left(-\frac{|y-x|^2}{4(t-s)}\right) g_i(y,s) \prod_{j=1}^m \Big(u_j^{(k-1)}(y,s)\Big)^{p_{ij}} \\ &\qquad = \frac{1}{\{4\pi(t-s)\}^{n/2}} \exp\left(-\frac{|y-x|^2}{4(t-s)}\right) g_i(y,s) \prod_{j=1}^m \Big(\overline{u}_j(y,s)\Big)^{p_{ij}} \end{aligned} \tag{3.3}$$

for each $(y,s) \in \mathbf{R}^n \times [0,\infty)$. Moreover, it follows from (G_2) and (2.11) that there exists some positive constant C such that

$$\begin{aligned}
&\Big|\frac{1}{\{4\pi(t-s)\}^{n/2}} \exp\Big(-\frac{|y-x|^2}{4(t-s)}\Big) g_i(y,s) \prod_{j=1}^{m} \Big(u_j^{(k-1)}(y,s)\Big)^{p_{ij}}\Big| \\
&\qquad \le C(4\pi)^{-n/2} \frac{1}{(t-s)^{n/2}} \exp\Big(-\frac{|y-x|^2}{4(t-s)}\Big)|y|^l (s+1)^{\theta-\lambda P_i}
\end{aligned} \tag{3.4}$$

for each $k = 2,3,4,\cdots$, where $P_i := \sum_{j=1}^m p_{ij}$. Furthermore, we obtain the following result.

Lemma 3.1. *For any fixed point* $(x,t) \in \mathbf{R}^n \times [0,\infty)$,

$$I := \int_0^t \int_{\mathbf{R}^n} \frac{1}{(t-s)^{n/2}} \exp\Big(-\frac{|y-x|^2}{4(t-s)}\Big)|y|^l (s+1)^{\theta-\lambda P_i} dyds < \infty. \tag{3.5}$$

Thus by applying the Lebesgue convergence theorem for (3.1), we obtain Lemma 1.2 from (3.3), (3.4) and (3.5). □

Now we show Lemma 3.1.

Proof of Lemma 3.1. Noting $\theta - \lambda P_i < 0$ for any $i = 1,2,\cdots,m$ from (1.6), (1.7), $l > -2$ and $\theta \ge 0$ and setting $\tau := t-s$, we have

$$\begin{aligned}
I &\le \int_0^t \int_{\mathbf{R}^n} \frac{1}{(t-s)^{n/2}} \exp\Big(-\frac{|y-x|^2}{4(t-s)}\Big)|y|^l dyds \\
&= \int_0^t \int_{\mathbf{R}^n} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y-x|^2}{4\tau}\Big)|y|^l dyd\tau.
\end{aligned}$$

First, let $x = 0$. Then by setting $z := y/\sqrt{\tau}$ and noting $l > -2$ and $n \ge 3$, we get

$$\begin{aligned}
I &\le \int_0^t \int_{\mathbf{R}^n} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y|^2}{4\tau}\Big)|y|^l dyd\tau \\
&= \int_0^t \int_{\mathbf{R}^n} \exp\Big(-\frac{|z|^2}{4}\Big)|z|^l \tau^{l/2} dzd\tau \\
&= \int_0^t \tau^{l/2} d\tau \int_{\mathbf{R}^n} |z|^l \exp\Big(-\frac{|z|^2}{4}\Big) dz \\
&= \frac{2}{2+l} t^{1+l/2} \int_{S^{n-1}} d\omega \int_0^\infty r^{l+n-1} \exp\Big(-\frac{r^2}{4}\Big) dr = C_1 t^{1+l/2},
\end{aligned}$$

where $C_1 = C_1(l,n)$ is some positive constant.

Next, let $x \neq 0$. Here, note that if $|y| \leq |x|/2$, then for any $\tau > 0$

$$\exp\Big(-\frac{|y-x|^2}{4\tau}\Big) \leq \exp\Big(-\frac{|y|^2}{4\tau}\Big) \tag{3.6}$$

holds. Now we set

$$I_1 := \int_0^t \int_{|y|\leq|x|/2} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y-x|^2}{4\tau}\Big)|y|^l dyd\tau$$

and

$$I_2 := \int_0^t \int_{|y|>|x|/2} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y-x|^2}{4\tau}\Big)|y|^l dyd\tau.$$

Similarly to the case $x = 0$, we have

$$\begin{aligned} I_1 &\leq \int_0^t \int_{|y|\leq|x|/2} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y|^2}{4\tau}\Big)|y|^l dyd\tau \\ &\leq \int_0^t \int_{\mathbf{R}^n} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y|^2}{4\tau}\Big)|y|^l dyd\tau = C_1 t^{1+l/2} \end{aligned}$$

by using (3.6). Moreover, if $-2 < l < 0$, then $|y| > |x|/2$ implies $|y|^l < (|x|/2)^l$, so, by setting $z := (y-x)/\sqrt{\tau}$, we obtain

$$\begin{aligned} I_2 &< \int_0^t \int_{|y|>|x|/2} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y-x|^2}{4\tau}\Big)\Big(\frac{|x|}{2}\Big)^l dyd\tau \\ &\leq \int_0^t \int_{\mathbf{R}^n} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y-x|^2}{4\tau}\Big)\Big(\frac{|x|}{2}\Big)^l dyd\tau \\ &= \Big(\frac{|x|}{2}\Big)^l \int_0^t d\tau \int_{\mathbf{R}^n} \exp\Big(-\frac{|z|^2}{4}\Big)dz = C_2|x|^l t, \end{aligned}$$

where $C_2 = C_2(l,n)$ is some positive constant. Furthermore, if $l \geq 0$, then, by setting $z := (y-x)/\sqrt{\tau}$, we have

$$\begin{aligned} I_2 &\leq \int_0^t \int_{\mathbf{R}^n} \frac{1}{\tau^{n/2}} \exp\Big(-\frac{|y-x|^2}{4\tau}\Big)|y|^l dyd\tau \\ &= \int_0^t \int_{\mathbf{R}^n} \exp\Big(-\frac{|z|^2}{4}\Big) \left|\sqrt{\tau}z + x\right|^l dzd\tau \\ &\leq \int_0^t \int_{\mathbf{R}^n} \exp\Big(-\frac{|z|^2}{4}\Big) \cdot 2^l \left(\tau^{l/2}|z|^l + |x|^l\right) dzd\tau \\ &= 2^l\Big\{\int_0^t \tau^{l/2} d\tau \int_{\mathbf{R}^n} |z|^l \exp\Big(-\frac{|z|^2}{4}\Big)dz + |x|^l \int_0^t d\tau \int_{\mathbf{R}^n} \exp\Big(-\frac{|z|^2}{4}\Big)dz\Big\} \\ &= C_3 t^{1+l/2} + C_4|x|^l t, \end{aligned}$$

where $C_3 = C_3(l, n)$ and $C_4 = C_4(l, n)$ are some positive constants. Thus we conclude Lemma 3.1. □

References

[1] Y. Aoyagi, K. Tsutaya, and Y. Yamauchi, *Global existence of solutions for a reaction-diffusion system*, Differential Integral Equations, 20 (2007), 1321–1339.

[2] M. Escobedo and H. A. Levine, *Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations*, Arch. Rational Mech. Anal., 129 (1995), 47–100.

[3] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Tokyo Sect. IA Math., 13 (1966), 109–124.

[4] M. Hirose, *Existence of global solutions for a semilinear parabolic Cauchy problem,* Differential Integral Equations, 21 (2008), 623–652.

[5] M. Hirose, *Existence of global solutions for the Cauchy problem of some parabolic equations,* Adv. Math. Sci. Appl., 18 (2008), 489–516.

[6] T.-Y. Lee and W.-M. Ni, *Global existence, large time behavior and life span of solutions of nonlinear parabolic Cauchy problem*, Trans. Amer. Math. Soc., 333 (1992), 365–371.

[7] R. G. Pinsky, *Existence and nonexistence of global solutions for* $u_t = \Delta u + a(x)u^p$ *in* R^d, J. Differential Equations, 133 (1997), 152–177.

[8] X. Wang, On the Cauchy problem for reaction-diffusion equations, Trans. Amer. Math. Soc., 337 (1993), 549–590.

[9] E. Yanagida and S. Yotsutani, *Existence of positive radial solutions to* $\Delta u + K(|x|)u^p = 0$ *in* R^n, J. Differential Equations, 115 (1995), 477–502.