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BLOW-UP CRITERION VIA SCALING INVARIANT QUANTITIES WITH EFFECT ON COEFFICIENT GROWTH IN KELLER-SEGEL SYSTEM

Yoshie Sugiyama
Department of Mathematics, Tsuda College
2-1-1, Tsuda-chou, Kodaira-shi, Tokyo, 187-8577, Japan

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Abstract. We consider the effects on a blow-up phenomena of the Keller-Segel system (KS) in terms of the mass and second moment of initial data in connection with three coefficients γ, α, χ. In particular, for $\gamma = 0$, our criterion on blow-up of solutions coincides with the quantity of the scaling invariant class associated with the Keller-Segel system. We also show that the size of the $L^{\frac{N}{2}}$-norm plays an important role in construction of the time global and blow-up solutions of (KS). Furthermore, we give essential examples of small-L^1 initial data which yield blow-up solutions. Consequently, we give the answer to the conjecture by Childress-Percus [2] for $N \geq 3$; *i.e.,* that even though the L^1-norm of the initial data is small, the blow-up solutions of (KS) exist in the case of $N \geq 3$. This implies that the smallness of the L^1-norm of the initial data does not give us any criterion on the existence of global solutions except when $N = 2$.

1. Introduction

We consider the semi-linear Keller-Segel system of parabolic-elliptic type in $\mathbb{R}^N$, $N \geq 3$:

$$\text{(KS)} \quad \begin{cases} u_t = \nabla \cdot \big(\nabla u - \chi u \nabla v\big), & x \in \mathbb{R}^N,\ t > 0, \\ 0 = \Delta v - \gamma v + \alpha u, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $u(x,t)$ and $v(x,t)$ denote the unknown functions describing the density of amoebae and the concentration of the chemical attractant, respectively, while u_0 is the given initial data; $\alpha, \chi > 0,\ \gamma \geq 0$ are the given constant coefficients. This equation is called the Keller-Segel model [6] describing the motion of chemotaxis molds.

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Childress-Percus [2] treated (KS) with the second equation replaced by $v_t = \Delta v - \gamma v + u$ under the homogeneous Neumann condition in bounded domains. They proposed the following questions (i), (ii) and (iii):

(1) In the case of $N = 1$, the solution of (KS) exists globally in time without any restriction on the size of the initial data.

(2) In the case of $N = 2$, the following alternatives hold: there exists a constant π^* such that

(i) If $\int u_0\,dx < \pi^*$, then the solution of (KS) exists globally in time and is bounded;

(ii) If $\int u_0\,dx > \pi^*$, then $u(x,t)$ blows up in a finite time.

(3) In the case of $N \geq 3$, there exist blow-up solutions of (KS) even though the L^1-norm of the initial data is small.

Motivated by these questions of Childress-Percus, in this paper, we give a positive answer to (3) above for (KS) with $\gamma = 0$ in the whole space $\mathbb{R}^N$. Indeed, except for $N = 2$, we shall show that the size of the L^1-norm of the initial data does not give us any contribution to obtaining the global solution of (KS). In addition, we consider the case of $N \geq 3$ and investigate the effect of the coefficients for all $\gamma \geq 0$ and $\alpha, \chi > 0$ in (KS) on blow-up phenomena. We make it clear how the coefficients γ, α and χ have influence on the blow-up phenomena of solutions to (KS) in connection with $\|u_0\|_{L^1}$ and the second moment $\||x|^2 u_0\|_{L^1}$. In conclusion, we illustrate that the large γ prevents solutions from blowing up (see Remark 1 (i)–(ii) below in detail). From a mathematical point of view, it seems to be sufficient to deal with (KS) with $\gamma = 0$. For instance, in the case of $\gamma = 0$, (KS) has a scaling property, which governs the principal structure of (KS). Conversely, in the original model proposed by Keller-Segel [6], the effect of γ plays such an essential role for the chemo-attractant that a mathematical investigation is required from the biological and the chemical point of view to treat (KS) for all $\gamma \geq 0$.

Furthermore, we establish a criterion on the finite time blow-up of solutions to (KS) for the case $N \geq 3$. As we stated above, the L^1-norm gives a criterion on global existence and blow-up of solutions for (KS) in the 2-D case but not in the higher-dimension cases. On the other hand, it is known that the $L^{\frac{N}{2}}$-norm plays an important role for global existence. Indeed, the author [11], Corrias-Perthame-Zaag [1] and Kozono-Sugiyama [7] constructed a global-in-time strong solution for every small initial data u_0 in $L^{\frac{N}{2}}(\mathbb{R}^N)$. Thus, we are led to the question whether the large initial data in $L^{\frac{N}{2}}(\mathbb{R}^N)$ gives a criterion on blow-up solutions.

In order to give the answer to the above question, firstly we introduce the criterion for $\gamma = 0$ in terms of the L^1-norm and the second moment of initial data $R(u_0)$:

$$R(u_0) = \frac{\|u_0\|_{L^1}^{\frac{N}{N-2}}}{\||x|^2 u_0\|_{L^1}}. \tag{1.1}$$

Assume that $\gamma = 0$. We can show that there exits $\delta = \delta(\alpha, \chi, N, \|u_0\|_{L^1})$ with the following properties (a) and (b):

(a) if $R(u_0) < \delta$, then there exists a global solution u of (KS);

(b) if $R(u_0) > \delta$, then every solution u of (KS) blows up within a finite time.

Next, we construct $u_0^*(x)$ such that

$$\|u_0^*\|_{L^{\frac{N}{2}}} = \beta_N R(u_0^*), \tag{1.2}$$

for some β_N depending only on N.

By virtue of (b) and (1.2) above, we find that there exist infinitely many initial data u_0^* with large $L^{\frac{N}{2}}$-norm which guarantee the blow-up of solutions to (KS). This implies that the initial space $L^{\frac{N}{2}}(\mathbb{R}^N)$ may be a candidate for giving a criterion on blow-up of the solution. Our example of initial data u_0^* should be large in $L^{\frac{N}{2}}(\mathbb{R}^N)$ and simultaneously small in $L^1(\mathbb{R}^N)$. Therefore, it turns out that the L^1-norm of the initial data never gives the criterion on global existence for $N \geq 3$.

Concerning the criterion in terms of $R(u_0)$, we consider only the case of $\gamma = 0$. This is natural since there is a scaling property to (KS) only in the case of $\gamma = 0$. In fact, if (u, v) is a pair of solution to (KS), then the family $(u_\lambda, v_\lambda)_{\lambda>0}$ also solves (KS), where $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t), v_\lambda(x,t) = v(\lambda x, \lambda^2 t)$. Therefore, the quantity $R(u_0)$ needs to have the property that $R(u_{0,\lambda}) = R(u_0)$ for all functions u_0 and all $\lambda > 0$, where $u_{0,\lambda}(x) := \lambda^2 u_0(\lambda x)$. This implies that the quantity $R(u_0)$ is invariant under the above change of scale. Notice that $\|u_{0,\lambda}\|_{L^{\frac{N}{2}}} = \|u_0\|_{L^{\frac{N}{2}}}$ for all $u_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and all $\lambda > 0$. Hence, $R(u_0)$ and $\|u_0\|_{L^{\frac{N}{2}}}$ are typical quantities for criterion on global existence and blow-up of solutions.

Before stating our result, we first recall the definition of strong solutions to (KS).

Definition. Let $u_0 \in L^1 \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ with $u_0 \geq 0$. A pair of non-negative measurable functions (u, v) on $\mathbb{R}^N \times (0, T)$ is called a strong solution of (KS) on $[0, T)$ if

(i) $u \in C([0,T); L^1 \cap L^{\frac{N}{2}}(\mathbb{R}^N)) \cap C((0,T); H^{2,r}(\mathbb{R}^N)) \cap C^1((0,T); L^r(\mathbb{R}^N))$,

(ii) $v \in C((0,T); H^{2,r}(\mathbb{R}^N))$ if $\gamma > 0$, $D^k v \in C((0,T); L^{r_k}(\mathbb{R}^N))$ if $\gamma = 0$ for some r with $\frac{N}{3} < r < \frac{N}{2}$, where r_k is defined by $\frac{1}{r_k} = \frac{1}{r} - \frac{2-k}{N}$, $\quad k = 0, 1, 2$,

(iii) (u, v) satisfies (KS) for almost every (x,t) in $\mathbb{R}^N \times (0,T)$.

Concerning the existence of local and global strong solutions, and its blow-up in a finite time, we have the following.

Proposition 1.1. (i) *For every* $u_0 \in L^1 \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ *with* $u_0 \geq 0$, *there exist* $T > 0$ *and a strong solution* (u,v) *of (KS) on* $[0,T)$.

(ii) *Suppose that* $T_{\max}$ *is the maximal existence time; i.e., that, for every* $T' > T_{\max}$ *the strong solution* (u,v) *on* $[0, T_{\max})$ *cannot be extended in the class of* (i) *and* (ii) *on* $[0,T')$ *in Definition. Then, we have*

$$\limsup_{t \to T_{\max}} \|u(t)\|_{L^p} \quad = \quad \infty \qquad \textit{for all } \frac{N}{2} < p \leq \infty.$$

(iii) *There exists a constant* $\lambda > 0$ *such that, if* $\|u_0\|_{L^{\frac{N}{2}}} \leq \lambda$, *then we have a unique strong solution* (u,v) *of (KS) on* $(0,\infty)$.

For the proof, see [7], [11], and [9].

Let us define a family $\{g_\gamma\}_{\gamma \geq 0}$ of functions on $(0,\infty)$ by

$$g_\gamma(s) \quad = \quad 2(N+1)\omega_{N-1} \cdot s^{N-2} e^{\sqrt{\gamma} s}, \qquad s > 0, \tag{1.3}$$

where $\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. Since $g_\gamma(s)$ is strictly increasing on $(0,\infty)$, we may define the inverse function $g_\gamma^{-1}(t)$ for $t > 0$. In particular, for $\gamma = 0$, we have an explicit expression $g_0^{-1}(t) = \left(\frac{1}{2(N+1)\omega_{N-1}}\right)^{\frac{1}{N-2}} \cdot t^{\frac{1}{N-2}}$.

Our results read as follows.

Theorem 1.1. *Let* $N \geq 3$ *and* $0 \leq \gamma < \infty$, $0 < \alpha, \chi < \infty$. *Let* $u_0 \in L^1 \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ *and* $|x|^2 u_0 \in L^1(\mathbb{R}^N)$ *with* $u_0 \geq 0$. *Suppose that*

$$\||x|^2 u_0\|_{L^1} \quad \leq \quad \frac{1}{8(N+1)} \|u_0\|_{L^1} \Big(g_\gamma^{-1}(\alpha\chi \|u_0\|_{L^1}) \Big)^2. \tag{1.4}$$

Then the strong solution of (KS) *blows up within a finite time; that is, the maximal existence time* $T_{\max} < \infty$. *In particular, in the case of* $\gamma = 0$, *if*

$$R(u_0) \quad \geq \quad 8(N+1) \Big(2(N+1)\omega_{N-1} \cdot \frac{1}{\alpha\chi} \Big)^{\frac{2}{N-2}}, \tag{1.5}$$

then we have $T_{\max} < \infty$, *where* $R(u_0)$ *is defined by* (1.1).

Remark 1. (i) It is easy to see that, if $0 \le \gamma_1 \le \gamma_2$, then

$$g_{\gamma_2}^{-1}(t) \quad \le \quad g_{\gamma_1}^{-1}(t) \qquad \text{for all } t > 0.$$

Accordingly, for fixed α and χ, if u_0 satisfies (1.4) for some γ, then u_0 fulfills (1.4) necessarily for all $0 < \gamma' \le \gamma$. This implies that the blow-up phenomena occurs more often for small γ. Moreover, we find that, for large γ, the solution (u, v) of (KS) blows up in a finite time provided the mass of initial data concentrates near the origin.

(ii) Since $g_\gamma^{-1}(t)$ is monotone increasing for $t > 0$, we see that, if u_0 satisfies (1.4) for some α_1 and χ_1, then u_0 fulfills (1.4) necessarily for α_2 and χ_2 such that $\alpha_1\chi_1 \le \alpha_2\chi_2$. This implies that, for fixed $\gamma > 0$, the blow-up phenomena occurs more often for large α and χ. Therefore, we see that the growth of the non-linear term has an influence on the blow-up of solutions.

(iii) The condition (1.5) is invariant under the change of scale associated with (KS) for $\gamma = 0$. Indeed, we have $R(u_{0,\lambda}) = R(u_0)$ for all $\lambda > 0$, where $u_{0,\lambda}(x) = \lambda^2 u_0(\lambda x)$.

(iv) The blow-up phenomena of solutions to the drift equations was obtained by Kurokiba-Ogawa [8].

In Theorem 1.1, it turns out that, for $\gamma = 0$, the ratio $\frac{\|u_0\|_{L^1}^{\frac{N}{N-2}}}{\||x|^2 u_0\|_{L^1}}$ governs the blow-up of solutions. On the other hand, it follows from Proposition 1.1 (iii) that, for every small initial data u_0 in $L^{\frac{N}{2}}(\mathbb{R}^N)$, there exists a global solution to (KS). Hence it is an interesting question whether the solution $u(t)$ blows up within a finite time provided $\|u_0\|_{L^{\frac{N}{2}}}$ is sufficiently large. On the other hand, since we have obtained the blow-up solution for every initial data u_0 with large $R(u_0)$, it seems to be also an interesting question whether the solution exists globally in time provided $R(u_0)$ is sufficiently small. The following theorem gives the answer to these questions. In addition, we make the conjecture by [2] more clear, *i.e.,* that even though the L^1-norm of the initial data is small, there exist blow-up solutions of (KS) in the case of $N \ge 3$. This implies that the smallness of the L^1-norm of the initial data does not give us any criterion on the existence of a global solution except when $N = 2$.

Theorem 1.2. *Let $N \ge 3$ and $0 < \alpha, \chi < \infty$. Suppose that $\gamma = 0$.*

(*i*) (*blow-up for large $L^{\frac{N}{2}}$ data and small L^1 data*) *There exist sequences $\{u_{0,m}\}_{m=1}^\infty$ and $\{u_{0,m}^*\}_{m=1}^\infty$ of non-negative functions in $L^1 \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ with $|x|^2 u_{0,m}, |x|^2 u_{0,m}^* \in L^1(\mathbb{R}^N)$ such that the following properties* (1) *and* (2) *hold:*

(1)

$$C_N \cdot \frac{1}{\alpha\chi} \le \|u_{0,1}\|_{L^{\frac{N}{2}}} < \|u_{0,2}\|_{L^{\frac{N}{2}}} < \cdots, \qquad (1.6)$$
$$\textit{with } \lim_{m\to\infty} \|u_{0,m}\|_{L^{\frac{N}{2}}} = \infty,$$

where $C_N := 2^{2N-1} \cdot N^{N-1-\frac{2}{N}}(N+1)(N+2)^{-\frac{N^2-2N+4}{2N}} \cdot (\omega_{N-1})^{\frac{2}{N}}$, *and*

$$\lim_{m\to\infty} \|u^*_{0,m}\|_{L^1} = 0; \qquad (1.7)$$

(2) *For each* $m = 1, 2, \cdots$, *both strong solutions* $u_m(t)$ *and* $u^*_m(t)$ *of (KS) with* $u_m|_{t=0} = u_{m,0}$ *and* $u^*_m|_{t=0} = u^*_{m,0}$ *blow up in a finite time.*

(*ii*) (*global existence for small* $R(u_0)$ *data*)
There exists a sequence $\{u_{0,m}\}_{m=1}^\infty$ *of non-negative functions in* $L^1 \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ *with* $|x|^2 u_{0,m} \in L^1(\mathbb{R}^N)$ *such that the following properties* (1) *and* (2) *hold:*

(1)

$$R(u_{0,1}) > R(u_{0,2}) > \cdots, \qquad \textit{with } \lim_{m\to\infty} R(u_{0,m}) = 0, \qquad (1.8)$$

where $R(u_0)$ *is the same as* (1.1).

(2) *For each* $m = 1, 2, \cdots$, *there exists a unique strong solution* $u_m(t)$ *of (KS) on* $[0, \infty)$ *with* $u_m|_{t=0} = u_{0,m}$.

Remark 2. (i) In the case of $\gamma = 0$ and $N \ge 3$, the blow-up phenomena occurs even though the initial data is sufficiently small in $L^1(\mathbb{R}^N)$. This implies that the global existence theorem of solutions to (KS) for small initial data in $L^1(\mathbb{R}^N)$ does not hold except when $N = 2$.

(ii) Instead of the L^1-norm in the 2-D case, both the quantities $R(u_0)$ and the $L^{\frac{N}{2}}$-norm might be expected to provide the threshold number in higher-dimensional cases.

(iii) The choice of g_γ in (1.3) has some freedom. In fact, if we define g_γ by

$$g_\gamma(s) := 2(N+\delta)\omega_{N-1} \cdot s^{N-2} e^{\sqrt{\gamma} s},$$

with sufficiently small δ, then hypothesis (1.4) can be relaxed. Even though Theorem 1.2 holds for $N \ge 3$, by taking $N = 2$ in (1.6) formally, we see that such a relaxation makes it possible to cause C_2 to be sufficiently close to 8π. This inspires the threshold number $\frac{8\pi}{\alpha\chi}$ of the L^1-initial data in the 2-D case.

2. PROOF OF THE MAIN RESULTS

Let us first recall the Bessel potential $G_\gamma = (-\Delta + \gamma)^{-1}$. For $f \in L^p(\mathbb{R}^N)$, $N \geq 3$, the solution $u \in W^{2,p}(\mathbb{R}^N)$ of $(-\Delta + \gamma)v = f$ can be expressed as

$$v(x) = \int_{\mathbb{R}^N} G_\gamma(x-y) \cdot f(y)\, dy,$$

where

$$G_\gamma(x) \quad = \quad \gamma^{\frac{N}{2}-1} \cdot \nu_N \cdot e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\sqrt{\gamma}|x|s} \cdot \Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}} ds, \quad (2.1)$$

with the constant ν_N given by

$$\nu_N = \Big(2^{\frac{N+1}{2}} \cdot \pi^{\frac{N-1}{2}} \cdot \Gamma\Big(\frac{N-1}{2}\Big)\Big)^{-1}.$$

For details, see, *e.g.,* E.M. Stein [10, Chapter V, Section 6.5].

The following lemma is fundamental, and it plays an important role in the proof of Theorem 1.1.

Lemma 2.1. *Let $N \geq 3$. Then, it holds that*

$$x \cdot \nabla G_\gamma(x) \leq -\frac{1}{w_{N-1}} \cdot \frac{1}{|x|^{N-2}} \cdot e^{-\sqrt{\gamma}|x|} \qquad \textit{for all } x \in \mathbb{R}^N.$$

Proof. By the explicit representation of G_γ as in (2.1), we have

$$\begin{aligned}
x \cdot \nabla G_\gamma(x) &= -\nu_N \gamma^{\frac{N}{2}-1} \sqrt{\gamma}|x| e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\sqrt{\gamma}|x|s}(1+s)\Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}} ds \\
&\leq -\nu_N \gamma^{\frac{N}{2}-1} \sqrt{\gamma}|x| e^{-\sqrt{\gamma}|x|} \frac{1}{2^{\frac{N-3}{2}}} \int_0^\infty e^{-\sqrt{\gamma}|x|s} s^{N-2}\, ds \\
&= -\frac{\nu_N}{2^{\frac{N-3}{2}}} \gamma^{\frac{N}{2}-1} \frac{1}{(\sqrt{\gamma}|x|)^{N-2}} e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\xi} \xi^{N-2}\, d\xi \\
&= -\frac{\nu_N}{2^{\frac{N-3}{2}}} \Gamma(N-1) \gamma^{\frac{N}{2}-1} \frac{1}{(\sqrt{\gamma}|x|)^{N-2}} e^{-\sqrt{\gamma}|x|} \\
&= -\frac{1}{2\pi^{\frac{N}{2}}} \Gamma\Big(\frac{N}{2}\Big) \gamma^{\frac{N}{2}-1} \frac{1}{(\sqrt{\gamma}|x|)^{N-2}} e^{-\sqrt{\gamma}|x|} \\
&= -\frac{1}{\omega_{N-1}} \frac{1}{|x|^{N-2}} e^{-\sqrt{\gamma}|x|}.
\end{aligned}$$

This proves Lemma 2.1.

2.1. **Proof of Theorem 1.1.** For the proof of Theorem 1.1, we investigate the behavior of $\int_{\mathbb{R}^N} |x|^2 u(x,t)\,dx$ as $t \to T_{\max}$.

Lemma 2.2. *Let $N \geq 3$ and $0 \leq \gamma < \infty$, $0 < \alpha, \chi < \infty$. Let $u_0 \in L^1 \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ and $|x|^2 u_0 \in L^1(\mathbb{R}^N)$ with $u_0 \geq 0$. Suppose that (u,v) is the strong solution of* (KS) *on $[0,T)$. Then, we have the following assertions* (*i*) *and* (*ii*) :

(*i*) *$M(t) := \int_{\mathbb{R}^N} |x|^2 u(x,t)\,dx < +\infty$ for all $t \in [0,T)$;*

(*ii*) *Taking $R = g_\gamma^{-1}(\alpha\chi\|u_0\|_{L^1})$ with g_γ as in* (1.3), *we define the linear function $F(M)$ of $M \geq 0$ by*

$$F(M) \quad := \quad 2\|u_0\|_{L^1}\Big(\frac{2}{\omega_{N-1}} \cdot \alpha\chi \cdot R^{-N} \cdot e^{-\sqrt{\gamma}\cdot R} \cdot M - 1\Big). \tag{2.2}$$

Then, $M(t)$ is subject to the estimate

$$M(t) \quad \leq \quad M(0) + \int_0^t F(M(s))\,ds \quad \text{for } 0 < t < T.$$

Postponing the proof of Lemma 2.2, we proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. We shall argue by contradiction. Suppose that $T_{\max} = \infty$. Let us first investigate $F(M)$ defined by (2.2). Taking $M_* = \frac{R^2}{4(N+1)} \cdot \|u_0\|_{L^1}$, we have $F(M_*) = 0$. Indeed, by the definition of R, we have $g_\gamma(R) = \alpha\chi\|u_0\|_{L^1}$. By (1.3) and (2.2), the root M_* of $F = 0$ is expressed as

$$\begin{aligned} M_* = \frac{\omega_{N-1}}{2} \cdot \frac{R^N e^{\sqrt{\gamma}R}}{\alpha\chi} &= \frac{R^2}{4(N+1)} \cdot 2(N+1)\omega_{N-1} R^{N-2} e^{\sqrt{\gamma}R} \cdot \frac{1}{\alpha\chi} \\ &= \frac{R^2}{4(N+1)} \cdot g_\gamma(R) \cdot \frac{1}{\alpha\chi} = \frac{R^2}{4(N+1)} \cdot \|u_0\|_{L^1}. \end{aligned}$$

Obviously, the hypothesis (1.4) is equivalent to

$$M(0) = \||x|^2 u_0\|_{L^1} \leq \frac{M_*}{2}.$$

Since F is strictly increasing on $[0,\infty)$,

$$F(M(0)) \quad \leq \quad F(M_*/2) < 0. \tag{2.3}$$

Now defining a function $H(t)$ by

$$H(t) \quad := \quad M(0) + \int_0^t F\big(M(s)\big)\,ds, \qquad 0 < t < \infty, \tag{2.4}$$

we obtain from Lemma 2.2 that

$$M(t) \quad \le \quad H(t), \qquad \text{for all } 0 < t < \infty. \tag{2.5}$$

On the other hand, we have the following claim:

$$H(t) \quad \le \quad H(0) \qquad \text{for all } 0 < t < \infty. \tag{2.6}$$

For the moment, let us assume (2.6). Then, it follows from (2.3)–(2.6) and the monotonicity of F that

$$H'(t) = F(M(t)) \le F(H(t)) \le F(H(0)) = F(M(0)) \le F(M_*/2),$$

which yields again by (2.5) that

$$M(t) \le H(t) \le H(0) + F(M_*/2)t = M(0) + F(M_*/2)t, \qquad 0 < t < \infty.$$

Since $F(\frac{M_*}{2}) < 0$, this implies that

$$M(t) < 0 \qquad \text{for } -\frac{M(0)}{F(M_*/2)} < t < \infty,$$

which contradicts the non-negativity of u.

Now it only remains to prove (2.6). A similar argument can be seen in [8]. By (2.3), we have

$$H'(0) = F(M(0)) < 0,$$

which yields

$$H(s) \le H(0) \qquad \text{for } s \in [0, T_*) \quad \text{with some } 0 < T_* \le \infty.$$

Suppose that $T_* < \infty$. We may assume that

$$T_* = \sup \{ t > 0 : H(s) \le H(0) \quad \text{for all } s \in (0, t) \}.$$

Since H is a continuous function on $[0, \infty)$, we have $H(T_*) = H(0)$. It follows from (2.3)–(2.5) and the monotonicity of F that

$$H'(T_*) \quad = \quad F(M(T_*)) \le F(H(T_*)) = F(H(0)) = F(M(0)) < 0,$$

which yields $H(s) \le H(0)$ for all $0 < s < T'$ with some $T' > T_*$. This contradicts the definition of T_* and we obtain (2.6). This proves Theorem 1.1.

Now it remains to prove Lemma 2.2.

2.2. **Proof of Lemma 2.2.** (i) We define the localized weight function ψ by

$$\psi(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le 1, \\ 1 - 2(r-1)^2 & \text{for} \quad 1 < r \le \frac{3}{2}, \\ 2(2-r)^2 & \text{for} \quad \frac{3}{2} < r < 2, \\ 0 & \text{for} \quad r \ge 2, \end{cases}$$

and define ψ_ℓ by $\psi_\ell(x) := \psi(\frac{|x|}{\ell})$ for $x \in \mathbb{R}^N$ and $\ell = 1, 2, 3, \cdots$. There exists a positive constant c_1 depending only on N such that

$$|\nabla \psi_\ell(x)| \ \le \ \frac{c_1}{\ell}(\psi_\ell(x))^{\frac{1}{2}}, \quad |\Delta \psi_\ell(x)| \le \frac{c_1}{\ell^2}, \qquad x \in \mathbb{R}^N, \tag{2.7}$$

for all $\ell = 1, 2, \cdots$.

Multiplying the first equation in (KS) by $|x|^2\psi_\ell(x)$ and then integrating by parts, we have

$$\frac{d}{dt}\int u|x|^2\psi_\ell \, dx = \int u\Delta\big(|x|^2\psi_\ell\big)\, dx + \chi \int u\nabla v \cdot \nabla\big(|x|^2\psi_\ell\big)\, dx. \tag{2.8}$$

Since $\psi^{'}(r) \le 0$ for all $r \ge 0$ and since $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$ for all $t \in (0, T)$, we have by (2.7) that

$$\begin{aligned} \int u\Delta\big(|x|^2\psi_\ell\big)\, dx &= \int u\Big((\Delta|x|^2)\psi_\ell + 2x \cdot \nabla\psi_\ell + |x|^2\Delta\psi_\ell\Big)\, dx \\ &\le \int u\Big(2N\psi_\ell + 2\frac{|x|}{\ell}\psi'(\frac{|x|}{\ell})\Big)dx + c_1\int_{\ell \le |x| \le 2\ell} u|x|^2\frac{1}{\ell^2}dx \\ &\le 2N\int u\psi_\ell\, dx + 8c_1\int_{\ell\le|x|\le 2\ell} u\, dx \\ &\le (2N + 8c_1)\|u_0\|_{L^1}. \end{aligned} \tag{2.9}$$

On the other hand, by (2.7) we have

$$|x|^2|\nabla\psi_\ell(x)| \le c_1\frac{|x|}{\ell}\cdot|x|(\psi_\ell(x))^{\frac{1}{2}} \le 2c_1|x|(\psi_\ell(x))^{\frac{1}{2}} \quad \text{for all } \ell \le |x| \le 2\ell.$$

Thus, since $\mathrm{supp}\nabla\psi_\ell \subset \{\ell \le |x| \le 2\ell\}$, we have by the Schwarz inequality that

$$\begin{aligned} &\chi\int u\nabla v\cdot\nabla\big(|x|^2\psi_\ell\big)\, dx \\ &\le 2\chi\int u\nabla v\cdot x\psi_\ell\, dx + \chi\int_{\ell\le|x|\le 2\ell}|u\nabla v\cdot|x|^2\nabla\psi_\ell|\, dx \end{aligned}$$

$$\leq 2\chi\|\nabla v\|_{L^\infty(\mathbb{R}^N\times(0,T))}\cdot\|u_0\|_{L^1}^{\frac{1}{2}}\cdot\Big(\int u|x|^2(\psi_\ell)^2\,dx\Big)^{\frac{1}{2}}$$
$$+\,4c_1\chi\cdot\|\nabla v\|_{L^\infty(\mathbb{R}^N\times(0,T))}\cdot\|u_0\|_{L^1}^{\frac{1}{2}}\Big(\int_{\ell\leq|x|\leq 2\ell}u|x|^2\psi_\ell\,dx\Big)^{\frac{1}{2}}$$
$$\leq 4\chi(c_1+1)\|\nabla v\|_{L^\infty(\mathbb{R}^N\times(0,T))}\cdot\|u_0\|_{L^1}^{\frac{1}{2}}\Big(\int u|x|^2\psi_\ell\,dx\Big)^{\frac{1}{2}}. \tag{2.10}$$

By (2.8)–(2.10) and the Young inequality, we have

$$\frac{d}{dt}\int u|x|^2\psi_\ell\,dx\leq C\|u_0\|_{L^1}+\int u|x|^2\psi_\ell\,dx,$$

for all $0<t<T$, and all $\ell=1,2,\cdots$, where C is a constant depending on $\|\nabla v\|_{L^\infty(\mathbb{R}^N\times(0,T))}$ but not on ℓ. Notice that $\|\nabla v\|_{L^\infty(\mathbb{R}^N\times(0,T))}\leq\|\nabla G_\gamma\|_{L^1(\mathbb{R}^N)}\|u\|_{L^\infty(\mathbb{R}^N\times(0,T))}$. Hence, it follows from the Gronwall inequality that

$$\int u|x|^2\psi_\ell\,dx\leq\Big(\int u_0|x|^2\,dx+C\|u_0\|_{L^1}T\Big)\cdot e^T,$$

for all $0<t<T$, and all $\ell=1,2,\cdots$. Letting $\ell\to\infty$, we have

$$\int u(x,t)|x|^2\,dx\leq\Big(\int u_0|x|^2\,dx+C\|u_0\|_{L^1}T\Big)\cdot e^T, \tag{2.11}$$

for all $0<t<T$.

(ii) Again by (2.8)–(2.10), we have

$$\frac{d}{dt}\int u|x|^2\psi_\ell\,dx\leq 2N\int u\psi_\ell\,dx+4c_1\int_{\ell\leq|x|\leq 2\ell}u\,dx+2\chi\int u\nabla v\cdot x\psi_\ell\,dx$$
$$+\,2c_1\chi\cdot\|\nabla v\|_{L^\infty(\mathbb{R}^N\times(0,T))}\cdot\|u_0\|_{L^1}^{\frac{1}{2}}\Big(\int_{\ell\leq|x|\leq 2\ell}u|x|^2\psi_\ell\,dx\Big)^{\frac{1}{2}}\text{ for }t\in(0,T),$$

which yields

$$\int u|x|^2\psi_\ell\,dx-\int u_0|x|^2\psi_\ell\,dx \tag{2.12}$$
$$\leq 2N\int_0^t\int u\psi_\ell\,dxds+4c_1\cdot T\int_{\ell\leq|x|\leq 2\ell}u\,dx+2\chi\int_0^t\int u\nabla v\cdot x\psi_\ell\,dxds$$
$$+\,2c_1\chi\cdot\|\nabla v\|_{L^\infty(\mathbb{R}^N\times(0,T))}\cdot\|u_0\|_{L^1}^{\frac{1}{2}}\int_0^t\Big(\int_{\ell\leq|x|\leq 2\ell}u|x|^2\psi_\ell\,dx\Big)^{\frac{1}{2}}ds,$$

for $t \in (0,T)$. By virtue of (2.11), letting $\ell \to \infty$ in (2.12), we obtain

$$\int u(t)|x|^2\,dx - \int u_0|x|^2\,dx \le 2\int_0^t \int \Big(Nu(s) + \chi u(s)\nabla v(s)\cdot x\Big)\,dxds$$
$$=: 2\int_0^t I(s)\,ds \qquad \text{for } t \in (0,T). \tag{2.13}$$

Since $v = \alpha G_\gamma * u$, we have

$$I(s) = N\|u_0\|_{L^1} + \frac{\alpha\chi}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} u(x,s)u(y,s)\nabla G_\gamma(x-y,s)\cdot(x-y)\,dxdy.$$

We follow the argument which was employed in Corrias-Perthame-Zaag [1]. For every $R > 0$, it follows from Lemma 2.1 that

$$\begin{aligned}
I(s) &\le N\|u_0\|_{L^1} - \frac{\alpha\chi}{2\omega_{N-1}}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} u(x,s)u(y,s)\frac{1}{|x-y|^{N-2}}e^{-\sqrt{\gamma}|x-y|}\,dxdy\\
&\le N\|u_0\|_{L^1} - \frac{\alpha\chi}{2\omega_{N-1}}\int\int_{|x-y|\le R} u(x,s)u(y,s)\cdot\frac{1}{R^{N-2}}\cdot e^{-\sqrt{\gamma}\cdot R}\,dxdy\\
&= N\|u_0\|_{L^1} - \frac{\alpha\chi}{2\omega_{N-1}}\cdot\frac{1}{R^{N-2}}\cdot e^{-\sqrt{\gamma}\cdot R}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} u(x,s)u(y,s)\,dxdy\\
&\quad + \frac{\alpha\chi}{2\omega_{N-1}}\cdot\frac{1}{R^{N-2}}\cdot e^{-\sqrt{\gamma}\cdot R}\int\int_{|x-y|\ge R} u(x,s)u(y,s)\,dxdy\\
&\le N\|u_0\|_{L^1} - \frac{\alpha\chi}{2\omega_{N-1}}\cdot\frac{1}{R^{N-2}}\cdot e^{-\sqrt{\gamma}\cdot R}\cdot\|u_0\|_{L^1}^2\\
&\quad + \frac{\alpha\chi}{2\omega_{N-1}}\cdot\frac{1}{R^{N}}\cdot e^{-\sqrt{\gamma}\cdot R}\int\int_{|x-y|\ge R} u(x,s)u(y,s)|x-y|^2\,dxdy\\
&\le \|u_0\|_{L^1}\Big(N - \frac{\alpha\chi}{2\omega_{N-1}}\|u_0\|_{L^1}\cdot\frac{1}{R^{N-2}}\cdot e^{-\sqrt{\gamma}\cdot R}\Big)\\
&\quad + \frac{2}{\omega_{N-1}}\cdot\alpha\chi\|u_0\|_{L^1}\cdot R^{-N}\cdot e^{-\sqrt{\gamma}\cdot R}\int_{\mathbb{R}^N} u(x,s)|x|^2\,dx.
\end{aligned}$$

Now, we take $R > 0$ so that

$$N - \frac{\alpha\chi}{2\omega_{N-1}}\|u_0\|_{L^1}\cdot\frac{1}{R^{N-2}}\cdot e^{-\sqrt{\gamma}\cdot R} = -1,$$

which is equivalent to $g_\gamma(R) = \alpha\chi\|u_0\|_{L^1}$, so that

$$I(s) \le \|u_0\|_{L^1}\Big(-1 + \frac{2}{\omega_{N-1}}\cdot\alpha\chi\cdot R^{-N}\cdot e^{-\sqrt{\gamma}\cdot R}\int_{\mathbb{R}^N} u(s)|x|^2\,dx\Big).$$

Thus we have by (2.13) that

$$\begin{aligned}&\int u(t)|x|^2\,dx-\int u_0|x|^2\,dx\\&\le 2\|u_0\|_{L^1}\int_0^t\Big(\frac{2}{\omega_{N-1}}\cdot\alpha\chi\cdot R^{-N}\cdot e^{-\sqrt{\gamma}\cdot R}\int u(s)|x|^2\,dx-1\Big)\,ds,\end{aligned}$$

for all $0<t<T$. Defining the linear function $F(M)$ for $M\ge 0$ by (2.2), we obtain the desired estimate for $M(t)=\int|x|^2u(x,t)\,dx$. This completes the proof of Lemma 2.2.

2.3. **Proof of Theorem 1.2.** We give the relation between the quantity $R(u_0)=\frac{\|u_0\|_{L^1}^{\frac{N}{N-2}}}{\||x|^2u_0\|_{L^1}}$ in (1.5) for the finite time blow-up and $\|u_0\|_{L^{\frac{N}{2}}}$ in Proposition 1.1 (iii) for the time global existence. (see (1.1).)

Proposition 2.1. *Let* $N\ge 3$. *We take the function* $u_{0,1}$ *by* $u_{0,1}(x):=A\big(1-\frac{|x|^N}{b^N}\big)_+$ *with two parameters* A *and* b *of positive constants. Then, it holds that*

$$R(u_{0,1})=\beta_N\cdot\|u_{0,1}\|_{L^{\frac{N}{2}}}^{\frac{2}{N-2}},\tag{2.14}$$

where

$$\beta_N:=2^{\frac{-2(N+2)}{N(N-2)}}N^{\frac{2(2-N(N-1))}{N(N-2)}}(N+1)(N+2)^{\frac{N^2-2N+4}{N(N-2)}}(\omega_{N-1})^{\frac{2}{N}}.$$

Proof. By a direct calculation, we have

$$\|u_{0,1}\|_{L^1}=\frac{\omega_{N-1}}{2N}\cdot Ab^N,\quad \|u_{0,1}\|_{L^{\frac{N}{2}}}=\Big(\frac{2\omega_{N-1}}{N(N+2)}\Big)^{\frac{2}{N}}\cdot Ab^2,\tag{2.15}$$

$$\||x|^2u_{0,1}\|_{L^1}=\frac{N\omega_{N-1}}{2(N+1)(N+2)}\cdot Ab^{N+2}.\tag{2.16}$$

Hence, it holds that

$$\begin{aligned}R(u_{0,1})&=\frac{(N+1)(N+2)}{N^{\frac{2(N-1)}{N-2}}}\Big(\frac{\omega_{N-1}}{2}\Big)^{\frac{2}{N-2}}\cdot(Ab^2)^{\frac{2}{N-2}}\\&=\Big(\Big(\frac{2\omega_{N-1}}{N(N+2)}\Big)^{\frac{2}{N}}\cdot Ab^2\Big)^{\frac{2}{N-2}}\cdot\beta_N=\|u_{0,1}\|_{L^{\frac{N}{2}}}^{\frac{2}{N-2}}\cdot\beta_N,\end{aligned}\tag{2.17}$$

which yields (2.14).

Proof of Theorem 1.2. (i) For each $m = 1, 2, \cdots$, we now choose $u_{0,m}$ by $u_{0,m}(x,t) = mu_{0,1}(x,t)$. Since relation (2.14) is invariant under multiplication of the constant by the function, we have

$$R(u_{0,m}) = \beta_N \cdot \|u_{0,m}\|_{L^{\frac{N}{2}}}^{\frac{2}{N-2}} \qquad \text{for all } m \geq 1. \tag{2.18}$$

In addition,

$$\|u_{0,m}\|_{L^{\frac{N}{2}}} = m\|u_{0,1}\|_{L^{\frac{N}{2}}} \geq \|u_{0,1}\|_{L^{\frac{N}{2}}} \quad \text{for all } m \geq 1. \tag{2.19}$$

Now we take A and b so large that

$$\|u_{0,1}\|_{L^{\frac{N}{2}}} \geq C_N \cdot \frac{1}{\alpha\chi},$$

which is guaranteed by (2.15). Then we see that the sequence $\{u_{0,m}\}_{m=1}^{\infty}$ has the property (1.6). Moreover, it follows from (2.18) and (2.19) that

$$\begin{aligned} R(u_{0,m}) &\geq \beta_N \cdot \|u_{0,1}\|_{L^{\frac{N}{2}}}^{\frac{2}{N-2}} \geq \beta_N \cdot \Big(C_N \cdot \frac{1}{\alpha\chi}\Big)^{\frac{2}{N-2}} \\ &= 8(N+1)\Big(2(N+1)\omega_{N-1} \cdot \frac{1}{\alpha\chi}\Big)^{\frac{2}{N-2}} \qquad \text{for all } m \geq 1, \end{aligned}$$

which implies that $\{u_{0,m}\}_{m=1}^{\infty}$ fulfills the blow-up criterion (1.5). By Theorem 1.1, we observe that $\{u_{0,m}\}_{m=1}^{\infty}$ is the desired sequence of initial data.

Next, we construct the sequence $\{u^*_{0,m}\}_{m=1}^{\infty}$ of initial data with the property (1.7) such that, for each $m = 1, 2, \cdots$, the corresponding solution $u_m(t)$ with $u_m|_{t=0} = u^*_{0,m}$ blows up within a finite time. We will first illustrate that for every $\varepsilon > 0$ there are a function $u_{0,1}$ with $\|u_{0,1}\|_{L^1} < \varepsilon$ and a solution u of (KS) with $u|_{t=0} = u_{0,1}$ such that u blows up within a finite time. Since $\|u_{0,1}\|_{L^1}$ and $\|u_{0,1}\|_{L^{\frac{N}{2}}}$ are represented as in (2.15), the choice of $u_{0,1}$ satisfying

$$\|u_{0,1}\|_{L^1} \leq \varepsilon \quad \text{and} \quad \|u_{0,1}\|_{L^{\frac{N}{2}}} \geq C_N \frac{1}{\alpha\chi} \tag{2.20}$$

can be reduced to the problem of finding A and b in such a way that

$$\Big(\frac{N(N+2)}{2\omega_{N-1}}\Big)^{\frac{2}{N}} C_N \cdot \frac{1}{\alpha\chi} \cdot b^{-2} < A < \frac{2N}{\omega_{N-1}} \varepsilon b^{-N}. \tag{2.21}$$

It is easy to see that, for every $\varepsilon > 0$, there exists such a pair (A, b) satisfying (2.21). On the other hand, as we have seen above, relation (2.14) and the second inequality of (2.20) yield the blow-up criterion (1.5) on $u_{0,1}$. Therefore it follows from Theorem 1.1 that there is a strong solution u

of (KS) with $u|_{t=0} = u_{0,1}$ which blows up in a finite time. Now taking $\varepsilon = \frac{1}{m}$, $m = 1, 2, \cdots$, we obtain the desired sequence $\{u^*_{0,m}\}_{m=1}^{\infty}$ satisfying (1.7).

(ii) Next, we construct a sequence $\{u_{0,m}\}_{m=1}^{\infty}$ of initial data with (1.8) such that, for each $m = 1, 2, \cdots$, there exists a strong solution u_m of (KS) on $[0, \infty)$ with $u_m|_{t=0} = u_{0,m}$. To this end, we choose $u_{0,m}$ by $u_{0,m}(x,t) = \dfrac{u_{0,1}(x,t)}{m}$. Since

$$R(u_{0,m}) = \frac{1}{m^{\frac{2}{N-2}}} R(u_{0,1}) \le R(u_{0,1}) \qquad \text{for all } m \ge 1, \tag{2.22}$$

we have by (2.18) and (2.22) that

$$R(u_{0,1}) \quad \ge \quad \beta_N \cdot \|u_{0,m}\|_{L^{\frac{N}{2}}}^{\frac{2}{N-2}} \qquad \text{for all } m \ge 1. \tag{2.23}$$

Hence, taking A and b sufficiently small, we have by (2.17) and (2.23) that

$$\|u_{0,m}\|_{L^{\frac{N}{2}}} \quad \le \quad \lambda \qquad \text{for all } m \ge 1,$$

where λ is the same constant as in Proposition 1.1 (iii). Now it is easy to see that $\{u_{0,m}\}_{m=1}^{\infty}$ has the desired properties (1) and (2). This proves Theorem 1.2.

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