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ASYMPTOTIC PROFILE OF BLOW-UP SOLUTIONS OF KELLER-SEGEL SYSTEMS IN SUPER-CRITICAL CASES

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Abstract. We deal with $(\mathrm{KS})_m$ below for the super-critical cases of $q \geq m+\frac{2}{N}$ with $N \geq 2,\ m \geq 1,\ q \geq 2$. Based on an ε-regularity theorem in [20], we prove that the set S_u of blow-up points of the weak solution u consists of finitely many points if $u^{\frac{N(q-m)}{2}} \in C_w([0,T]; L^1(\mathbb{R}^N))$. Moreover, we show that $u^{\frac{N(q-m)}{2}}$ forms a *delta*-function singularity at the blow-up time. Simultaneously, we give a sufficient condition on u such that $u^{\frac{N(q-m)}{2}} \in C_w([0,T]; L^1(\mathbb{R}^N))$. Our condition exhibits a scaling invariant class associated with $(\mathrm{KS})_m$.

1. Introduction

We consider the following reaction-diffusion equation:

$$(\mathrm{KS})_m \quad \begin{cases} \partial_t u &= \Delta u^m - \nabla \cdot \left(u^{q-1}\nabla v\right), & x \in \mathbb{R}^N,\ t>0, \\ 0 &= \Delta v - \gamma v + u, & x \in \mathbb{R}^N,\ t>0, \\ u(x,0) &= u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Throughout this paper, we assume that $N \geq 2$, and that m, q, and γ are the constants satisfying $m \geq 1,\ q \geq 2,\ \gamma > 0$, respectively. The initial data u_0 is a non-negative function satisfying $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$ with $u_0^m \in H^1(\mathbb{R}^N)$. This equation is often called the Keller-Segel model describing the motion of the chemotaxis molds. Here, $u(x,t)$ and $v(x,t)$ denote the density of amoebae and the concentration of the chemo-attractant, respectively. We refer to Keller-Segel [10], and Horstman [9].

In $(\mathrm{KS})_m$, it is known that the exponent $q = m + \frac{2}{N}$ divides the situation regarding whether the solution of $(\mathrm{KS})_m$ exists globally in time or blows up in a finite time. Indeed, if $q < m + \frac{2}{N}$, then $(\mathrm{KS})_m$ is globally solvable without any restriction on the size of the initial data u_0. On the other hand, if $q \geq m + \frac{2}{N}$, then the global existence result is obtained for small initial

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data u_0 in $L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)$. It is also clarified in [18] that for $q \geq m + \frac{2}{N}$ there are infinitely many initial functions u_0 in $L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)$ such that the solution $u(t)$ of $(\mathrm{KS})_m$ blows up in a finite time. Therefore, the exponent $q = m + \frac{2}{N}$ can be regarded as the critical exponent of $(\mathrm{KS})_m$.

In the critical case of $q = m + \frac{2}{N}$, it was shown in [19] and [20] that the set S_u of blow-up points of the weak solution u consists of finitely many points under variant conditions on u. Moreover, it was proved that the solution u behaves like the Dirac δ-function at the blow-up points. More concretely, if u blows up at $t = T$, then it follows that

$$u(\cdot, t) \quad \longrightarrow \quad \sum_{i=1}^{k} M_i \delta_{x_i}(\cdot) + f(\cdot) \qquad \text{as} \;\; t \to T,$$

in the sense of distributions in $\mathbb{R}^N$, where $\{M_i\}_{i=1}^{\infty}$ are the positive constants with $M_i \geq \varepsilon_0$ $(i = 1, 2, \cdots, k)$, $f \in L^1(\mathbb{R}^N)$ and δ_{x_i} denotes the *delta*-function supported at x_i, $i = 1, 2, \cdots, k$. This kind of result was first obtained in [15] and [8] etc. for the 2-D semi-linear case; *i.e.,* when $(N, m, q) = (2, 1, 2)$. Our results in [19] and [20] are generalizations of their results for all critical cases of $q = m + \frac{2}{N}$ including the degenerate type of $m > 1$.

In [21], for the super-critical cases of $q \geq m + \frac{2}{N}$, an ε-regularity theorem was studied. Indeed, we showed that if the solution u of $(\mathrm{KS})_m$ satisfies

$$\sup_{0<t<T} \int_{B(x_0, 2\rho)} u^{\frac{N(q-m)}{2}}(x, t)\,dx \quad < \quad \varepsilon_0, \tag{1.1}$$

for some $x_0 \in \mathbb{R}^N$ and $\rho > 0$, then it holds that

$$\sup_{(x,t) \in B(x_0, \rho) \times (0,T)} u(x, t) \quad < \quad C,$$

where C depends only on $N, m, q, \gamma, \rho, \|u_0\|_{L^1(\mathbb{R}^N)}$ and $\|u_0\|_{L^\infty(\mathbb{R}^N)}$ but not on x_0. In our generalized case, the space $L^\infty(0, \infty; L^{\frac{N(q-m)}{2}}(\mathbb{R}^N))$ is also a scaling invariant class associated with $(\mathrm{KS})_m$.

The aim of this paper is to prove that the set S_u of blow-up points of the weak solution u consists of finitely many points under the hypothesis that $u^{\frac{N(q-m)}{2}} \in C_w([0, T]; L^1(\mathbb{R}^N))$. Moreover, we show that $u^{\frac{N(q-m)}{2}}$ forms a *delta*-function singularity at the time T, which implies more precisely that

$$u^{\frac{N(q-m)}{2}}(\cdot, t) \quad \longrightarrow \quad \sum_{i=1}^{k} M_i \delta_{x_i}(\cdot) + f(\cdot) \qquad \text{as} \;\; t \to T,$$

in the sense of distributions in $\mathbb{R}^N$, where $\{M_i\}_{i=1}^{\infty}$ are the positive constants with $M_i \geq \varepsilon_0$ $(i = 1, 2, \cdots, k)$, $f \in L^1(\mathbb{R}^N)$ and δ_{x_i} denotes the *delta*-function supported at x_i $(1 \leq i \leq k)$, with $S_u = \{x_1, x_2, \cdots, x_k\}$.

Finally, we show that $u^{\frac{N(q-m)}{2}}$ belongs to the class $C_w([0,T]; L^1(\mathbb{R}^N))$ if $u \in L^{q_*+q-1}(\mathbb{R}^N \times (0,T))$ and if $\nabla u^{\frac{m+q_*-1}{2}} \in L^2(\mathbb{R}^N \times (0,T))$; that is, u is in the scaling invariant class associated with $(\mathrm{KS})_m$.

2. Results

Throughout this paper, we impose the following assumption on m, q and the initial data u_0.

Assumption (i) The powers m and q of non-linearity satisfy

$$\begin{cases} m \geq 1, & 2 \leq q < m+2 \quad \text{for } N = 2, \\ m \geq 2 - \frac{2}{N}, & 2 \leq q < m+1 \quad \text{for } N \geq 3, \end{cases}$$

and the coefficient $\gamma > 0$.

(ii) The initial data u_0 is a non-negative function satisfying

$$u_0 \in L^1 \cap L^\infty(\mathbb{R}^N) \quad \text{with } u_0^m \in H^1(\mathbb{R}^N).$$

Our definition of a weak solution now reads as follows.

Definition 1 *Let the Assumption hold. A pair (u, v) of non-negative functions defined in $\mathbb{R}^N \times [0,T)$ is called a weak solution of $(\mathrm{KS})_m$ on $[0,T)$ if*

(i) $u \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^\infty(0,T'; L^\infty(\mathbb{R}^N))$;

(ii) $\nabla u^{m-\frac{q}{2}+\frac{1}{2}} \in L^2(0,T'; L^2(\mathbb{R}^N))$, $(u^{m-q+2})_t \in L^1(0,T'; L^1(\mathbb{R}^N))$;

(iii) $v \in L^\infty(0,T'; H^1(\mathbb{R}^N))$ *for all* T' *with* $0 < T' < T$;

(iv) (u, v) *satisfies the following identities:*

$$\int_0^T \int_{\mathbb{R}^N} \left(\nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \cdot \partial_t \varphi \right) dx\, dt = \int_{\mathbb{R}^N} u_0(x) \varphi(x,0) dx,$$

$$\int_{\mathbb{R}^N} \left(\nabla v \cdot \nabla \psi + \gamma v \cdot \psi - u \cdot \psi \right) dx = 0 \qquad a.a.\ t \in [0,T),$$

for all $\varphi \in H^1(0,T; L^2(\mathbb{R}^N)) \cap L^2(0,T; H^1(\mathbb{R}^N))$ satisfying $\varphi(\cdot, t) = 0$ for all $t \in [T', T]$ with some $0 < T' < T$, and all $\psi \in H^1(\mathbb{R}^N)$.

Concerning the local in time existence of weak solutions to $(\mathrm{KS})_m$, the following result can be shown by a slight modification of the argument developed by the author [16, Theorem 1.1].

Proposition 2.1. ([17]) (Local existence of a weak solution and its uniform L^∞-bound) *Let the Assumption hold. Then there exist T_0 and a weak solution (u,v) of (KS)$_m$ on $[0,T_0)$ with the mass conservation law:*

$$\|u(t)\|_{L^1(\mathbb{R}^N)} \quad = \quad \|u_0\|_{L^1(\mathbb{R}^N)} \qquad \text{for all } 0 \le t < T_0. \tag{2.1}$$

Such an interval T_0 of local existence can be taken as

$$T_0 = (\|u_0\|_{L^\infty(\mathbb{R}^N)} + 2)^{-q},$$

and the weak solution $u(t)$ above satisfies the following estimate:

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \quad \le \quad \|u_0\|_{L^\infty(\mathbb{R}^N)} + 2 \qquad \text{for all } t \in [0,T_0). \tag{2.2}$$

In [20], we proved that if the local mass concentration of u is less than the sufficiently small constant ε, then u becomes locally bounded. Indeed, we have the following.

Proposition 2.2. ([20]) (ε-regularity) *Let the Assumption hold. Then there exists a positive number ε_0 depending only on N and m with the following property: Suppose that (u,v) is an arbitrary weak solution of (KS)$_m$ on $[0,T)$ in Definition 1 with the mass conservation law (2.1) with $T = T_0$. If u satisfies*

$$\sup_{0<t<T} \int_{B(x_0,\rho_0)} u^{\frac{N(q-m)}{2}}(x,t)\, dx \quad < \quad \varepsilon_0, \tag{2.3}$$

for some $x_0 \in \mathbb{R}^N$ and $\rho_0 > 0$, then it holds that

$$\sup_{(x,t)\in B(x_0,\frac{\rho_0}{2})\times(0,T)} u(x,t) \quad < \quad C, \tag{2.4}$$

where $C = C(N,m,\gamma,\|u_0\|_{L^1\cap L^\infty},T,\rho_0)$ is a constant independent of x_0.

Definition 2. *Let (u,v) be the weak solution of (KS)$_m$ on $[0,T)$ in Definition 1.*

(i) (blow-up time) *We say that u blows up at the time $T < \infty$ if*

$$\limsup_{t\to T-0} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty. \tag{2.5}$$

Such a T is called a blow-up time of u.

(ii) (blow-up point) *Let T be a blow-up time of u. We call $x_0 \in \mathbb{R}^N$ a blow-up point of u at the time T if for every sequence $\{t_n\}_{n=1}^\infty \subset (0,T)$ with $t_n \to T$ as $n \to \infty$, there exists a sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ with $x_n \to x_0$ such that $u(x_n,t_n) \to \infty$ as $n \to \infty$.*

We establish the finiteness of the number of blow-up points. Indeed, the following holds.

Theorem 2.1. (Finiteness of the blow-up points) *Let the Assumption hold. Suppose that (u, v) is an arbitrary weak solution of $(KS)_m$ on $[0, T)$ with the mass conservation law (2.1) with $T = T_0$. Let T be the blow-up time of the weak solution u of $(KS)_m$. Assume that u has the additional properties*

$$u \in L^\infty(0, T; L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)), \tag{2.6}$$

and

$$\int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t)\psi(x)\, dx \text{ is a continuous function on } t \in [0, T], \tag{2.7}$$

for every $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi(x) = \psi(|x - x_0|)$ for some $x_0 \in \mathbb{R}^N$. Then, the number k of the blow-up points of u of $(KS)_m$ at the time T is finite. More precisely, k is bounded by

$$k \quad \le \quad \frac{\|u_0\|_{L^1(\mathbb{R}^N)}}{\varepsilon_0},$$

where ε_0 is the same constant given by Theorem 2.2.

Remark 1. If (u, v) solves $(\mathrm{KS})_m$ with $\gamma = 0$, then (u_λ, v_λ) is also a solution for all $\lambda > 0$, where

$$\begin{cases} u_\lambda(x,t) & := & \lambda^2 u\big(\lambda^{q-m}x, \lambda^{2(q-1)}t\big), \\ v_\lambda(x,t) & := & \lambda^{2(m-q+1)} v\big(\lambda^{q-m}x, \lambda^{2(q-1)}t\big). \end{cases} \tag{2.8}$$

It follows from a direct calculation of (2.8) that

$$\|u_\lambda\|_{L^s(0,\infty;L^p(\mathbb{R}^N))} = \lambda^{2\left(1-(\frac{q_*}{p}+\frac{q-1}{s})\right)}\|u\|_{L^s(0,\infty;L^p(\mathbb{R}^N))}, \tag{2.9}$$

for all $\lambda > 0$, and for all $1 \le p, s \le \infty$, where $q_* = \frac{N(q-m)}{2}$. Hence, the space $L^s(0, \infty; L^p(\mathbb{R}^N))$ is called the scaling invariant class associated with u of $(\mathrm{KS})_m$ provided $\frac{q_*}{p} + \frac{q-1}{s} = 1$. It should be noted that the solution u is the scaling invariance in

$$L^\infty(0, \infty; L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)) \quad \text{and} \quad L^{q_*+q-1}(0, \infty; L^{q_*+q-1}(\mathbb{R}^N)),$$

since it holds that

$$\begin{aligned} \sup_{0<t<\infty} \|u_\lambda(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)} &= \sup_{0<t<\infty} \|u(t)\|_{L^{\frac{N(q-m)}{2}}(\mathbb{R}^N)}, \\ \|u_\lambda\|_{L^{q_*+q-1}(0,\infty;L^{q_*+q-1}(\mathbb{R}^N))} &= \|u(t)\|_{L^{q_*+q-1}(0,\infty;L^{q_*+q-1}(\mathbb{R}^N))}, \end{aligned} \tag{2.10}$$

for all $\lambda > 0$. In addition, it follows from a direct calculation of (2.8) that

$$\|\nabla u_\lambda^{\frac{m+q_*-1}{2}}\|_{L^s(0,\infty;L^p(\mathbb{R}^N))} = \lambda^{2(\frac{q_*+q-1}{2}-\frac{q_*}{p}-\frac{q-1}{s})}\|\nabla u^{\frac{m+q_*-1}{2}}\|_{L^s(0,\infty;L^p(\mathbb{R}^N))}, \tag{2.11}$$

for all $\lambda > 0$, and for all $1 \le p, s \le \infty$. Hence, the space $L^s(0,\infty;L^p(\mathbb{R}^N))$ for $\nabla u^{\frac{m+q_*-1}{2}}$ is also scaling invariant provided $\frac{q_*}{p} + \frac{q-1}{s} = \frac{q_*+q-1}{2}$.

For the super critical cases of $q \ge m + \frac{2}{N}$, we show that $u^{\frac{N(q-m)}{2}}(x,t)$ forms the δ-*function* singularity at $\{x_i\}_{i=1}^k$ and at the time T with the mass $\{M_i\}_{i=1}^k$.

Theorem 2.2. (δ-function singularity) *Let the Assumption hold. Suppose that (u,v) is an arbitrary weak solution of $(KS)_m$ on $[0,T)$ with the mass conservation* (2.1) *with $T = T_0$. Let T be the blow-up time of a weak solution u of $(KS)_m$. Let $\{x_i\}_{i=1}^k$ be the blow-up points of u of $(KS)_m$ at the time T. Suppose that ε_0 is the constant given by Proposition* 2.2. *Assume* (2.6) *and* (2.7) *in Theorem* 2.1. *Then, there exist a sequence $\{t_n\}_{n=1}^\infty \subset (0,T)$ with $\lim_{n\to\infty} t_n = T$ and a function f in $L^1(\mathbb{R}^N)$ such that*

$$u^{\frac{N(q-m)}{2}}(\cdot,t_n) \quad \longrightarrow \quad \sum_{i=1}^k M_i\delta_{x_i}(\cdot) + f(\cdot) \quad \text{as} \quad n \to \infty,$$

in the sense of distributions in $\mathbb{R}^N$; i.e.,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t_n)\psi(x)\,dx = \sum_{i=1}^k M_i\psi(x_i) \;+\; \int_{\mathbb{R}^N} f(x)\psi(x)\,dx, \tag{2.12}$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. In particular, if, in addition to (2.6),

$$u^{\frac{N(q-m)}{2}} \in C_w([0,T];L^1(\mathbb{R}^N)), \tag{2.13}$$

then we have (2.12).

We give a sufficient condition which assures (2.7) in Theorem 2.1.

Theorem 2.3. *In addition to the Assumption, let $q \ge m + \frac{2}{N}(2-\frac{2}{N})$. Suppose that (u,v) is an arbitrary weak solution of $(KS)_m$ on $[0,T)$ with the mass conservation law* (2.1) *with $T = T_0$. Suppose that u has the property that*

$$u \in L^{q_*+q-1}(0,T;L^{q_*+q-1}(\mathbb{R}^N)), \tag{2.14}$$

and

$$\nabla u^{\frac{m+q_*-1}{2}} \in L^2(0,T;L^2(\mathbb{R}^N)), \tag{2.15}$$

where $q_* = \frac{N(q-m)}{2}$. *Then, for each* $\psi \in C_0^\infty(\mathbb{R}^N)$, *the function*

$$\int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t)\psi(x)\, dx,$$

is continuous on $[0,T]$.

Remark 2. The additional hypothesis of $q \geq m + \frac{2}{N}(2 - \frac{2}{N})$ comes from the validity of the energy estimate for u, which seems to be so redundant that we should remove it.

This paper is organized as follows: In Section 3, we prepare several lemmas. Section 4 is devoted to the proof of the finiteness of the set S_u of blow-up points of the weak solution u under the hypothesis $u^{\frac{N(q-m)}{2}} \in C_w([0,T]; L^1(\mathbb{R}^N))$. In Section 5, under the same assumption, we show that $u^{\frac{N(q-m)}{2}}$ forms a delta-function singularity at the blow-up time. In Section 6, we give a sufficient condition on u such that $u^{\frac{N(q-m)}{2}} \in C_w([0,T]; L^1(\mathbb{R}^N))$.

3. Preliminaries

In what follows, we abbreviate simply as

$$\|\cdot\|_r = \|\cdot\|_{L^r(\mathbb{R}^N)}, \qquad \int \cdot\, dx := \int_{\mathbb{R}^N} \cdot\, dx, \qquad 1 < r < \infty,$$

and C denotes the constant which may change from line to line. In particular, $C = C(*, \cdots, *)$ denotes a constant depending only on the variables appearing in the parenthesis.

Let us first introduce a cut-off function η with several properties. The proof was given in [19].

Lemma 3.1. *Let* $\rho_0 > 0$ *and* $\delta > 0$. *Let* $\eta(x) = \eta(|x|)$ *be defined as*

$$\eta(x) := \begin{cases} 1 & \text{for} \quad 0 \leq |x| < \rho_0, \\ \exp(1 - \frac{\delta}{\rho_0 + \delta - |x|}) & \text{for} \quad \rho_0 \leq |x| < \rho_0 + \delta, \\ 0 & \text{for} \quad |x| \geq \rho_0 + \delta. \end{cases}$$

Then, it holds that

$$|\nabla \eta(x)| \leq \frac{c}{a^2\delta} \cdot \eta(x)^{1-a}, \tag{3.1}$$

$$|\Delta \eta(x)| \leq \frac{c}{a^4\delta^2} \cdot \eta(x)^{1-a}, \tag{3.2}$$

for all $x \in \mathbb{R}^N$ *and all* $0 < a < 1$, *where* c *is an absolute positive constant. For such* η *and* $x_0 \in \mathbb{R}^N$, *we define also a cut-off function* η_{x_0} *by* $\eta_{x_0}(x) =$

$\eta(x-x_0)$. *In what follows, we will freely identify* η *with* η_{x_0} *and call* η *a cut-off function centered at* x_0 *with parameters* ρ *and* δ.

The following lemma is regarded as an interpolation inequality.

Lemma 3.2. *Let* $1 \le p_1 \le p_2 \le p_3 \le \infty$ *and let* λ *be defined as*

$$\lambda \quad = \quad \frac{p_3}{p_2} \cdot \frac{p_2 - p_1}{p_3 - p_1}.$$

Suppose that u *and* $v \ge 0$ *satisfy* $u \in L^{p_1}(\mathbb{R}^N)$ *and* $uv^{\frac{1}{\lambda}} \in L^{p_3}(\mathbb{R}^N)$. *Then, it holds that* $uv \in L^{p_2}(\mathbb{R}^N)$ *with the estimate*

$$\|uv\|_{p_2} \quad \le \quad \|u\|_{p_1}^{1-\lambda} \cdot \|uv^{\frac{1}{\lambda}}\|_{p_3}^{\lambda}.$$

The proof is established by means of variants of the standard Hölder inequality. So, we may omit it.

The following representation formula is well known. See, *e.g.,* E.M.Stein [22, Chapter V, Section 6.5], N.Aronszajn and K.T.Smith [1, page 415], S.T.Kuroda [12, page 58].

Lemma 3.3. *For* $f \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$, *let us consider the problem*

$$\text{(E)} \qquad -\Delta z + \gamma z = f \quad \text{in } \mathbb{R}^N.$$

Then the function $z(x)$ *given by*

$$z(x) \quad = \quad \int_{\mathbb{R}^N} G(x-y) f(y)\, dy \tag{3.3}$$

belongs to $W^{2,p}(\mathbb{R}^N)$ *and satisfies* (E), *where* $G(x)$ *is the kernel of the Bessel potential with the expression*

$$G(x) \quad = \quad \gamma^{\frac{N}{2}-1} \cdot a_N \cdot e^{-\sqrt{\gamma}|x|} \int_0^\infty e^{-\sqrt{\gamma}|x|s} \cdot \Big(s + \frac{s^2}{2}\Big)^{\frac{N-3}{2}}\, ds, \tag{3.4}$$

with the constant a_N *given by*

$$a_N \quad = \quad 2^{-1} \cdot (2\pi)^{-\frac{N-1}{2}} / \Gamma\Big(\frac{N-1}{2}\Big).$$

By using the Steklov averages (See *e.g.,* [4].), we have several integral identities for weak solutions. The proof is given in [21, Appendix A].

Lemma 3.4. *Let the Assumption hold. Suppose that* (u, v) *is an arbitrary weak solution of* $(KS)_m$ *on* $[0, T)$ *in Definition* 1. *Let* T' *be an arbitrary positive number such as* $T' < T$. *Then we have the following identities (i) and (ii).*

(i) It holds that

$$\int u(x,t)\psi(x)\,dx = \int_0^t \int \big(-\nabla u^m \cdot \nabla\psi + u^{q-1}\nabla v \cdot \nabla \psi\big)\,dx\,ds + \int u_0(x)\psi(x)\,dx, \quad 0 \le t \le T'; \tag{3.5}$$

(ii) For $r \ge 2 - \frac{2}{N}$, it holds that

$$\frac{1}{r}\int u^r(x,t)\psi(x)\,dx = \int_0^t (I_1 + I_2)(s)\,ds + \frac{1}{r}\int u_0^r(x)\psi(x)\,dx, \quad 0 \le t \le T', \tag{3.6}$$

for all $\psi \in H^1(\mathbb{R}^N)$, where I_1 and I_2 are defined by

$$I_1 := -\int \nabla u^m \cdot \nabla\big(u^{r-1}\cdot\psi\big)\,dx, \quad I_2 := \int u^{q-1}\nabla v \cdot \nabla\big(u^{r-1}\cdot\psi\big)\,dx.$$

4. Proof of Theorem 2.1

We first prepare the following lemma, which is essentially due to [19, Section 5] for the critical case of $q = m + \frac{2}{N}$. See also [15].

Lemma 4.1. *Let the Assumption hold. Suppose that (u, v) is an arbitrary weak solution of $(KS)_m$ on $[0, T)$ with the mass conservation law* (2.1) *with $T = T_0$. Let T be the blow-up time of the weak solution u of $(KS)_m$. We assume* (2.6) *and* (2.7). *Let $\{x_1, x_2, \cdots, x_k\}$ be the blow-up points of u of $(KS)_m$ at the time T. We choose a positive constant d so that*

$$B(x_i, 2d) \cap B(x_j, 2d) = \phi \quad \text{for all } i, j = 1, 2, \cdots, k. \tag{4.1}$$

Then, it holds that

$$k\varepsilon_0 < \sum_{n=1}^{k} \liminf_{t\to T} \int_{B(x_n,d)} u^{\frac{N(q-m)}{2}}(x,t)\cdot \eta_n(x)\,dx \tag{4.2}$$

$$\le \sup_{0<t<T} \int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(t)\,dx, \tag{4.3}$$

where η_n is the cut-off function centered at x_n with $\rho_0 = \frac{d}{2}$ and $\delta = \frac{d}{2}$ as in Lemma 3.1.

Proof. As an immediate consequence of Proposition 2.2, we have

$$\sup_{0<t<T} \int_{B(x_n,\rho)} u^{\frac{N(q-m)}{2}}(x,t)\,dx \ \ge\ \varepsilon_0 \quad \text{for all } \rho > 0,$$

and for all $n = 1, 2, \cdots, k$, where ε_0 is the same constant given by Proposition 2.2. Hence, it holds that

$$\limsup_{t\to T} \int_{B(x_n,\frac{d}{2})} u^{\frac{N(q-m)}{2}}(x,t)\ dx \geq \varepsilon_0 \qquad \text{for all } n = 1, 2, \cdots, k. \tag{4.4}$$

Since $\eta_n(x) = \eta(|x-x_n|)$ for $n = 1, 2, \cdots, k$, we see by (2.7) that the function $\int u^{\frac{N(q-m)}{2}}(x,t)\eta_n(x)\ dx$ is continuous on $[0, T]$, which yields with (4.4) that

$$\begin{aligned}\varepsilon_0 &< \limsup_{t\to T} \int_{B(x_n,\frac{d}{2})} u^{\frac{N(q-m)}{2}}(x,t)\, dx \leq \limsup_{t\to T} \int_{B(x_n,d)} u^{\frac{N(q-m)}{2}}(x,t)\cdot\eta_n(x)\, dx \\ &= \liminf_{t\to T} \int_{B(x_n,d)} u^{\frac{N(q-m)}{2}}(x,t)\cdot\eta_n(x)\ dx.\end{aligned}$$

This implies (4.2). On the other hand, for arbitrary $\varepsilon > 0$, there exists $\mu_n = \mu_n(\varepsilon)$ such that

$$\begin{aligned}&\liminf_{\tau\to T} \int_{B(x_n,d)} u^{\frac{N(q-m)}{2}}(x,\tau)\cdot\eta_n(x)\ dx - \varepsilon \\ &\leq \|u^{\frac{N(q-m)}{2}}(s)\eta_n\|_{L^1(B(x_n,d))} \ \leq\ \|u^{\frac{N(q-m)}{2}}(s)\|_{L^1(B(x_n,d))},\end{aligned} \tag{4.5}$$

for all $T - \mu_n < s < T$. Now let us define $\mu_k := \min_{1\leq n\leq k} \mu_n$. Then, it follows from (4.1) and (4.5) that

$$\begin{aligned}&\sum_{n=1}^{k} \Big(\liminf_{t\to T} \int_{B(x_n,d)} u^{\frac{N(q-m)}{2}}(x,t)\cdot\eta_n(x)\ dx - \varepsilon \Big) \\ &\leq \sum_{n=1}^{k} \|u^{\frac{N(q-m)}{2}}(T-\mu_k/2)\|_{L^1(B(x_n,d))} \ \leq\ \|u^{\frac{N(q-m)}{2}}(T-\mu_k/2)\|_{L^1(\mathbb{R}^N)} \\ &= \sup_{0<t<T} \int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t)\ dx.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily taken, we obtain (4.3). Thus, we complete the proof of Lemma 4.1.

Proof of Theorem 2.1. From (2.6), we can take $k \in \mathbb{N}$ such that

$$\sup_{0<t<T} \int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t)\ dx \ < \ k\varepsilon_0,$$

where ε_0 is the same constant as in Proposition 2.2. Suppose that there exist infinitely many blow-up points $\{x_i\}_{i=1}^{\infty}$. Then, we can choose $d > 0$ and k

blow-up points $\{x_1, x_2, \cdots, x_k\} \subset \{x_i\}_{i=1}^{\infty}$ satisfying (4.1). For the above d and $\{x_i\}_{i=1}^{k}$, it follows from Lemma 4.1 that

$$\begin{aligned} k\varepsilon_0 &< \sum_{n=1}^{k} \liminf_{t\to T} \int_{B(x_n, \frac{3d}{2})} u^{\frac{N(q-m)}{2}}(x,t)\eta_n(x)\ dx \\ &\le \sup_{0<t<T} \int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t)\, dx < k\varepsilon_0, \end{aligned}$$

which causes a contradiction. Hence we conclude that the number of the blow-up points at the time T is finite.

5. Proof of Theorem 2.2

The proof is given as a super-critical version of that in [19, Section 5: Proof of Thorem 2.6]. Let T be the blow-up time of u. By Theorem 2.1, we may assume that u blows up at k points $x_1, \cdots, x_k$. Let us define $M_{i,r},\ 1 \le i \le k$ by

$$M_{i,r} := \lim_{t\to T} \int_{B(x_i,r)} u^{\frac{N(q-m)}{2}}(x,t)\eta_i(x)\ dx \qquad \text{for } r > 0, \tag{5.1}$$

where η_i is the same cut-off function centered at x_0 with $\rho_0 = \delta = \frac{r}{2}$ as in Lemma 3.1 such that supp $\eta_i \subset B(x_i, r)$. It should be noted that the limit in (5.1) exists on account of the assumption (2.7). Since $M_{i,r}$ is monotone decreasing in r and bounded from below by ε_0 for all $i = 1, 2, \cdots, k$, there exists the limit of $M_{i,r}$ as $r \to 0$, *i.e.*,

$$M_i \ := \ \lim_{r\to 0} M_{i,r} \ < \ \infty \qquad \text{for all } i = 1, 2, \cdots, k. \tag{5.2}$$

We determine the regular part $f(x)$ of $u(x,t)$ as $t \to T$ in the following lemma. The proof is essentially due to [19, Proof of Lemma 5.2].

Lemma 5.1. *Let all assumptions in Theorem* 2.2 *hold. Then, there exist a sequence* $\{t_n\}_{n=1}^{\infty} \subset (0,T)$ *with* $\lim_{n\to\infty} t_n \to T$ *and a function* $\hat{f} \in L^1(\mathbb{R}^N)$ *such that*

$$u^{\frac{N(q-m)}{2}}(x,t_n) \to \hat{f}(x) \ < \ \infty \qquad \textit{as } n\to\infty \quad \textit{for a.a. } x \in \mathbb{R}^N. \tag{5.3}$$

For a moment, we assume this lemma and continue the proof of Theorem 2.2. We shall now show that, for every $\psi \in C_0^{\infty}(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{B(0,\rho)} u^{\frac{N(q-m)}{2}}(x,t_n)\psi(x)\, dx = \sum_{i=1}^{k} M_i \psi(x_i) + \int_{B(0,\rho)} \hat{f}(x)\psi(x)\, dx,$$

where ρ is taken so that supp $\psi \cup \{x_1, \cdots, x_k\} \subset B(0, \rho)$. Let us take the cut-off functions $\eta_i(x)$, $i = 1, \cdots, k$ as in (5.1). Since $1 - \eta_i(x) = 0$ for all $x \in B(x_i, \frac{r}{2})$, we have by a direct calculation that

$$\begin{aligned}
&\int_{B(0,\rho)} u^{\frac{N(q-m)}{2}}(x,t)\psi(x)\,dx - \sum_{i=1}^{k} M_i\psi(x_i) - \int_{B(0,\rho)} \hat{f}(x)\psi(x)\,dx \\
&= \int_{B(0,\rho)\setminus\bigcup_{i=1}^{k} B(x_i,r)} \big(u^{\frac{N(q-m)}{2}}(x,t) - \hat{f}(x)\big)\psi(x)\,dx \\
&- \sum_{i=1}^{k} \int_{B(x_i,r)} \hat{f}(x)\psi(x)\,dx + \sum_{i=1}^{k} \int_{B(x_i,r)} u^{\frac{N(q-m)}{2}}(x,t)\eta_i(x)\,dx \cdot \psi(x_i) \\
&- \sum_{i=1}^{k} M_i\psi(x_i) - \sum_{i=1}^{k} \int_{B(x_i,r)} u^{\frac{N(q-m)}{2}}(x,t)\eta_i(x)\,dx \cdot \psi(x_i) \\
&+ \sum_{i=1}^{k} \int_{B(x_i,r)} u^{\frac{N(q-m)}{2}}(x,t)\psi(x)\,dx \\
&= \int_{B(0,\rho)\setminus\bigcup_{i=1}^{k} B(x_i,r)} \big(u^{\frac{N(q-m)}{2}}(x,t) - \hat{f}(x)\big)\psi(x)\,dx \\
&- \sum_{i=1}^{k} \int_{B(x_i,r)} \hat{f}(x)\psi(x)\,dx + \sum_{i=1}^{k} \Big(\int_{B(x_i,r)} u^{\frac{N(q-m)}{2}}(x,t)\eta_i(x)\,dx - M_i\Big)\psi(x_i) \\
&+ \sum_{i=1}^{k} \int_{B(x_i,r)\setminus B(x_i,\frac{r}{2})} \big(u^{\frac{N(q-m)}{2}}(x,t) - \hat{f}(x)\big)\psi(x) \cdot \big(1 - \eta_i(x)\big)\,dx \\
&+ \sum_{i=1}^{k} \int_{B(x_i,r)\setminus B(x_i,\frac{r}{2})} \hat{f}(x)\psi(x) \cdot \big(1 - \eta_i(x)\big)\,dx \\
&+ \sum_{i=1}^{k} \int_{B(x_i,r)} u^{\frac{N(q-m)}{2}}(x,t)\eta_i(x) \cdot \big(\psi(x) - \psi(x_i)\big)\,dx. \qquad (5.4)
\end{aligned}$$

Since

$$\sup_{0<t<T} \int_{B(0,\rho)\setminus B(x_i,r)} u^{\frac{N(q-m)}{2}}(x,t)\; dx < \infty,$$

for all $i = 1, 2, \cdots, k$, and since $\hat{f} \in L^1(\mathbb{R}^N))$, we have by (5.3) and the Lebesgue dominated convergence theorem that

$$\Big|\int_{B(0,\rho)\setminus\bigcup_{i=1}^k B(x_i,r)}\big(u^{\frac{N(q-m)}{2}}(x,t_n)-\hat{f}(x)\big)\psi(x)\ dx\Big|\ \to\ 0,$$
$$\sum_{i=1}^k\Big|\int_{B(x_i,r)\setminus B(x_i,\frac{r}{2})}\big(u^{\frac{N(q-m)}{2}}(x,t_n)-\hat{f}(x)\big)\psi(x)\cdot\big(1-\eta_i(x)\big)\ dx\Big|\ \to\ 0$$

as $n\to\infty$. Substituting $t=t_n$ in (5.4) and then letting $n\to\infty$, we obtain from (5.1) that

$$\begin{aligned}
&\limsup_{n\to\infty}\Big|\int_{B(0,\rho)}u^{\frac{N(q-m)}{2}}(x,t_n)\psi(x)\,dx-\sum_{i=1}^k M_i\psi(x_i)-\int_{B(0,\rho)}\hat{f}(x)\psi(x)\,dx\Big|\\
&\le\sum_{i=1}^k\int_{B(x_i,r)}\hat{f}(x)\ dx\cdot\max_{x\in B(0,\rho)}|\psi(x)|\ +\ \sum_{i=1}^k|M_{i,r}-M_i||\psi(x_i)|\\
&+\sum_{i=1}^k\int_{B(x_i,r)}\hat{f}(x)\ dx\cdot\max_{x\in B(0,\rho)}|\psi(x)|\\
&+\sum_{i=1}^k\sup_{0<t<T}\int_{B(x_i,r)}u^{\frac{N(q-m)}{2}}(x,t)\ dx\cdot\max_{x\in B(x_i,r)}|\psi(x)-\psi(x_i)|\\
&=:F(r).
\end{aligned}\tag{5.5}$$

Since $\psi\in C_0^\infty(B(0,\rho))$ and since $\hat{f}\in L^1(\mathbb{R}^N)$, we have by (5.2) that $\lim_{r\to0}F(r)=0$. Since the left-hand side of (5.5) is independent of r, we conclude that

$$\lim_{n\to\infty}\Big|\int_{B(0,\rho)}u^{\frac{N(q-m)}{2}}(x,t_n)\psi(x)\,dx-\sum_{i=1}^k M_i\psi(x_i)-\int_{B(0,\rho)}\hat{f}(x)\psi(x)\,dx\Big|=0,$$

which completes the proof of Theorem 2.2.

Now, it remains to prove Lemma 5.1. The proof is given by a modification of that in [19, Section 5: Proof of Lemma 5.2]. We here give the modified parts essentially.

First of all, we define Ω_r and Ω_r' by

$$\Omega_r:=\mathbb{R}^N\setminus\bigcup_{i=1}^k B(x_i,r),\qquad \Omega_r':=\mathbb{R}^N\setminus\bigcup_{i=1}^k B(x_i,\frac{r}{2}),$$

respectively, where $r>0$ is taken so small that $B(x_i,r)\cap B(x_j,r)=\phi$ for $1\le i\ne j\le k$. From an argument similar to [19, Section 5: Proof of Lemma

5.2], we obtain the following regularities on ∇u^m and $\partial_t u^m$:

$$\nabla u^m \in L^2(0,T;L^2(\Omega_r)), \quad \partial_t u \in L^2(0,T;H^1(\Omega_r)^*). \tag{5.6}$$

We shall now construct $\hat{f}$ in (5.3). Since $m > 1$ and $u \in L^\infty(\Omega'_r \times (0,T))$, we see by (5.6) that

$$\partial_t u^m \quad = \quad m u^{m-1} \partial_t u \ \in \ L^2(0,T;H^1(\Omega_r)^*).$$

Hence, by the well-known interpolation argument (see Lions-Magenus [13]), we conclude that $u^m \in C([0,T];L^2(\Omega_r))$. Hence, there exist a function $f_r \in L^2(\Omega_r)$ and a sequence $\{t_n\}_{n=1}^\infty \subset (0,T)$ with $\lim_{n\to\infty} t_n = T$ such that $u(x,t_n) \to f_r(x)$ for almost all $x \in \Omega_r$ as $n \to \infty$. Let $\{r_\ell\}_{\ell=1}^\infty$ satisfy $r_1 > r_2 > \cdots > r_\ell \to 0$. Then, for every $x \in \mathbb{R}^N \setminus \bigcup_{i=1}^k \{x_i\}$, there exists ℓ such that $x \in \Omega_{r_\ell}$. We define the function $f(x)$ on $\mathbb{R}^N \setminus \bigcup_{i=1}^k \{x_i\}$ by $f(x) = f_{r_\ell}(x)$. It is easy to see that $f(x)$ is well defined on $\mathbb{R}^N \setminus \bigcup_{i=1}^k \{x_i\}$ since $f_{r_\ell}(x) = f_{r_p}(x)$ for all $p \ge \ell$. Furthermore, by the usual diagonal argument, we can choose a sequence $\{t_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} t_n = T$ such that $u(x,t_n) \to f(x)$ for almost all $x \in \mathbb{R}^N \setminus \bigcup_{i=1}^k \{x_i\}$ as $n \to \infty$, which implies that $u^{\frac{N(q-m)}{2}}(x,t_n) \to f^{\frac{N(q-m)}{2}}(x) =: \hat{f}(x)$ for almost all $x \in \mathbb{R}^N \setminus \bigcup_{i=1}^k \{x_i\}$ as $n \to \infty$. We next show that

$$\hat{f} \in L^1(\mathbb{R}^N). \tag{5.7}$$

Since

$$u(x,t_n) \quad \le \quad \sup_{0<t<T} u(x,t) \qquad \text{for all } x \in \Omega_r,$$

and since

$$\sup_{0<t<T} u(\cdot,t) \in \ L^\infty(\mathbb{R}^N \setminus \bigcup_{i=1}^k B(x_i,r)),$$

choosing r as small as in the proof of Lemma 5.1, we have by the Lebesgue dominated convergence theorem that

$$\begin{aligned} \int_{\Omega_r} \hat{f}(x)\, dx &= \lim_{n\to\infty} \int_{\Omega_r} u^{\frac{N(q-m)}{2}}(x,t_n)\, dx \\ &\le \sup_{0<t<T} \int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t)\, dx \qquad \text{for all } r > 0. \end{aligned} \tag{5.8}$$

Letting $r \to 0$ in (5.8), we have that

$$\int_{\mathbb{R}^N} \hat{f}(x)\ dx \quad \leq \quad \sup_{0<t<T} \int_{\mathbb{R}^N} u^{\frac{N(q-m)}{2}}(x,t)\ dx.$$

This proves Lemma 5.1.

6. Proof of Theorem 2.3

Since $\frac{N(q-m)}{2} \geq 2 - \frac{2}{N}$, due to the fact that $q \geq m + \frac{2}{N}(2-\frac{2}{N})$, using (3.6) with $r = \frac{N(q-m)}{2}$ in Lemma 3.4, we have

$$\begin{aligned} &\int u^{\frac{N(q-m)}{2}}(x,t)\psi(x)\ dx - \int u^{\frac{N(q-m)}{2}}(x,s)\psi(x)\ dx \\ &= \frac{N(q-m)}{2}\Big((f_1(t)-f_1(s)) \ + \ (f_2(t)-f_2(s))\Big), \quad 0 \leq s \leq t \leq T' \end{aligned} \tag{6.1}$$

with $T' < T$ for all $\psi \in H^1(\mathbb{R}^N)$, where

$$\begin{aligned} f_1(t) &= -\int_0^t \int \nabla u^m(x,\tau) \cdot \nabla (u^{\frac{N(q-m)}{2}-1}(x,\tau)\psi(x))\ dxd\tau, \\ f_2(t) &= \int_0^t \int (u^{q-1}\nabla v)(x,\tau) \cdot \nabla (u^{\frac{N(q-m)}{2}-1}(x,\tau)\psi(x))\ dxd\tau. \end{aligned}$$

Let us denote $q_* = \frac{N(q-m)}{2}$ for simplification. Since

$$\begin{aligned} \nabla u^m \cdot \nabla u^{\frac{N(q-m)}{2}-1} &= \nabla u^m \cdot \nabla u^{\frac{N(q-m)}{2}-1} \\ &= m(q_*-1)(\frac{2}{m+q_*-1})^2|\nabla u^{\frac{m+q_*-1}{2}}|^2, \end{aligned}$$

we have

$$\begin{aligned} f_1(t) - f_1(s) &= -m(q_*-1)(\frac{2}{m*q_*-1})^2 \int_s^t \int |\nabla u^{\frac{m+q_*-1}{2}}(x,\tau)|^2\psi(x)\ dxd\tau \\ &\quad - \int_s^t \int \nabla u^m(x,\tau) \cdot u^{q_*-1}(x,\tau) \cdot \nabla\psi(x)\ dxd\tau \\ &= -m(q_*-1)(\frac{2}{m+q_*-1})^2 \int_s^t \int |\nabla u^{\frac{m+q_*-1}{2}}(x,\tau)|^2\psi(x)\ dxd\tau \\ &\quad + \frac{m}{m+q_*-1}\int_s^t \int u^{m+q_*-1}(x,\tau)\Delta\psi(x)\ dxd\tau, \end{aligned}$$

for all $0 < s \le t < T'$. Since $\psi \in C_0^\infty(\mathbb{R}^N)$, it holds that

$$|f_1(t) - f_1(s)| \le C \int_s^t \int_B |\nabla u^{\frac{m+q_*-1}{2}}(x,\tau)|^2 \, dx d\tau \qquad (6.2)$$
$$+ C \int_s^t \int_B u^{m+q_*-1}(x,\tau) \, dx d\tau,$$

for all $0 < s \le t < T'$, where $C = C(N, m, q, \sup_{x \in \mathrm{supp}\psi} \psi(x), \sup_{x \in \mathrm{supp}\psi} |\Delta\psi(x)|)$ and the ball B is taken so that supp $\psi \subset B$. By (2.14), (2.15) and (6.2), we see $f_1 \in C([0,T])$ since C is independent of T'. On the other hand, it holds that

$$f_2(t) - f_2(s) = \int_s^t \int (u^{q-1}\nabla v)(x,\tau) \cdot \nabla(u^{q_*-1}(x,\tau)\psi(x)) \, dx d\tau$$
$$= \frac{q_* - 1}{q_* + q - 2} \int_s^t \int \nabla u^{q_*+q-2}(x,\tau) \cdot \nabla v \cdot \psi(x) \ dx d\tau$$
$$+ \int_s^t \int u^{q_*+q-2}(x,\tau) \nabla v(x,\tau) \cdot \nabla\psi(x) \ dx d\tau$$
$$= -\frac{q_* - 1}{q_* + q - 2} \int_s^t \int u^{q_*+q-2}(x,\tau) \cdot \Delta v \cdot \psi(x) \ dx d\tau$$
$$+ \Big(1 - \frac{q_* - 1}{q_* + q - 2}\Big) \int_s^t \int u^{q_*+q-2}(x,\tau) \nabla v(x,\tau) \cdot \nabla\psi(x) \ dx d\tau$$
$$= -\gamma \cdot \frac{q_* - 1}{q_* + q - 2} \int_s^t \int u^{q_*+q-2}(x,\tau) \cdot v(x,\tau) \cdot \psi(x) \ dx d\tau$$
$$+ \ \frac{q_* - 1}{q_* + q - 2} \int_s^t \int u^{q_*+q-1}(x,\tau) \cdot \psi(x) \ dx d\tau$$
$$+ \frac{q - 1}{q_* + q - 2} \int_s^t \int u^{q_*+q-2}(x,\tau) \nabla v(x,\tau) \cdot \nabla\psi(x) \ dx d\tau. \qquad (6.3)$$

Using the Hölder and Hardy-Littlewood-Sobolev inequalities, we have by $\frac{N(q_*+q-1)}{N+q_*+q-1} < N$ that

$$\frac{q - 1}{q_* + q - 2} \int_s^t \int u^{q_*+q-2}(x,\tau) \nabla v(x,\tau) \cdot \nabla\psi(x) \ dx d\tau$$
$$\le C \int_s^t \|u^{q_*+q-2}(\tau)\|_{\frac{q_*+q-1}{q_*+q-2}} \|\nabla v(\tau)\|_{q_*+q-1} \ d\tau$$

$$\leq C \int_s^t \|u(\tau)\|_{q_*+q-1}^{q_*+q-2} \|u(\tau)\|_{\frac{N(q_*+q-1)}{N+q_*+q-1}} \; d\tau,$$

with $C = C(N, \gamma, m, q, \sup_{x \in \mathrm{supp}\psi} |\nabla\psi(x)|)$. Since $1 \leq \frac{N(q_*+q-1)}{N+q_*+q-1} \leq q_* + q - 1$, implied by the facts that $q \geq 2$ and $q_* \geq 1$, it follows from the interpolation inequality that

$$\begin{aligned}
&\frac{q-1}{q_*+q-2} \int_s^t \int u^{q_*+q-2}(x,\tau) \nabla v(x,\tau) \cdot \nabla\psi(x) \; dx d\tau \\
&\leq C \int_s^t \|u(\tau)\|_{q_*+q-1}^{q_*+q-2} \|u_0\|_1^{\frac{q_*+q-1}{N(q_*+q-2)}} \|u(\tau)\|_{q_*+q-1}^{1-\frac{q_*+q-1}{N(q_*+q-2)}} \; d\tau \\
&\leq C|t-s| + C \int_s^t \|u(\tau)\|_{q_*+q-1}^{q_*+q-1} \; d\tau,
\end{aligned} \tag{6.4}$$

with $C = C(N, \gamma, m, q, \|u_0\|_1, \sup_{x \in \mathrm{supp}\psi} |\nabla\psi(x)|)$. By a similar argument as above for (6.4), together with the Hausdorff Young inequality, it holds that

$$\begin{aligned}
&-\gamma \cdot \frac{q_*-1}{q_*+q-2} \int_s^t \int u^{q_*+q-2}(x,\tau) \cdot v(x,\tau) \cdot \psi(x) \; dx d\tau \qquad (6.5)\\
&\leq C \int_s^t \|u(\tau)\|_{q_*+q-1}^{q_*+q-2} \|v(\tau)\|_{q_*+q-1} \; d\tau \leq C \int_s^t \|u(\tau)\|_{q_*+q-1}^{q_*+q-1} \; d\tau,
\end{aligned}$$

with $C = C(N, \gamma, m, q, \sup_{x \in \mathrm{supp}\psi} \psi(x))$. From (6.3)–(6.5), we obtain that

$$|f_2(t) - f_2(s)| \leq C|t-s| + C \int_s^t \|u(\tau)\|_{q_*+q-1}^{q_*+q-1} d\tau, \tag{6.6}$$

for all $0 < s \leq t < T'$ with

$$C = C(N, \gamma, m, q, \|u_0\|_1, \sup_{x \in \mathrm{supp}\psi} \psi(x), \sup_{x \in \mathrm{supp}\psi} |\nabla\psi(x)|, T),$$

which implies that $f_2 \in C([0,T])$ since C is independent of T'. Now the desired continuity follows from (6.1). This proves Theorem 2.3.

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