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GLOBAL SOLUTIONS TO THE CAUCHY PROBLEM FOR A SYSTEM OF DAMPED WAVE EQUATIONS

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Abstract. The Cauchy problem to the system of nonlinear damped wave equations is treated. Several authors have shown existence and asymptotic behavior of global solutions to the above problem when the space dimension is not greater than three. We will show the existence and asymptotic behavior of global solutions to the problem with rapidly decaying initial data when the space dimension is greater than three, where we apply estimates in weighted Sobolev spaces of the above solution operator. Moreover, using the theory of modulation spaces introduced by Feitinger [4], we will also show the existence and asymptotic behavior of global solutions to the problem with slowly decaying initial data.

1. Introduction

In this paper we are concerned with global existence and asymptotic behavior of solutions to the Cauchy problem for a system of the nonlinear damped wave equations

$$\partial_t^2 u_j - \Delta u_j + \partial_t u_j = F_j(u), \quad t > 0, x \in \mathbb{R}^n, \quad 1 \le j \le N, \tag{1.1}$$

with initial data

$$u_j(0,x) = \varphi_j(x),\ \partial_t u_j(0,x) = \psi_j(x), \quad x \in \mathbb{R}^n, \quad 1 \le j \le N, \tag{1.2}$$

where $u = (u_1, \cdots, u_N)$ are unknown, nonlinear terms, $F_j(u)$ is a function of C^1 class such that

$$|F_j(u)| \le C\,|u_1|^{\sigma_{j,1}} \cdot \cdots \cdot |u_N|^{\sigma_{j,N}}, \tag{1.3}$$

$$\left|\frac{\partial}{\partial u_k} F_j(u)\right| \le C\sigma_{j,k}\,|u_1|^{\sigma_{j,1}} \cdot \cdots \cdot |u_k|^{\sigma_{j,k}-1} \cdot \cdots \cdot |u_N|^{\sigma_{j,N}} \tag{1.4}$$

for $k = 1, \cdots, N$, $\partial_t = \partial/\partial t$ and Δ is the Laplace operator in $\mathbb{R}^n$.

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We denote $\partial_k = \partial/\partial x_k$ $(k = 1, \cdots, n)$, $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ of non-negative integers. We use function spaces $L^p = L^p(\mathbb{R}^n)$ and

$$H^{s,p} = H^{s,p}(\mathbb{R}^n) = \{u \in \mathcal{S}'(R^n) : (1-\Delta)^{s/2}u \in L^p\}, \quad p \in [1,\infty], s \in \mathbb{R},$$

equipped with norms $\|u\|_{L^p}$ and $\|u\|_{H^{s,p}} = \left\|(1-\Delta)^{s/2}u\right\|_{L^p}$, respectively, where $\mathcal{S}'(\mathbb{R}^n)$ is a set of tempered distributions on $\mathbb{R}^n$. It is known that $H^{s,p}$ is a Banach space, and the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s,p}$ when $p \neq \infty$.

We also use the modulation space $M_{p,q}^s = M_{p,q}^s(\mathbb{R}^n)$ $(p, q \in [1,\infty], s \in \mathbb{R})$ introduced by Feitinger [4]. In the next section we will state the preliminary and known facts on modulation spaces. We denote by $\mathcal{L}(X,Y)$ the set of bounded linear operators from a Banach space X to a Banach space Y.

When $N = 1$, Matsumura [13] showed the asymptotic behavior of the solution to the Cauchy problem (1.1)–(1.2). It was proved that $\sigma_F := 1+2/n$ is the critical exponent to Problem (1.1)–(1.2) by Todorova-Yordanov [24] and Zhang [27] (cf. [6, 7, 8, 12, 13, 14, 18]). When $\sigma_{1,1} > \sigma_F$ and the initial data are rather rapidly decaying and sufficiently small, then (1.1)–(1.2) admits a unique global solution and the solutions behave as ones to the corresponding heat equation when $t \to \infty$. On the other hand, when $1 < \sigma_{1,1} \leq \sigma_F$, then the solution of Problem (1.1) with the nonlinear term replaced by $|u_1|^{\sigma_{1,1}}$ may blow-up in finite time even if the initial data is small.

When $N = 2$ and $n = 1, 3$, Sun and Wang [22] have proved that Problem (1.1)–(1.2), with nonlinear terms

$$F_1(u) = |u_2|^{\sigma_1}, \quad F_2(u) = |u_1|^{\sigma_2}, \tag{1.5}$$

admits global solution under the conditions

$$\max\left(\frac{\sigma_1+1}{\sigma_1\sigma_2-1}, \frac{\sigma_2+1}{\sigma_1\sigma_2-1}\right) < \frac{n}{2},\ \sigma_1 \geq 1,\ \sigma_2 \geq 1,\ \sigma_1\sigma_2 > 1. \tag{1.6}$$

Moreover, they have shown that any non-negative and non-trivial solutions to the problem blow up in finite time when (1.6) does not holds.

When $N = 2$, $1 \leq n \leq 3$ and $1 \leq p < \infty$, Narazaki [16] proved that Problem (1.1)–(1.2) and (1.5) with initial data $(\varphi_j, \psi_j) \in (H^{1,p} \cap H^{1,\infty}) \times (L^p \cap L^\infty)$ admits global solutions that satisfy estimates

$$\|u_j(t,\cdot)\|_{L^q} \leq C(1+t)^{n/2q-(\sigma_j+1)/(\sigma_1\sigma_2-1)} \quad \forall q \in [p,\infty],$$

under the condition

$$\max\left(\frac{\sigma_1+1}{\sigma_1\sigma_2-1},\frac{\sigma_2+1}{\sigma_1\sigma_2-1}\right)<\frac{n}{2p},\ \sigma_1\geq 1,\ \sigma_2\geq 1,\ \sigma_1\sigma_2>1. \tag{1.7}$$

When $1 \leq n \leq 3$, Ogawa-Takeda [20] proved that the Cauchy problem for a general system of damped wave equations (1.1)–(1.2) admits a global solution; moreover, they showed that the global solution satisfies the decay estimate described in Theorem 1 below, provided that the equation

$$\begin{pmatrix} \sigma_{1,1}, & \cdots & \sigma_{1,N} \\ \vdots & \cdots & \vdots \\ \sigma_{N,1}, & \cdots & \sigma_{N,N} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \vdots \\ \nu_N \end{pmatrix}=\begin{pmatrix} 1+\nu_1 \\ \vdots \\ 1+\nu_N \end{pmatrix} \tag{1.8}$$

admits a solution $(\nu_1,\cdots,\nu_N)$ such that

$$0<\min(\nu_1,\cdots,\nu_N)\leq\max(\nu_1,\cdots,\nu_N)<\tfrac{n}{2}. \tag{1.9}$$

Takeda [23] proved that the solutions of (1.1)–(1.2) blow up in finite time when condition (1.9) does not hold. It should be noted that $\nu_j=(\sigma_j+1)/(\sigma_1\sigma_2-1)$ $(j=1,2)$ for nonlinear terms (1.5).

The aim of this paper is to remove the restriction $n\leq 3$. We will show that the Cauchy problem (1.1)–(1.2) admits global solutions when $n\geq 4$ under certain conditions. Moreover, we show the asymptotic behavior of the above solution. We consider first the case where the initial data are rather rapidly decaying, and then we consider the case where initial data are slowly decaying, i.e., the case where initial data do not belong to L^1.

$\mathcal{F}[f](\xi)=\hat{f}(\xi)$ is the Fourier transformation of f with respect to x:

$$\mathcal{F}[f](\xi)=\hat{f}(\xi)=\left(\tfrac{1}{2\pi}\right)^{n/2}\int_{R^n}e^{ix\cdot\xi}f(x)\,dx.$$

$\mathcal{F}^{-1}$ is the inverse Fourier transformation, and $\langle x\rangle:=\sqrt{1+|x|^2}$.

The first result of this paper is as follows.

Theorem 1. *Let N be a positive integer, let $F_j(u)$ $(1\leq j\leq N)$ be a function of C^1 class that satisfies* (1.3)–(1.4)*, and let $\sigma_{j,k}$ $(1\leq j,k\leq N)$ be constants such that $\sigma_{j,k}\geq 1$ when $\sigma_{j,k}\neq 0$. Assume that ν_j are positive constants that satisfy* (1.8)–(1.9)*. Moreover, assume that*

$$1<\sigma_j:=\sum_{k=1}^{N}\sigma_{j,k}\leq 1+\frac{2}{\max(0,n-4)},\qquad \forall j\in[1,N]. \tag{1.10}$$

Let m be a non-negative even integer that satisfies

$$m > \max_{1\le j\le N} \frac{n}{2}\cdot\frac{2-\sigma_j}{\sigma_j - 1} \tag{1.11}$$

when $\min_{1\le j\le N}\sigma_j < 2$, and let $m = 0$ when $\min_{1\le j\le N}\sigma_j \ge 2$.

Then, there exists a positive constant ϵ such that, if the initial data satisfies $(\langle x\rangle^m\varphi_j, \langle x\rangle^m\psi_j) \in \left(L^1\cap H^{2,2}\right)\times\left(L^1\cap H^{1,2}\right)$ for $1\le j\le N$, and

$$I := \sum_{j=1}^N \left(\|\langle x\rangle^m\varphi_j\|_{L^1} + \|\langle x\rangle^m\varphi_j\|_{H^{2,2}} + \|\langle x\rangle^m\psi_j\|_{L^1} + \|\langle x\rangle^m\psi_j\|_{H^{1,2}}\right) \le \epsilon,$$

then Problem (1.1)–(1.2) *admits a unique global solution $u = (u_j)_{1\le j\le N}$ such that $u_j \in C\left([0,\infty); H^{2,2}\right)\cap C^1\left([0,\infty); H^{1,2}\right)\cap C^2\left([0,\infty); L^2\right)$, and it satisfies*

$$\|\langle x\rangle^\mu u_j(t,\cdot)\|_{L^2} \le C(1+t)^{n/4-\nu_j+\mu/2} I,$$

$$\left\|\langle x\rangle^\mu \partial_t^k\partial_x^\alpha u_j(t,\cdot)\right\|_{L^2} \le C(1+t)^{n/4-\nu_j+\mu/2-1/2} I, \quad (1\le k+|\alpha|\le 2)$$

for any $\mu\in[0,m]$.

Theorem 1 is a consequence of Proposition 2 stated in section 3. From Proposition 2 below one obtains the next Corollary.

Corollary 1.1. *Under the notation in Theorem* 1, *the solution $(u_1,\cdots,u_N)$ obtained in Theorem* 1 *satisfies $u_j(t,\cdot) = v_j(t,\cdot) + w_j(t,\cdot)$ for some $v_j \in C([0,\infty); H^{2,1}\cap H^{2,\infty})$ and $w_j \in C([0,\infty); H^{2,2})$ satisfy*

$$\|v_j(t,\cdot)\|_{L^p} \le C(1+t)^{n/2p-\nu_j} I, \quad \|w_j(t,\cdot)\|_{H^{2,2}} \le C(1+t)^{n/4-3/2-\nu_j} I,$$

for any $p\in[1,\infty]$ when $n = 4, 5$, and for any $p\in[1,\infty)$ when $n = 6$. Moreover,

$$\|v_j(t,\cdot)\|_{L^\infty} \le C(1+t)^{-\nu_j}\log(t+2) I$$

when $n = 6$.

When $\min_{1\le j\le N}\sigma_j = \min_{1\le j\le N}\sum_{k=1}^N \sigma_{j,k} > 1+2/n$, one obtains sharper decay estimates than the ones obtained in Theorem 1.

Proposition 1. *Under the notation in Theorem* 1, *in addition, assume that*

$$\min_{1\le j\le N}\sigma_j > 1+\tfrac{2}{n}.$$

Then the solution $(u_1,\cdots,u_N)$ obtained in Theorem 1 *satisfies*

$$\|\langle x\rangle^\mu u_j(t,\cdot)\|_{L^2} \le C(1+t)^{-n/4+\mu/2} I,$$

$$\sum_{1\le k+|\alpha|\le 2}\left\|\langle x\rangle^{\mu}\partial_t^k\partial_x^\alpha u_j(t,\cdot)\right\|_{L^2}\le C(1+t)^{-n/4+\mu/2-1/2}I$$

for any $\mu\in[0,m]$.

Now let us consider (1.1)–(1.2) with slowly decaying initial data. When $1\le n\le 3$, using L^p-L^p $(1\le p\le\infty)$ estimates of the solution operator for (1.1)–(1.2), several authors have shown the existence of global solutions of (1.1)–(1.2) [9, 17, 16]. On the other hand, it seems that L^p-L^p $(p\gg 1)$ estimates of the solution operator for (1.1)–(1.2) do not hold when $n\ge 4$. To avoid the above difficulties we apply the theory of modulation spaces.

Theorem 2. *Let* $N\ge 2$ *be an integer,* $\sigma_{j,k}$ $(1\le j\le N)$ *be a non-negative integer, and let* $F_j(u)=Cu_1^{\sigma_{j,1}}\cdot\cdots\cdot u_N^{\sigma_{j,N}}$ $(1\le j\le N)$. *Let* $p\in[1,\infty)$, *and let* m *be a nonnegative integer. Assume that positive constants* $\nu_1,\cdots,\nu_N$ *determined by* (1.8) *satisfy*

$$\max(\nu_1,\cdots,\nu_N)<\frac{n}{2p}. \tag{1.12}$$

Then, there exists a positive constant ϵ *such that, if*

$$(\varphi_j,\psi_j)\in M_{p,1}^{m+1}\times M_{p,1}^{m}\quad(1\le j\le N),$$

and

$$\sum_{j=1}^{N}\left(\|\varphi_j\|_{M_{p,1}^{m+1}}+\|\psi_j\|_{M_{p,1}^{m}}\right)\le\epsilon,$$

Problem (1.1)–(1.2) *admits a unique global solution* $u=(u_j)_{1\le j\le N}$ *in the class*

$$u_j\in\bigcap_{l=0}^{m+1}C^k\left([0,\infty);M_{p,1}^{m+1-k}\right)\quad(1\le j\le N),$$

and it satisfies

$$\left\|\partial_t^k u_j(t,\cdot)\right\|_{M_{q,1}^{m+1-k}}\le C(1+t)^{n/2q-\nu_j-k}\sum_{j=1}^{N}\left(\|\varphi_j\|_{M_{p,1}^{m+1}}+\|\psi_j\|_{M_{p,1}^{m}}\right)$$

for any $t\ge 0$, $q\in[p,\infty]$, $j\in\{1,\cdots,N\}$ *and* $k\in\{0,1,\cdots,m+1\}$.

In Theorem 1 we consider Problem (1.1)–(1.2) with fast decaying initial data as $|x|\to\infty$. On the other hand, in Theorem 2, we consider the problem with slowly decaying initial data as $|x|\to\infty$. When we study the slowly decaying problem in the usual Sobolev space $H^{m,p}$ $(p>2)$, we may encounter difficulties caused by derivative-loss of solutions to linear wave equation. To avoid this kind of difficulty, we apply the theory of modulation spaces in Theorem 2.

Recently several authors have studied the damped wave equation with variable coefficients. In the above research it is assumed that the initial data are compactly supported, or the initial data decays very fast ([10, 11, 19, 21, 25]).

Remark 1.1. Let $p \in [1, \infty)$, let μ be a real constant, and let $\delta(p) = 0$ when $p \in (1, \infty)$ and let $\delta(1)$ be any positive constant. Lemma 2.1(3), (7) below shows that $H^{n+\mu+\delta(p),p} \subset M^{\mu}_{p,1} \subset H^{\mu,p}$. Hence Theorem 2 shows that there exists a positive constant $\bar{\epsilon}$ such that if the initial data satisfies $(\varphi_j, \psi_j) \in H^{m+1+n+\delta(p),p} \times H^{m+n+\delta(p),p}$ for $j \in \{1, \cdots, N\}$, and

$$\bar{I} := \sum_{j=1}^{N} \left(\|\varphi_j\|_{H^{m+1+n+\delta(p),p}} + \|\psi_j\|_{H^{m+n+\delta(p),p}} \right) \leq \bar{\epsilon},$$

then Problem (1.1)–(1.2) admits a unique global solution $u = (u_j)_{1 \leq j \leq N}$ in the class

$$u_j \in \bigcap_{k=0}^{m+1} C^k \left([0, \infty); H^{m+1-k,p} \right) \qquad \forall j \in \{1, \cdots, N\},$$

and it satisfies

$$\left\| \partial_t^k u_j(t, \cdot) \right\|_{H^{m+1-k,q}} \leq C(1+t)^{n/2q-\nu_j-k} \bar{I}, \quad \forall t \geq 0$$

for any $q \in [p, \infty]$, $j \in \{1, \cdots, N\}$ and $k \in \{0, 1, \cdots, m+1\}$.

2. PRELIMINARIES

In this section we show several preliminaries. For the convenience of readers we will sketch a definition and fundamental properties of modulation spaces. Let

$$Q_0 = \left\{ \xi := (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n : \xi_j \in [-\tfrac{1}{2}, \tfrac{1}{2}), j = 1, \cdots, n \right\},$$

and choose $\vartheta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \vartheta(\xi) \leq 1$, $\vartheta(\xi) \geq c > 0$ for some constant c when $\xi \in Q_0$, $\vartheta(\xi) = 0$ when $|\xi| \geq \sqrt{n}$, and

$$\sum_{\gamma \in \mathbb{Z}^n} \vartheta_\gamma(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n, \qquad \vartheta_\gamma(\xi) := \vartheta(\xi - \gamma).$$

For any $\gamma \in \mathbb{Z}^n$ define operators $\Box_\gamma$ by $\Box_\gamma = \mathcal{F}^{-1}\vartheta_\gamma\mathcal{F}$. For any $p, q \in [1, \infty]$, $s \in \mathbb{R}$, the modulation space $M^s_{p,q}$ is the set of any $f \in \mathcal{S}'$ such that

$$\|f\|_{M^s_{p,q}} := \Big(\sum_{\gamma\in\mathbb{Z}^n} \langle\gamma\rangle^{sq} \|\Box_\gamma f\|^q_{L^p} \Big)^{\frac{1}{q}} < \infty, \quad \langle\gamma\rangle = \sqrt{1+|\gamma|^2}.$$

$M^0_{p,q}$ will be denoted as $M_{p,q}$ for simplicity. The next lemma is well known.

Lemma 2.1 ([1, 2, 3, 5])**.** *Let s, s_1, s_2, μ be real constants, and let $p, q, p_1, p_2, q_1, q_2 \in [1, \infty]$.*

(1) *$M^s_{p,q}$ with norm $\|\cdot\|_{M^s_{p,q}}$ is a Banach space, and it satisfies $\mathcal{S} \subset M^s_{p,q} \subset \mathcal{S}'$;*
(2) *when $p, q \in [1, \infty)$, $\mathcal{S}$ is dense in $M^s_{p,q}$;*
(3) *$J_\mu := (1-\Delta)^\mu$ is an isomorphism from $M^s_{p,q}$ to $M^{s-\mu}_{p,q}$;*
(4) *$(M^s_{p,q})' = M^{-s}_{p',q'}$ when $1/p + 1/p' = 1/q + 1/q' = 1$;*
(5) *$M^{s_1}_{p_1,q_1} \subseteq M^{s_2}_{p_2,q_2}$ when $s_1 \geq s_2$, $p_1 \leq p_2$, $q_1 \leq q_2$;*
(6) *$M^{s_1}_{p,q_1} \subseteq M^{s_2}_{p,q_2}$ when $q_1 > q_2$, $s_1 - s_2 > n\,(1/q_2 - 1/q_1)$;*
(7) *$H^{\mu n\theta_1(p,q),p} \subset M_{p,q} \subset H^{\mu n\theta_2(p,q),p}$ when $\mu > 1$ with*

$$\theta_1(p,q) = \max\left(0, \tfrac{1}{q} - \min\left(\tfrac{1}{p}, 1-\tfrac{1}{p}\right)\right),$$

$$\theta_2(p,q) = \min\left(0, \tfrac{1}{q} - \max\left(\tfrac{1}{p}, 1-\tfrac{1}{p}\right)\right);$$

(8) *$M_{p,1} \subset L^p \cap L^\infty$, $M_{2,2} = L^2$.*

From the definition of $M^s_{p,q}$ one easily obtains the next lemma.

Lemma 2.2. *Let $0 < R < \infty$ be a constant, and set $L^p_R := \{f \in L^p : \hat{h}(\xi) = 0, |\xi| > R\}$, $M^s_{p,q,R} := \{f \in M^s_{p,q} : \hat{h}(\xi) = 0, |\xi| > R\}$. Then $L^p_R = M^s_{p,q,R}$ with equivalent norm.*

The following lemmas play an essential role in the proof of Theorem 2.

Lemma 2.3 (Fourier multiplier in modulation spaces, [3, 5])**.** *Let $\mu \in C^k(\mathbb{R}^n)$, $k = [n/2] + 1$, $p, q \in [1, \infty]$; then it follows that*

$$\left\|\mathcal{F}^{-1}(\mu\hat{f})\right\|_{M_{p,q}} \leq C_{p,q} \sum_{|\alpha|\leq k} \left\|\partial^\alpha_\xi \mu\right\|_{L^\infty} \|f\|_{M_{p,q}} \quad \forall g \in M_{p,q}.$$

The next lemma gives an estimate of nonlinear terms in the modulation space.

Lemma 2.4 ([2]). *Let k be a positive integer, and let $p, p_1, \cdots, p_k \in [1,\infty]$ be such that $1/p = 1/p_1 + \cdots + 1/p_k$. Assume that $f_j \in M_{p_j,1}$ for $j = 1, \cdots, k$. Then $f_1 \cdot \cdots \cdot f_k \in M_{p,1}$, and they satisfy*

$$\|f_1 \cdot \cdots \cdot f_k\|_{M_{p,1}} \le C \|f_1\|_{M_{p_1,1}} \cdot \cdots \cdot \|f_k\|_{M_{p_k,1}}.$$

Now we show estimates of solutions to the following linear damped wave equation:

$$\partial_t^2 U - \Delta U + \partial_t U = 0, \quad U(0,x) = U_0(x), \quad \partial_t U(0,x) = U_1(x) \tag{2.1}$$

$t > 0$, $x \in \mathbb{R}^n$. Introduce the operators $S_0(t)$ and $S_1(t)$ defined by

$$S_0(t)f(x) = \mathcal{F}^{-1}\left(e^{-t/2}\cos t\sqrt{|\xi|^2 - 1/4}\hat{f}(\xi)\right)(x), \quad t > 0,$$

$$S_1(t)f(x) = \mathcal{F}^{-1}\left(e^{-t/2}\frac{\sin t\sqrt{|\xi|^2 - 1/4}}{\sqrt{|\xi|^2 - 1/4}}\hat{f}(\xi)\right)(x), \quad t > 0.$$

Let $U_0, U_1 \in \mathcal{S}'$, and let $U(t,x)$ be the solution of (2.1) in the distribution sense, then it follows that

$$U(t,x) = S_0(t)U_0 + S_1(t)\left(\tfrac{U_0}{2} + U_1\right).$$

The following lemmas give estimates for the solutions of (2.1).

Lemma 2.5 (Estimates of weighted norm of low frequency part. See [14, 15]). *Let $1 \le p_1 \le p_2 \le \infty$, $j = 0, 1$, $a > 0$ and m be a nonnegative integer. Assume that*

$$\langle x\rangle^m f \in L^{p_1}, \qquad \hat{f}(\xi) = 0 \quad (|\xi| \ge a).$$

Then, for any integer $k \ge 0$ and any index of non-negative integers α,

$$\langle x\rangle^m \partial_t^k \partial_x^\alpha S_j(t) f \in L^{p_2}, \quad \forall t > 0,$$

$$\begin{aligned}&\left\|\langle x\rangle^\mu \partial_t^k \partial_x^\alpha S_j(t) f\right\|_{L^{p_2}} \\ &\le C(1+t)^{-(n/2)(1/p_1 - 1/p_2) - k - |\alpha|/2}\left((1+t)^\mu \|f\|_{L^{p_1}} + \|\langle x\rangle^\mu f\|_{L^{q_1}}\right) \quad \forall t > 0,\end{aligned}$$

for any $j = 0, 1$ and $\mu \in [0, m]$.

From Lemmas 2.2, 2.5 one obtains estimates of the low frequency part of $U(t,x)$ in the modulation space.

Corollary 2.1 (Estimates of low frequency part in the modulation space). *Let $j = 0, 1$, $1 \le p_1 \le p_2 \le \infty$ and $a > 0$. Assume that $f \in M_{p_1,1}$,*

$\hat{f}(\xi) = 0 \quad (|\xi| \geq a)$. *Then, for any integer* $k \geq 0$ *and any index of non-negative integers* α,

$$\partial_t^k \partial_x^\alpha S_j(t) f \in M_{p_2,1} \quad \forall t \geq 0,$$

and

$$\left\| \partial_t^k \partial_x^\alpha S_j(t) f \right\|_{M_{p_2,1}} \leq C(1+t)^{-(n/2)(1/p_1-1/p_2)-k-|\alpha|/2} \|f\|_{M_{p_1,1}} \quad \forall t \geq 0.$$

The next lemma gives estimates of the high frequency part to $U(t,x)$ in weighted Sobolev spaces.

Lemma 2.6 (Estimates of weighted norm of the high frequency part. See [14, 15])**.** *Let* $a > 0$, $m \geq 0$ *be an integer, and let* $U(t,x)$ *be the solution of* (2.1). *Assume that*

$$\langle x \rangle^m U_0 \in H^{2,2}, \quad \langle x \rangle^m U_1 \in H^{1,2},$$

and

$$\hat{U}_0(\xi) = \hat{U}_1(\xi) = 0 \quad (|\xi| \leq a).$$

Then, there exists a positive constant $\delta = \delta(a)$ *such that for* $k = 0, 1$, *estimates*

$$\|\langle x \rangle^m U(t,\cdot)\|_{H^{k+1,2}} \leq Ce^{-\delta t} \left(\|\langle x \rangle^m U_0\|_{H^{k+1,2}} + \|\langle x \rangle^m U_1\|_{H^{k,2}} \right) \quad \forall t \geq 0$$

hold.

Remark 2.1. Let $\mu \geq 0$, $1 \leq p_1 \leq p_2 \leq \infty$, and let $\chi_1(\xi) \in C^\infty$ be a cut-off function such that $\chi_1(\xi) = 1$ when $|\xi| \leq 1$, $\chi_1(\xi) = 0$ when $|\xi| \geq 2$. Assume that $f \in L^{p_1}$ such that $\langle x \rangle^\mu f \in L^{p_1}$ and $\hat{f}(\xi) = 0$ when $|\xi| \geq a$. Then it follows that $\hat{f}(\xi) = \chi_1(\xi)\hat{f}(\xi)$ and

$$\partial_x^\alpha f(x) = c \int_{R^n} f(x-y) \partial_y^\alpha \left(\mathcal{F}^{-1} \chi_1 \right)(y)\, dy.$$

Since

$$\langle x \rangle^\mu \leq C \left(\langle x-y \rangle^\mu + \langle y \rangle^\mu \right), \quad \langle x \rangle^\mu \partial_x^\alpha \mathcal{F}^{-1} \chi \in L^1,$$

Young's inequality shows that

$$\|\langle x \rangle^\mu \partial_x^\alpha f\|_{L^{p_2}} \leq C \|\langle x \rangle^\mu f\|_{L^{p_1}}.$$

From Lemma 2.3 one obtains the next lemma.

Lemma 2.7 (Estimates of the high frequency part in the modulation space)**.** *Let* $p, q \in [1, \infty]$, $s \in \mathbb{R}$, k *be a non-negative integer. Assume that* $f \in M^s_{p,q}$ *and* $\hat{f}(\xi) = 0$ *when* $|\xi| \le 2$. *Then, the following estimate holds for any* $\delta < 1/2$*:*

$$\left\| \partial_t^k S_0(t) f \right\|_{M^s_{p,q}} + \left\| \partial_t^k S_1(t) f \right\|_{M^{s+1}_{p,q}} \le C_{s,k,\delta} e^{-\delta t} \left\| f \right\|_{M^{s+k}_{p,q}}, \quad \forall t \in [0, \infty).$$

Proof. Let $\chi_1(\xi)$ be the cut-off function in Remark 2.1, and let $\chi_2(\xi) = 1 - \chi_1(\xi)$. Then easy calculation shows that

$$\sup_{\xi, |\alpha| \le [n/2]+1} \left| \partial_t^k \partial_\xi^\alpha \left(e^{-t/2} (1 + |\xi|^2)^{-k/2} \chi_2(\xi) \cos t \sqrt{|\xi|^2 - 1/4} \right) \right| \le C_{k,\alpha,\delta} e^{-\delta t},$$

$$\sup_{\xi, |\alpha| \le [n/2]+1} \left| \partial_t^k \partial_\xi^\alpha \left(e^{-t/2} (1 + |\xi|^2)^{(1-k)/2} \chi_2(\xi) \frac{\sin t \sqrt{|\xi|^2 - 1/4}}{\sqrt{|\xi|^2 - 1/4}} \right) \right| \le C_{k,\alpha,\delta} e^{-\delta t}$$

for $t \ge 0$ and any constant $\delta < 1/2$. Hence Lemma 2.3 shows that

$$\begin{aligned} &\left\| \partial_t^k S_0(t) (1 - \Delta)^{-k/2} f \right\|_{M_{p,q}} + \left\| \partial_t^k S_1(t) (1 - \Delta)^{(1-k)/2} f \right\|_{M_{p,q}} \\ &\le C_{k,\delta,q} e^{-\delta t} \left\| f \right\|_{M_{p,q}}, \end{aligned}$$

provided that $f \in M_{p,q}$ and $\hat{f}(\xi) = 0$ ($|\xi| \le 2$). Hence, from Lemma 2.1 (3), one obtains the desired estimate:

$$\begin{aligned} &\left\| \partial_t^k S_0(t) f \right\|_{M^s_{p,q}} + \left\| \partial_t^k S_1(t) f \right\|_{M^{s+1}_{p,q}} \\ &\le C \left(\left\| \partial_t^k (1 - \Delta)^{-k/2} S_0(t) (1 - \Delta)^{(k+s)/2} f \right\|_{M_{p,q}} \right. \\ &\qquad \left. + \left\| \partial_t^k (1 - \Delta)^{(1-k)/2} S_1(t) (1 - \Delta)^{(k+s)/2} f \right\|_{M_{p,q}} \right) \\ &\le C_{k,\delta} e^{-\delta t} \left\| (1 - \Delta)^{(k+s)/2} f \right\|_{M_{p,q}} \le C_{k,\delta,q} e^{-\delta t} \left\| f \right\|_{M^{k+s}_{p,q}}, \quad \forall t \ge 0 \end{aligned}$$

for any $\delta \in (0, 1/2)$. □

Let $f \in M^s_{p,q}$ ($p, q \in [1, \infty], s \in \mathbb{R}$), χ_1 and χ_2 be the cut-off functions in Lemma 2.7. Then it follows that $\hat{f} = \mathcal{F}^{-1} \chi_1 \hat{f} + \mathcal{F}^{-1} \chi_2 \hat{f}$. Moreover, from Lemma 2.2 one sees that

$$\mathcal{F}^{-1} \chi_1 \hat{f},\ \mathcal{F}^{-1} \chi_2 \hat{f} \in M^s_{p,q}, \quad \left\| F^{-1} \chi_1 \hat{f} \right\|_{M^s_{p,q}} + \left\| F^{-1} \chi_2 \hat{f} \right\|_{M^s_{p,q}} \le C \left\| f \right\|_{M^s_{p,q}}.$$

From Corollary 2.1 and Lemma 2.7 one obtains the next lemma.

Lemma 2.8 (Estimates in modulation spaces)**.** *Let $1 \le p_1 \le p_2 \le \infty$, k and m be non-negative integers. Then, for any $f \in M^{k+m}_{p_1,q}$, it follows that*

$$\left\|\partial_t^k S_j(t) f\right\|_{M^{m+j}_{p_2,q}} \le C(1+t)^{-(n/2)(1/p_1-1/p_2)-k} \|f\|_{M^{k+m}_{p_1,q}} \quad \forall t \in [0,\infty), j = 0, 1.$$

We need the next elementary lemma.

Lemma 2.9. *Let μ, μ_1, μ_2 and δ be constants. Then it follows that*

(1) *if $\mu_1 < 1$, then*

$$\int_{t/2}^{t} (1+t-\tau)^{-\mu_1}(1+\tau)^{-\mu_2}\, d\tau \le C(1+t)^{1-\mu_1-\mu_2}, \quad \forall t > 0,$$

(2) *if $\mu_2 < 1$, then*

$$\int_{0}^{t/2} (1+t-\tau)^{-\mu_1}(1+\tau)^{-\mu_2}\, d\tau \le C(1+t)^{1-\mu_1-\mu_2}, \quad \forall t > 0,$$

(3) *if $\delta > 0$, then*

$$\int_0^t e^{-\delta(t-\tau)}(1+\tau)^{-\mu}\, d\tau \le C(1+t)^{-\mu}, \quad \forall t > 0.$$

3. Proof of Theorem 1

Theorem 1 is a direct consequence of Proposition 2 below. One needs several notations to state Proposition 2. Let χ_1 be the cut-off function in Remark 2.1; $\chi_1 \in C^\infty(\mathbb{R}^n)$ is a radial function such that $\chi_1(\xi) = 1$ when $|\xi| \le 1$, and $\chi_1(\xi) = 0$ when $|\xi| \ge 2$. For $g \in \mathcal{S}'(\mathbb{R}^n)$ define $g^{(l)}, g^{(h)} \in \mathcal{S}'(\mathbb{R}^n)$ by

$$g^{(l)}(x) = \mathcal{F}^{-1}\left(\chi_1(\xi)\hat{g}(\xi)\right)(x), \qquad g^{(h)}(x) = g(x) - g^{(l)}(x).$$

Let $p_0, p_1 \in [1,\infty]$ be such that

$$p_0 = \begin{cases} 2n/(n-6) - \delta_0, & (n > 6), \\ 1/\delta_0, & (n = 6), \\ \infty, & (n = 4, 5), \end{cases} \tag{3.1}$$

$$p_1 = \begin{cases} 2n/(n-4) - \delta_0, & (n > 4), \\ 1/\delta_0, & (n = 4), \end{cases} \tag{3.2}$$

where δ_0 is a sufficiently small positive constant.

Let $X = \prod_1^N X_j$ be a Banach space equipped with the norm

$$\|(u_j)_{1\le j\le N}\|_X = \sum_{j=1}^N \|u_j\|_{X_j},$$

where X_j is a Banach space defined by

$$X_j = \Big\{ u_j \in C\left([0,\infty); H^{2,2}\right) : \langle x\rangle^m u_j^{(l)} \in C\left([0,\infty); L^1 \cap L^{p_0}\right), \\ \langle x\rangle^m u_j^{(h)} \in C\left([0,\infty); H^{2,2}\right), \|u_j\|_{X_j} < \infty \Big\}, \tag{3.3}$$

equipped with the norm

$$\begin{aligned} \|u_j\|_{X_j} := & \sup_{0<t, 0\le\mu\le m, 1\le p\le p_0} (1+t)^{-n/2p+\nu_j-\mu/2} \left\| \langle x\rangle^\mu u_j^{(l)}(t,\cdot) \right\|_{L^p} \\ & + \sup_{0<t,0\le\mu\le m,1\le p\le p_1} (1+t)^{-n/2p+\nu_j-\mu/2+1/2} \left\| \langle x\rangle^\mu \nabla u_j^{(l)}(t,\cdot) \right\|_{L^p} \\ & + \sup_{0<t,0\le\mu\le m,|\alpha|\le 2} (1+t)^{-n/4+\nu_j-\mu/2+3/2} \left\| \langle x\rangle^\mu \partial_x^\alpha u_j^{(h)}(t,\cdot) \right\|_{L^2} \end{aligned} \tag{3.4}$$

for $j \in \{1, \cdots, N\}$, where m is the integer in Theorem 1.

Proposition 2. *Under the same notation as in Theorem* 1, *there exists a positive constant ϵ such that if the initial data satisfies*

$$(\langle x\rangle^m \varphi_j, \langle x\rangle^m \psi_j) \in \left(L^1 \cap H^{2,2}\right) \times \left(L^1 \cap H^{1,2}\right) \quad \forall j \in \{1, \cdots, N\},$$

and $I \le \epsilon$, then Problem (1.1)–(1.2) *admits a unique global solution $u = (u_j)_{1\le j\le N} \in X$, and it satisfies $\|u\|_X \le CI$.*

Now we prove Proposition 2 in this section. Let $Y := \prod_{j=1}^N Y_j$ be a Banach space with norm $\|(f_j)_{1\le j\le N}\|_Y = \sum_{j=1}^N \|f_j\|_{Y_j}$, where Y_j is a Banach space defined by $Y_j = \{f_j \in C\left([0,\infty); L^1 \cap H^{1,2}\right) : \langle x\rangle^m f_j \in C([0,\infty); L^1 \cap H^{1,2}),$ $\|f_j\|_{Y_j} < \infty\}$ equipped with the norm

$$\begin{aligned} \|f_j\|_{Y_j} = & \sup_{t\ge 0, 1\le p\le 2, 0\le\mu\le m} (1+t)^{-n/2p+\nu_j-\mu/2+1} \|\langle x\rangle^\mu f_j(t,\cdot)\|_{L^p} \\ & + \sup_{t\ge 0,\mu\in[0,m]} (1+t)^{-n/4+\nu_j-\mu/2+3/2} \|\langle x\rangle^\mu \nabla f_j(t,\cdot)\|_{L^2} \end{aligned}$$

for $j \in \{1, \cdots, N\}$.

Remark 3.1. Assume that the estimates

$$\left\| \langle x\rangle^\mu \partial_x^\alpha v^{(h)}(t,x) \right\|_{L^2} \le C(1+t)^{n/4-\nu_j+\mu/2-1-|\alpha|/2}, \qquad t \ge 0,$$

hold for any $\mu \in [0,m]$ and α with $|\alpha| \le 1$. Since

$$\mathcal{F}\left(\langle x\rangle^\mu v^{(h)}(t,\cdot)\right)(\xi) = 0, \quad |\xi| \le 1, \quad \mu = 0, m,$$

it follows that

$$\begin{aligned}\left\|\langle x\rangle^{\mu} v^{(h)}(t,\cdot)\right\|_{L^2} &\le C\left\|\nabla\left(\langle x\rangle^{\mu} v^{(h)}(t,\cdot)\right)\right\|_{L^2}\\ &\le C\left(\mu\left\|\langle x\rangle^{\mu-1} v^{(h)}(t,\cdot)\right\|_{L^2}+\left\|\langle x\rangle^{\mu}\nabla v^{(h)}(t,\cdot)\right\|_{L^2}\right)\\ &\le C(1+t)^{n/4-\nu_j+\mu/2-3/2}, \qquad t\ge 0,\end{aligned}$$

for $\mu = 0, m$. Therefore, the Hölder inequality shows that

$$\left\|\langle x\rangle^{\mu} v^{(h)}(t,\cdot)\right\|_{L^2} \le C(1+t)^{n/4-\nu_j+\mu/2-3/2}, \quad t\ge 0,$$

for $\mu\in[0,m]$.

Let $\hat{X} = \prod_{j=1}^{N}\hat{X}_j$ and $\hat{Y} = \prod_{j=1}^{N}\hat{Y}_j$ be auxiliary Banach spaces with norms

$$\|(u_j)_{1\le j\le N}\|_{\hat{X}} = \sum_{j=1}^{N}\|u_j\|_{\hat{X}_j}, \quad \|(f_j)_{1\le j\le N}\|_{\hat{Y}} = \sum_{j=1}^{N}\|f_j\|_{\hat{Y}_j},$$

respectively, where $\hat{X}_j$ and $\hat{Y}_j$ are Banach spaces defined by

$$\begin{aligned}\hat{X}_j = \Big\{u_j\in C([0,\infty);H^{1,2}) : \langle x\rangle^{m} u_j^{(l)} \in C([0,\infty);L^1\cap L^{p_1}),\\ \langle x\rangle^{m} u_j^{(h)} \in C([0,\infty);H^{1,2}), \|u_j\|_{\hat{X}_1}<\infty\Big\},\end{aligned}$$

$$\hat{Y}_j = \{f_j\in C\left([0,\infty);L^1\cap L^2\right) : \langle x\rangle^{m} f_j\in C\left([0,\infty);L^1\cap L^2\right), \|f_j\|_{\hat{Y}_j}<\infty\},$$

equipped with norms

$$\begin{aligned}\|u_j\|_{\hat{X}_j} = &\sup_{0<t,0\le\mu\le m,1\le p\le p_1}(1+t)^{-n/2p+\nu_j-\mu/2}\left\|\langle x\rangle^{\mu} u_j^{(l)}(t,\cdot)\right\|_{L^p}\\ &+\sup_{0<t,0\le\mu\le m,|\beta|\le 1}(1+t)^{-n/4+\nu_j-\mu/2+1}\left\|\langle x\rangle^{\mu}\partial_x^{\beta} u_j^{(h)}(t,\cdot)\right\|_{L^2},\end{aligned}$$

and

$$\|f_j\|_{\hat{Y}_j} = \sup_{t\ge 0,1\le p\le 2,\mu\in[0,m]}(1+t)^{-n/2p+\nu_j-\mu/2+1}\left\|\langle x\rangle^{\mu} f_j(t,\cdot)\right\|_{L^p},$$

respectively, for $j = 1,\cdots,N$. One sees that $X\subset\hat{X}$ and $Y\subset\hat{Y}$ with continuous injection.

Then one obtains the following estimates.

Lemma 3.1. *Let* $u = (u_j)_{1\le j\le N}$, $v = (v_j)_{1\le j\le N} \in X$. *Under the same notation as in Theorem 1 we set*

$$F(u) = (F_j(u))_{1\le j\le N}, \quad F(v) = (F_j(v))_{1\le j\le N}.$$

Then $F(u) \in \hat{Y}$ *and* $F(v) \in \hat{Y}$, *and they satisfy*

$$\|F_j(u) - F_j(v)\|_{\hat{Y}_j} \le C\left(\|u\|_X^{\sigma_j-1} + \|v\|_X^{\sigma_j-1}\right)\|u-v\|_{\hat{X}}.$$

Proof. When $0 \le \nu \le m$, Sobolev's embedding theorem shows that

$$\|\langle x\rangle^\nu w_j\|_{L^{r_1}} \le \sum_{\rho=l,h}\left\|\langle x\rangle^\nu w_j^{(\rho)}\right\|_{L^{r_1}} \le C(1+t)^{n/2r_1-\nu_j+\nu/2}\|w_j\|_{X_j} \tag{3.5}$$

for $1/r_1 \in (1/2-2/n, 1/2)$, and

$$\|\langle x\rangle^\nu w_j\|_{L^{r_2}} \le \sum_{\rho=l,h}\left\|\langle x\rangle^\nu w_j^{(\rho)}\right\|_{L^{r_2}} \le C(1+t)^{n/2r_2-\nu_j+\nu/2}\|w_j\|_{\hat{X}_j} \tag{3.6}$$

for $1/r_2 \in (1/2-1/n, 1/2)$. Consider the case where $\sigma_{j,1} \ne 0$ ($\sigma_{j,1} \ge 1$). Since $\frac{\sigma_j-1}{2} + \frac{1}{2} > \frac{1}{2} \ge \left(\frac{1}{2}-\frac{2}{n}\right)(\sigma_j-1) + \left(\frac{1}{2}-\frac{1}{n}\right)$, there exists $r_1 > 0$, $r_2 > 0$ such that $1/r_1 \in (1/2-2/n, 1/2)$, $1/r_1 \in (1/2-1/n, 1/2)$ and $\frac{\sigma_j-1}{r_1} + \frac{1}{r_2} = \frac{1}{2}$. From (3.5)–(3.6) and Hölder's inequality one obtains

$$\begin{aligned}
&\left\|\langle x\rangle^\mu |u_1(t)|^{\sigma_{j,1}-1}|u_1(t)-v_1(t)||u_2(t)|^{\sigma_{j,2}}\cdots|u_N(t)|^{\sigma_{j,N}}\right\|_{L^2} \\
&\quad\le C\left\|P(x)^{\mu/\sigma_j}u_1\right\|_{L^{r_1}}^{\sigma_{j,1}-1}\left\|\langle x\rangle^{\mu/\sigma_j}(u_1-v_1)\right\|_{L^{r_2}}\prod_{k=2}^N\left\|\langle x\rangle^{\mu/\sigma_j}u_k(t)\right\|_{L^{r_1}}^{\sigma_{j,k}} \\
&\quad\le C(1+t)^{n/4-\nu_j+\mu/2-1}\|u\|_X^{\sigma_j-1}\|u-v\|_{\hat{X}}.
\end{aligned}$$

From a similar calculation to the one above one obtains that

$$\begin{aligned}
&\|\langle x\rangle^\mu(F_j(u(t,\cdot)) - F_j(v(t,\cdot)))\|_{L^2} \\
&\qquad\le C(1+t)^{n/4-\nu_j+\mu/2-1}\left(\|u\|_X^{\sigma_j-1} + \|v\|_X^{\sigma_j-1}\right)\|u-v\|_{\hat{X}}.
\end{aligned} \tag{3.7}$$

When $\sigma_j \ge 2$, a similar argument to the one used to obtain (3.7) gives

$$\begin{aligned}
&\|\langle x\rangle^\mu(F_j(u(t,\cdot)) - F_j(v(t,\cdot)))\|_{L^1} \\
&\qquad\le C(1+t)^{n/2-\nu_j+\mu/2-1}\left(\|u\|_X^{\sigma_j-1} + \|v\|_X^{\sigma_j-1}\right)\|u-v\|_{\hat{X}}.
\end{aligned} \tag{3.8}$$

Let us consider the case where $\sigma_j < 2$. Since $\sigma_{j,k} \ge 1$ when $\sigma_{j,k} \ne 0$, there exists $k \in \{1,\cdots,N\}$ such that $|F_j(u)| = |u_k|^{\sigma_j}$ and $\sigma_j\nu_k = \nu_j + 1$. Since there exists $r > 0$ such that $1/r \in (1/2, 1)$, $(\sigma_1 - 1)/2 + 1/r = 1$, it follows that

$$\begin{aligned} &\left\| \langle x \rangle^{\mu} \left(|u_k|^{\sigma_j - 1} + |v_k|^{\sigma_j - 1} \right) (u_k^{(l)} - v_k^{(l)}) \right\|_{L^1} \\ &\leq \left(\|u_k\|_{L^2}^{\sigma_j - 1} + \|v_k\|_{L^2}^{\sigma_j - 1} \right) \left\| \langle x \rangle^{\mu} (u_k^{(l)} - v_k^{(l)}) \right\|_{L^r} \\ &\leq C(1+t)^{n/2 - \nu_j + \mu/2 - 1} \left(\|u\|_X^{\sigma_j - 1} + \|v\|_X^{\sigma_j - 1} \right) \|u - v\|_{\hat{X}} . \end{aligned} \tag{3.9}$$

Let us choose a positive constant s such that

$$\frac{n(2-\sigma_j)}{2} < s \leq \frac{n(2-\sigma_j)}{2} + 2, \quad s + m \leq m\sigma_j,$$

then

$$\frac{2-\sigma_j}{2} + \frac{\sigma_j - 1}{2} + \frac{1}{2} = 1, \quad \frac{s+\mu}{\sigma_j} \leq m, \quad \langle x \rangle^{-s} \in L^{2/(2-\sigma_j)}.$$

Hence

$$\begin{aligned} &\left\| \langle x \rangle^{\mu} |u_k|^{\sigma_j - 1} \left(u_k^{(h)} - v_k^{(h)} \right) \right\|_{L^1} \\ = \quad &\left\| \langle x \rangle^{-s} \left| \langle x \rangle^{(s+\mu)/\sigma_j} u_k \right|^{\sigma_j - 1} \langle x \rangle^{(s+\mu)/\sigma_j} \left(u_k^{(h)} - v_k^{(h)} \right) \right\|_{L^1} \\ \leq \quad &\left\| \langle x \rangle^{-s} \right\|_{L^{2/(2-\sigma_j)}} \left\| \langle x \rangle^{(s+\mu)/\sigma_j} u_k \right\|_{L^2}^{\sigma_j - 1} \left\| \langle x \rangle^{(s+\mu)/\sigma_j} (u_k^{(h)} - v_k^{(h)}) \right\|_{L^2} \\ \leq \quad &C(1+t)^{n/2 - \nu_j + \mu/2 - 1} \|u\|_X^{\sigma_j - 1} \|u - v\|_{\hat{X}} ; \end{aligned} \tag{3.10}$$

one also obtains

$$\left\| \langle x \rangle^{\mu} |v_k|^{\sigma_j - 1} \left(u_k^{(h)} - v_k^{(h)} \right) \right\|_{L^1} \leq C(1+t)^{n/2 - \nu_j + \mu/2 - 1} \|v\|_X^{\sigma_j - 1} \|u - v\|_{\hat{X}} . \tag{3.11}$$

Since

$$|F_j(u) - F_j(v)| \leq C \sum_{\rho = l,h} \left| \langle x \rangle^{\mu} \left(|u_k|^{\sigma_j - 1} + |v_k|^{\sigma_j - 1} \right) (u_k^{(\rho)} - v_k^{(\rho)}) \right|,$$

(3.9)–(3.11) show that estimate (3.8) also holds when $\sigma_j < 2$. From (3.7) and (3.8) one obtains the desired estimate

$$\begin{aligned} &\| \langle x \rangle^{\mu} \left(F_j(u(t,\cdot)) - F_j(v(t,\cdot)) \right) \|_{L^p} \\ &\leq C(1+t)^{n/2p - \nu_j + \mu/2 - 1} \left(\|u\|_X^{\sigma_j - 1} + \|v\|_X^{\sigma_j - 1} \right) \|u - v\|_{\hat{X}} \end{aligned}$$

for any $p \in [1, 2]$, $\mu \in [0, m]$. □

Lemma 3.2. *Let $u \in X$. Under the notation in Lemma 3.1, $F(u) = (F_j(u))_{1\le j\le N} \in Y$, and it satisfies*

$$\|F_j(u)\|_{Y_j} \le C\,\|u\|_X^{\sigma_j} \quad (1 \le j \le N).$$

Proof. Let r_1, r_2 be positive constants such that

$$\frac{1}{2}-\frac{2}{n} \le \frac{1}{r_1} \le \frac{1}{2}, \quad \frac{1}{2}-\frac{1}{n} \le \frac{1}{r_2} \le \frac{1}{2}, \quad \frac{\sigma_j-1}{r_1}+\frac{1}{r_2}=\frac{1}{2}.$$

Since $H^{2,2} \subset L^{r_1}$, $H^{1,2} \subset L^{r_2}$, it follows that

$$\|u_k\|_{L^{r_1}} \le \left\|u_k^{(l)}\right\|_{L^{r_1}} + \left\|u_k^{(h)}\right\|_{L^{r_1}} \le C(1+t)^{n/2r_1-\nu_k}\,\|u\|_X\,,$$

for $k = 1, \cdots, N$, and when $\mu = 0, m$, $k = 1, \cdots, N$, it follows that

$$\begin{aligned}\|\langle x\rangle^\mu \nabla u_k\|_{L^{r_2}} &\le \left\|\langle x\rangle^\mu \nabla u_k^{(l)}\right\|_{L^{r_2}} + \left\|\langle x\rangle^\mu \nabla u_k^{(h)}\right\|_{L^{r_2}} \\ &\le \left\|\langle x\rangle^\mu \nabla u_k^{(l)}\right\|_{L^{r_2}} + C\left\|\langle x\rangle^\mu \nabla u_k^{(h)}\right\|_{H^{2,2}} \le C(1+t)^{n/2r_2-\nu_k-1/2}\,\|u\|_X\,.\end{aligned}$$

Hence one obtains

$$\begin{aligned}&\|\langle x\rangle^\mu \nabla F_j(u)\|_{L^2} \\ &\le C \sum_{1\le k\le N,\sigma_{j,k}\ne 0} \left\|\langle x\rangle^\mu |u_1|^{\sigma_{j,1}} \cdot \cdots \cdot |u_k|^{\sigma_{j,k}-1} \cdot \cdots \cdot |u_N|^{\sigma_{j,N}} \nabla u_k\right\|_{L^2} \\ &\le C \sum_{1\le k\le N,\sigma_{j,k}\ne 0} \|u_1\|_{L^{r_1}}^{\sigma_{j,1}} \cdot \cdots \cdot \|u_k\|_{L^{r_1}}^{\sigma_{j,k}-1} \cdot \cdots \cdot \|u_N\|_{L^{r_1}}^{\sigma_{j,N}}\,\|\langle x\rangle^\mu \nabla u_k\|_{L^{r_2}} \\ &\le C(1+t)^{n/4-\nu_j+\mu/2-3/2}\,\|u\|_X^{\sigma_j} \quad (\mu = 0, m).\end{aligned} \tag{3.12}$$

When $0 \le \mu \le m$, one obtains the desired estimate

$$\begin{aligned}\|\langle x\rangle^\mu \nabla F_j(u)\|_{L^2} &\le \|\nabla F_j(u)\|_{L^2}^{1-\mu/m}\,\|\langle x\rangle^m \nabla F_j(u)\|_{L^2}^{\mu/m} \\ &\le C(1+t)^{n/4-\nu_j+\mu/2-3/2}\,\|u\|_X^{\sigma_j}\,.\end{aligned}$$

From the above estimate and Lemma 3.1 with $v = 0$, one obtains the desired estimate. □

Problem (1.1)–(1.2) is equivalent to the following system of integral equations:

$$u_j(t,x) = U_j(t,x) + \int_0^t S_1(t-\tau) f_j(u(\tau,\cdot))\,d\tau, \tag{3.13}$$

where

$$U_j(t,x) = S_0(t)\varphi_j + S_1(t)\left(\tfrac{1}{2}\varphi_j + \psi_j\right), \tag{3.14}$$

for $j = 1, \cdots, N$. From Lemmas 2.5–2.6 one obtains the following lemmas.

Lemma 3.3. *Under the notation in Theorem* 1, $U = (U_j)_{1\le j\le N}$ *defined by* (3.14) *belongs to* X, *and it satisfies* $\|U\|_X \le CI$.

Proof. Since

$$\langle x\rangle^m f^{(l)} = c\mathcal{F}^{-1}\left((1-\Delta)^{m/2}\chi_1(\xi)\hat{f}\right), \quad \mathcal{F}^{-1}\left(\chi_1(\xi)\right)\in\mathcal{S},$$

it follows that

$$\left\|\langle x\rangle^m f^{(l)}\right\|_{L^1} \le C\left\|\langle x\rangle^m f\right\|_{L^1}, \quad \left\|\langle x\rangle^m f^{(l)}\right\|_{H^{k,2}} \le C\left\|\langle x\rangle^m f\right\|_{H^{k,2}}$$

for any nonnegative integer k. Hence Lemmas 2.5–2.6 and (1.9) show that $U_j \in C([0,\infty); H^{2,2})$, $\langle x\rangle^m U_j^{(l)} \in C([0,\infty); L^1\cap L^{p_0})$, $\langle x\rangle^m U_j^{(l)} \in C([0,\infty); H^{2,2})$, and

$$\begin{aligned}
&\sup_{0<t,0\le\mu\le m,1\le p\le p_0,|\alpha|\le 1}(1+t)^{-n/2p+\nu_j-\mu/2+|\alpha|/2}\left\|\langle x\rangle^\mu U_j^{(l)}(t,\cdot)\right\|_{L^p}\\
&\le C\sup_{0<t}(1+t)^{\nu_j-n/2}\left(\left\|\langle x\rangle^m\varphi_j^{(l)}\right\|_{L^1}+\left\|\langle x\rangle^m\varphi_j^{(l)}\right\|_{L^1}\right)\\
&\le C\left(\|\langle x\rangle^m\varphi_j\|_{L^1}\right)\le CI,
\end{aligned}$$

$$\begin{aligned}
&\sup_{0<t,0\le\mu\le m,1\le p\le p_0,|\alpha|\le 1}(1+t)^{-n/2p+\nu_j-\mu/2+|\alpha|/2}\left\|\langle x\rangle^\mu\left(S_0(t)\varphi_j\right)^{(l)}\right\|_{L^p}\\
&\le C\sup_{0<t}(1+t)^{\nu_j-n/2}\left\|\langle x\rangle^m\varphi_j^{(l)}\right\|_{L^1}\le C\|\langle x\rangle^m\varphi_j\|_{L^1}\le CI,
\end{aligned}$$

$$\begin{aligned}
&\sup_{0<t,0\le\mu\le m}(1+t)^{-n/4+\nu_j-\mu/2+3/2}\left\|\langle x\rangle^\mu U_j^{(h)}(t)\right\|_{H^{2,2}}\\
&\le C\left(\left\|\langle x\rangle^m\varphi_j^{(h)}\right\|_{H^{2,2}}+\left\|\langle x\rangle^m\psi_j^{(h)}\right\|_{H^{1,2}}\right)\\
&\le C\left(\|\langle x\rangle^m\varphi_j\|_{H^{2,2}}+\left\|\langle x\rangle^m\varphi_j^{(l)}\right\|_{H^{2,2}}\right)\le CI
\end{aligned}$$

for any $j = 1,\cdots < N$. From the above estimate one obtains the desired estimates $||U_j||_{X_j} \le CI$ $(j = 1,\cdots,N)$ and $||U||_X \le CI$. □

Lemma 3.4. *Let* G *be the operator defined by*

$$Gg = \int_0^t S_1(t-\tau)g(\tau,\cdot)\,d\tau, \qquad g\in\bigcup_{j=1}^N \hat{Y}_j.$$

Then $G \in \bigcap_{j=1}^N\left(\mathcal{L}(Y_j,X_j)\cap\mathcal{L}(\hat{Y}_j,\hat{X}_j)\right)$.

Proof. Let $p \in [1, p_0]$, $\mu \in \{0, m\}$, $g \in Y_j$ and $j \in \{1, \cdots, N\}$. One sees that

$$\begin{aligned}
&\left\| \langle x\rangle^{\mu} (Gg)^{(l)} (t, \cdot) \right\|_{L^p} \\
&\le \left\| \langle x\rangle^{\mu} \int_0^{t/2} S_1(t-\tau) g^{(l)}(\tau, \cdot)\, d\tau \right\|_{L^p} + \left\| \langle x\rangle^{\mu} \int_{t/2}^{t} S_1(t-\tau) g^{(l)}(\tau, \cdot)\, d\tau \right\|_{L^p} \\
&:= K_1(p, \mu) + K_2(p, \mu). \qquad (3.15)
\end{aligned}$$

Since $n/2 - \sigma_j - 1 > -1$ and

$$(1+t-\tau)^{\mu/2} + (1+\tau)^{\mu/2} \le 2(1+t)^{\mu/2}, \quad \tau \in [0, t],$$

Lemmas 2.5 and 2.9 show that

$$\begin{aligned}
K_1(p, \mu) &\le C \int_0^{t/2} (1+t-\tau)^{-(n/2)(1-1/q)} \qquad (3.16) \\
&\qquad \times \left\{ (1+t-\tau)^{\mu/2} \|g(\tau, \cdot)\|_{L^1} + \|\langle x\rangle^{\mu} g(\tau, \cdot)\|_{L^1} \right\} d\tau \\
&\le C(1+t)^{\mu/2} \int_0^{t/2} (1+t-\tau)^{-(n/2)(1-1/p)} (1+\tau)^{n/2-\nu_j-1}\, d\tau\, \|g\|_{Y_j} \\
&\le C(1+t)^{n/2p-\nu_j+\mu/2} \|g\|_{Y_j}, \qquad (3.17)
\end{aligned}$$

and

$$\begin{aligned}
K_2(1, \mu) &\le C \int_{t/2}^{t} \left((1+t-\tau)^{\mu/2} \|g(\tau, \cdot)\|_{L^1} + \|\langle x\rangle^{\mu} g(\tau, \cdot)\|_{L^1} \right) d\tau \qquad (3.18) \\
&\le C(1+t)^{\mu/2} \int_{t/2}^{t} (1+\tau)^{n/2-\nu_j-1}\, d\tau\, \|g\|_{Y_j} \le C(1+t)^{n/2q-\nu_j+\mu/2} \|g\|_{Y_j}.
\end{aligned}$$

Let us set $r = 2n/(n-2)$. Since $|\nabla \langle x\rangle^{m}| \le C\langle x\rangle^{m-1}$, from Sobolev's embedding theorem one obtains that

$$\|\langle x\rangle^{\mu} g(\tau, \cdot)\|_{L^r} \le C \|\nabla (\langle x\rangle^{\mu} g(\tau, \cdot))\|_{L^2} \le C(1+\tau)^{n/2r-\nu_j+\mu/2-1} \|g\|_{Y_j}. \qquad (3.19)$$

Lemmas 2.5, 2.9 and (3.19) show that

$$\begin{aligned}
K_2(p_0, \mu) &\le C \int_{t/2}^{t} (1+t-\tau)^{-(n/2)(1/r-1/p_0)} \\
&\qquad \times \left((1+t-\tau)^{\mu/2} \|g(\tau, \cdot)\|_{L^r} + \|\langle x\rangle^{\mu} g(\tau, \cdot)\|_{L^r} \right) d\tau \\
&\le C(1+t)^{\mu/2} \int_{t/2}^{t} (1+t-\tau)^{-(n/2)(1/r-1/p_0)} (1+\tau)^{n/2r-\nu_j-1}\, d\tau\, \|g\|_{Y_j}
\end{aligned}$$

$$\leq C(1+t)^{n/2p_0-\nu_j+\mu/2}\left\|g\right\|_{Y_j}, \tag{3.20}$$

because $-(n/2)(1/r-1/p_0)>-1$. From Hölder's inequality and (3.18), (3.20) one obtains that

$$K_2(p,\mu)\leq K_2(1,\mu)^{\theta}K_2(p_0,\mu)^{1-\theta}\leq C(1+t)^{n/2q-\nu_j+\mu/2}\left\|g\right\|_{Y_j} \tag{3.21}$$

for $1\leq p\leq p_0$, where $\theta=(p-p_0)/p(p_0-1)\in[0,1]$. (3.15), (3.17) and (3.21) show that

$$\left\|\langle x\rangle^{\mu}\left(Gg\right)^{(l)}(t,\cdot)\right\|_{L^p}\leq C(1+t)^{n/2p-\nu_j+\mu/2}\left\|g\right\|_{Y_j}$$

for $1\leq p\leq p_0$, $\mu=0,m$. Hence it follows that

$$\begin{aligned}\left\|\langle x\rangle^{\mu}\left(Gg\right)^{(l)}(t,\cdot)\right\|_{L^p}&\leq\left\|\langle x\rangle^{m}\left(Gg\right)^{(l)}(t,\cdot)\right\|_{L^p}^{\mu/m}\left\|\left(Gg\right)^{(l)}(t,\cdot)\right\|_{L^p}^{1-\mu/m}\\&\leq C(1+t)^{n/2p-\nu_j+\mu/2}\left\|g\right\|_{Y_j}\end{aligned} \tag{3.22}$$

for $1\leq p\leq p_0$, $\mu\in[0,m]$. A similar argument to the one above gives an estimate of $\|\langle x\rangle^{\mu}\nabla(Gf)^{l}(t,\cdot)\|_p$. Let us assume that $\mu\in\{0,m\}$ at first. Lemmas 2.5 and 2.9 again show that

$$\begin{aligned}&\left\|\langle x\rangle^{\mu}\nabla\left(Gg\right)^{(l)}(t,\cdot)\right\|_{L^1}\\&\leq C\int_0^t(1+t-\tau)^{-1/2}\left((1+t-\tau)^{\mu/2}\left\|g(\tau,\cdot)\right\|_{L^1}+\left\|\langle x\rangle^{\mu}g(\tau,\cdot)\right\|_{L^1}\right)d\tau\\&\leq C(1+t)^{n/2-\nu_j+\mu/2-1/2}\left\|g\right\|_{Y_j}\quad(\mu=0,m).\end{aligned} \tag{3.23}$$

It follows that

$$\left\|\langle x\rangle^{\mu}\nabla\left(Gg\right)^{(l)}(t,\cdot)\right\|_{L^{p_1}}\leq K_3(\mu)+K_4(\mu), \tag{3.24}$$

where

$$K_3(\mu)=\left\|\int_0^{t/2}\langle x\rangle^{\mu}\partial_{x_j}S_1(t-\tau)g^{(l)}(\tau,\cdot)\,d\tau\right\|_{L^{p_1}}, \tag{3.25}$$

$$K_4(\mu)=\left\|\int_{t/2}^{t}\langle x\rangle^{\mu}\partial_{x_j}S_1(t-\tau)g^{(l)}(\tau,\cdot)\,d\tau\right\|_{L^{p_1}}. \tag{3.26}$$

Since $-(n/2)(1/2-1/p_1)-1/2>-1$, Lemmas 2.5 and 2.9 show that

$$\begin{aligned}K_3(\mu)\leq C&\int_0^{t/2}(1+t-\tau)^{-(n/2)(1-1/p_1)-1/2}\\&\times\left((1+t-\tau)^{\mu/2}\left\|g(\tau,\cdot)\right\|_{L^1}+\left\|\langle x\rangle^{\mu}g(\tau,\cdot)\right\|_{L^1}\right)d\tau\end{aligned}$$

$$\leq C\int_0^{t/2}(1+t-\tau)^{-(n/2)(1-1/p_1)-1/2}(1+\tau)^{n/2-\nu_j-1}\,d\tau(1+t)^{\mu/2}\,\|g\|_{Y_j}$$
$$\leq C(1+t)^{n/2p_1-\nu_j+\mu/2-1/2}\,\|g\|_{Y_j}\,, \tag{3.27}$$

and

$$K_4(\mu)\leq C\int_{t/2}^{t}(1+t-\tau)^{-(n/2)(1/2-1/p_1)-1/2}$$
$$\times\left((1+t-\tau)^{\mu/2}\,\|g(\tau,\cdot)\|_{L^2}+\|\langle x\rangle^\mu g(\tau,\cdot)\|_{L^2}\right)\,d\tau$$
$$\leq C\int_{t/2}^{t}(1+t-\tau)^{-(n/2)(1/2-1/p_1)-1/2}(1+\tau)^{n/4-\nu_j-1}\,d\tau(1+t)^{\mu/2}\,\|g\|_{Y_j}$$
$$\leq C(1+t)^{n/2p_1-\nu_j+\mu/2-1/2}\,\|g\|_{Y_j}\,. \tag{3.28}$$

(3.24)–(3.28) show that

$$\left\|\langle x\rangle^\mu\nabla\,(Gg)^{(l)}\,(t,\cdot)\right\|_{L^{p_1}}\leq C(1+t)^{n/2p_1-\nu_j+\mu/2-1/2}\,\|g\|_{Y_j}\quad(\mu=0,m). \tag{3.29}$$

(3.23), (3.29) and Hölder's inequality show that

$$\left\|\langle x\rangle^\mu\nabla\,(Gg)^{(l)}\,(t,\cdot)\right\|_{L^{p}}\leq C(1+t)^{n/2p-\nu_j+\mu/2-1/2}\,\|g\|_{Y_j} \tag{3.30}$$

for any $p\in[1,p_1]$ and any $\mu\in[0,m]$. Lemmas 2.5 and 2.9 show that

$$\left\|\langle x\rangle^\mu\,(Gg)^{(h)}\,(t,\cdot)\right\|_{H^{2,2}}\leq C\int_0^t e^{-\delta(t-\tau)}\left\|\langle x\rangle^\mu\nabla g^{(h)}(t,\cdot)\right\|_{L^2}\,d\tau$$
$$\leq C\int_0^t e^{-\delta(t-\tau)}(1+\tau)^{n/4-\nu_j+\mu/2-3/2}\,d\tau\,\|g\|_{Y_j}$$
$$\leq C(1+t)^{n/4-\nu_j+\mu/2-3/2}\,\|g\|_{Y_j} \tag{3.31}$$

for $\mu\in[0,m]$. (3.22) and (3.30)–(3.31) give the desired estimate

$$\|Gg\|_{X_j}\leq C\,\|g\|_{Y_j}\quad(j=1,\cdots,N).$$

By a similar argument to the one above one also obtains the following estimate:

$$\|Gg\|_{\hat{X}_j}\leq C\,\|g\|_{\hat{Y}_j}\quad(j=1,\cdots,N).$$ □

Proof of Proposition 2. Let η be a small positive constant to be determined later, and let $X(\eta)$ be a closed subset of X defined by

$$X(\eta)=\{u=(u_j)_{1\leq j\leq N}\in X:\|u\|_X\leq\eta\}\,.$$

Let $U_j(t,\cdot)$ $(j = 1, \cdots, N)$ be the function defined by (3.14). When ϵ is sufficiently small, Lemma 3.3 shows that

$$(U_j)_{1\le j\le N} \in X, \quad \|(U_j)_{1\le j\le N}\|_X \le \frac{\eta}{2}.$$

Let us introduce the map $\Phi : X \to X$ defined by

$$\Phi(u) = (\Phi_1(u), \cdots, \Phi_N(u)) = (U_1 + GF_1(u), \cdots, U_N + GF_N(u)),$$

where $u = (u_j)_{1\le j\le N} \in X$. Lemmas 3.2 and 3.4 show that

$$\|\Phi(u)\|_X \le \frac{\eta}{2} + C\eta^\sigma, \quad \sigma := \min(\sigma_1, \cdots, \sigma_N), \quad \forall u \in X(\eta),$$

and

$$\|\Phi(u) - \Phi(v)\|_{\hat{X}} \le C\left(\|u\|_X^{\sigma-1} + \|v\|_X^{\sigma-1}\right)\|u-v\|_{\hat{X}} \le C\eta^{\sigma-1}\|u-v\|_{\hat{X}}$$

for any $u, v \in X(\eta)$. Hence one claims that

$$\Phi : X(\eta) \to X(\eta), \quad \|\Phi(u) - \Phi(v)\|_{\hat{X}} \le \frac{1}{2}\|u-v\|_{\hat{X}},$$

provided that η is sufficiently small. Let $\{(U_j^k)_{1\le j\le N}\}_{k=0,1,2,\cdots} \subset X(\eta)$ be the sequence defined by

$$(U_j^0)_{1\le j\le N} = (U_j)_{1\le j\le N}, \quad (U_j^{k+1})_{1\le j\le N} = \Phi((U_j^k)_{1\le j\le N}) \quad k = 0, 1, 2, \cdots.$$

Then it follows that

$$\begin{aligned}&\left\|(U_j^{k+1})_{1\le j\le N} - (U_j^k)_{1\le j\le N}\right\|_{\hat{X}} \\ &= \left\|\Phi\left((U_j^k)_{1\le j\le N}\right) - \Phi\left((U_j^{k-1})_{1\le j\le N}\right)\right\|_{\hat{X}} \le \left(\tfrac{1}{2}\right)^k \left\|(U_j^1 - U_j^0)_{1\le j\le N}\right\|_{\hat{X}},\end{aligned}$$

for $k = 1, 2, 3, \cdots$. Hence, there exists $u = (u_j)_{1\le j\le N} \in \hat{X}$ such that

$$\lim_{k\to\infty}(U_j^k)_{1\le j\le N} = u \quad \text{in } \hat{X}, \quad \Phi(u) = u.$$

Let $T \in (0,\infty)$, $\mu \in [0,m]$ and $j \in \{1, 2, \cdots, N\}$ be fixed. Since

$$\{\langle x\rangle^\mu (U_j^k)^{(h)}\}_{k=0,1,2,\cdots}$$

is bounded in $L^\infty(0,T;H^{2,2})$ and

$$\langle x\rangle^\mu u_j^{(h)} = \lim_{k\to\infty}\langle x\rangle^\mu (U_j^k)^{(h)}$$

in $C([0,T],H^{1,2})$, one sees that

$$\langle x\rangle^m u_j^{(h)} = \lim_{k\to\infty}\langle x\rangle^m (U_j^k)^{(h)}$$

weak* in $L^\infty(0,T,H^{2,2})$ and in $C([0,T],H^{s,2})$ when $s<2$.
This implies that

$$\sup_{0\le t\le T,|\alpha|\le 2}(1+t)^{-\mu/2-n/4+3/2}\left\|\langle x\rangle^\mu\partial_x^\alpha u_j^{(h)}(t,\cdot)\right\|_{L^2} \le \limsup_{k\to\infty}\sup_{0\le t,|\alpha|\le 2}(1+t)^{-\mu/2-n/4+3/2}\left\|\langle x\rangle^\mu\partial_x^\alpha (U_j^k)^{(h)}(t,\cdot)\right\|_{L^2} \tag{3.32}$$

for any $\mu\in[0,m]$, for any $j\in\{1,2,\cdots,N\}$ and for any $T\in(0,\infty)$.
Since, for any $j=1,2,\cdots,N$,

$$\langle x\rangle^m u_j^{(l)}\in C([0,T],H^{2,2}),\quad \langle x\rangle^m u_j^{(h)}\in C([0,T],H^s)\quad (s<2),$$

the argument in Lemma 3.2 shows that

$$\langle x\rangle^m F_j(u)\in C([0,T],H^{1,2})\quad (j=1,2,\cdots,N).$$

This implies that

$$\langle x\rangle^m u_j\in C([0,T],H^{2,2})\quad (j=1,2,\cdots,N)$$

for any $0<T<\infty$. Hence, one sees that

$$\langle x\rangle^m u_j\in C([0,\infty);H^{2,2})\quad (j=1,2,\cdots,N).$$

Moreover, from (3.32) one sees that $u=(u_j)_{1\le j\le N}\in X(\eta)$. Lemma 3.3 shows that it is possible to choose $\eta=CI$. Hence one obtains the desired estimates in Theorem 1. It is easy to show the uniqueness of solution to (1.1)–(1.2); we omit the proof. □

3.1. **Proof of Proposition 1: Brief sketch.** To prove Proposition 1, we introduce function spaces $\mathcal{X}=\prod_{j=1}^N\mathcal{X}_j$ equipped with the norm

$$\|(u_j)_{1\le j\le N}\|_{\mathcal{X}}=\sum_{j=1}^N\|u_j\|_{\mathcal{X}_j},$$

where $\mathcal{X}_j$ $(1\le j\le N)$ are Banach spaces defined by

$$\mathcal{X}_j=\big\{u_j\in C([0,\infty),H^{2,2}):$$
$$\langle x\rangle^m u_j^{(l)}\in C([0,\infty),L^1\cap L^{p_0}),\langle x\rangle^m u_j\in C([0,\infty),H^{2,2}),\|u_j\|_{\mathcal{X}_j}<\infty\Big\},$$
$$\|u_j\|_{\mathcal{X}_j}:=\sup_{0\le t,0\le\mu\le m}(1+t)^{-\mu/2}\Big\{\sup_{1\le p\le p_0}(1+t)^{n/2(1-1/p)}\left\|\langle x\rangle^\mu u_j^{(l)}(t,\cdot)\right\|_{L^p} + \sup_{1\le p\le p_1}(1+t)^{n/2(1-1/p)+1/2}\left\|\langle x\rangle^\mu\nabla u_j^{(l)}(t,\cdot)\right\|_{L^p}$$

$$+ (1+t)^{-n/4+n\sigma_j/2+1/2}\left\|\langle x\rangle^{\mu} u_j^{(h)}(t,\cdot)\right\|_{H^{2,2}}\Big\}.$$

Proposition 1 is a direct consequence of the next proposition.

Proposition 3. *Under the notation in Proposition* 1, *Problem* (1.1)–(1.2) *admits a global solution* $u = (u_j)_{1\le j\le N} \in \mathcal{X}$, *and it satisfies* $\|u\|_{\mathcal{X}} \le CI$.

Proposition 3 is a consequence of the following lemmas. To state these lemmas we introduce several function spaces $\mathcal{Y} = \prod_{j=1}^N \mathcal{Y}_j$, $\hat{\mathcal{X}} = \prod_{j=1}^N \hat{\mathcal{X}}_j$, $\hat{\mathcal{Y}} = \prod_{j=1}^N \hat{\mathcal{Y}}_j$ where $\mathcal{Y}_j$, $\hat{\mathcal{X}}_j$ and $\hat{\mathcal{Y}}_j$ are the following Banach spaces for $j = 1, \cdots, N$:

$$\mathcal{Y}_j := \{g_j \in C([0,\infty), L^1\cap H^{1,2}) : \langle x\rangle^m g_j \in C([0,\infty), L^1\cap H^{1,2}), \|g_j\|_{\mathcal{Y}_j} < \infty\},$$

$$\|g_j\|_{\mathcal{Y}_j} := \sup_{0\le t, 0\le\mu\le m} (1+t)^{-\mu/2}\Big\{\sup_{1\le p\le 2}(1+t)^{-n/2p+n\sigma_j/2}\|\langle x\rangle^{\mu} g_j\|_{L^p} + (1+t)^{-n/4+n\sigma_j/2+1/2}\|\langle x\rangle^{\mu}\nabla g_j\|_{L^2}\Big\},$$

$$\hat{\mathcal{X}}_j = \Big\{u_j \in C([0,\infty), H^{1,2}); \langle x\rangle^m u_j^{(l)} \in C([0,\infty), L^1\cap L^{p_1}), \langle x\rangle^m u_j \in C([0,\infty), H^{1,2}) : \|u_j\|_{\hat{\mathcal{X}}_j} < \infty\Big\},$$

where

$$\|u_j\|_{\hat{\mathcal{X}}_j} := \sup_{0\le t, 0\le\mu\le m}\Big\{\sup_{1\le p\le p_1}(1+t)^{n/2(1-1/p)-\mu/2}\left\|\langle x\rangle^{\mu} u_j^{(l)}\right\|_{L^p} + (1+t)^{-n/4+n\sigma_j/2-\mu/2}\left\|\langle x\rangle^{\mu}\nabla u_j^{(h)}\right\|_{L^2}\Big\},$$

$$\hat{\mathcal{Y}}_j = \Big\{g_j \in C([0,\infty), L^1\cap L^2) : \langle x\rangle^m g_j \in C([0,\infty), L^1\cap L^2), \|g_j\|_{\hat{\mathcal{Y}}_j} < \infty\Big\},$$

$$\|g_j\|_{\hat{\mathcal{Y}}_j} := \sup_{0\le t, 0\le\mu\le m, 1\le p\le 2}(1+t)^{-n/2p+n\sigma_j/2-\mu/2}\|\langle x\rangle^{\mu} g_j\|_{L^p}.$$

From Lemmas 2.5, 2.6 and 2.9 one obtains the next lemma.

Lemma 3.5. *Under the notation in Proposition* 1, *set*

$$F(v) = (F_j(v))_{1\le j\le N}, \quad F(w) = (F_j(w))_{1\le j\le N}, \quad \sigma = \min_{1\le j\le N}\sigma_N.$$

Then we have the following.

(1) *Let* $U = (U_j)_{1\le j\le N}$ *be the function defined by* (3.14). *Then* $U \in \mathcal{X}$ *and it satisfies* $\|U\|_{\mathcal{X}} \le CI$, $\|U\|_{\hat{\mathcal{X}}} \le C\tilde{I}$.

(2) *Let* $v, w \in \mathcal{X}$ *be such that* $||v||_{\mathcal{X}} \leq 1$ *and* $||w||_{\mathcal{X}} \leq 1$*, then*

$$\|F(v) - F(w)\|_{\hat{\mathcal{Y}}} \leq C\left(\|v\|_{\mathcal{X}} + \|w\|_{\mathcal{X}}\right)^{\sigma-1}\|v - w\|_{\hat{\mathcal{X}}}.$$

(3) *Let* $v \in \mathcal{X}$ *be such that* $||v||_{\mathcal{X}} \leq 1$*, then* $\|F(v)\|_{\mathcal{Y}} \leq C\|v\|_{\mathcal{X}}^{\sigma}$.

Lemma 3.6. *Let* G *be the operator defined by*

$$Gf = \int_0^t S_1(t-\tau)f(\tau,\cdot)\,d\tau, \quad f \in \bigcup_{j=1}^{N} \hat{\mathcal{Y}}.$$

Then $G \in \bigcap_{j=1}^{N}\left(\mathcal{L}(\mathcal{Y}_j, \mathcal{X}_j) \cap \mathcal{L}(\hat{\mathcal{Y}}_j, \hat{\mathcal{X}}_j)\right)$.

Proof of Proposition 3: Brief sketch. Let η be a positive constant, and let $\mathcal{X}(\eta)$ be a closed subset of $\mathcal{X}$ defined by

$$\mathcal{X}(\eta) = \left\{(u,v) \in \mathcal{X} : \|u\|_{\mathcal{X}_1} \leq \eta, \|v\|_{\mathcal{X}_2} \leq \eta\right\}.$$

Let Φ be the operator defined in Theorem 1. When ϵ is sufficiently small, Lemmas 3.5 and 3.6 show that it is possible to choose η such that Φ is a mapping on $\mathcal{X}(\eta)$ such that

$$\|\Phi(u_1,v_1) - \Phi(u_2,v_2)\|_{\hat{\mathcal{X}}} \leq \tfrac{1}{2}\|(u_1,v_1) - (u_2,v_2)\|_{\hat{\mathcal{X}}}$$

for any $(u_1,v_1), (u_2,v_2) \in \mathcal{X}(\eta)$. Therefore one claims that there exists a $(u,v) \in \hat{\mathcal{X}}$ such that $\Phi(u,v) = (u,v)$ which is a desired solution. Repeating arguments in the proof of Proposition 2, one also obtains that $(u,v) \in \mathcal{X}(\eta)$. □

4. Proof of Theorem 2

In this section we give a short sketch of the proof of Theorem 2. For $j = 1, \cdots, N$, let $\bar{X}_j$ and $\bar{Y}_j$ be the following Banach spaces:

$$\bar{X}_j = \left\{u_j \in \bigcap_{k=0}^{m+1} C^k([0,\infty); M_{p,1}^{m+1-k}) : \|u_j\|_{\bar{X}_j} < \infty\right\},$$

$$\bar{Y}_j = \left\{g_j \in \bigcap_{k=0}^{m+1} C^k([0,\infty); M_{p,1}^{m+1-k}) : \|g_j\|_{\bar{Y}_j} < \infty\right\},$$

equipped with norms

$$\|u_j\|_{\bar{X}_j} = \sum_{k=0}^{m+1} \sup_{t\geq 0, p\leq q\leq\infty} (1+t)^{-n/2q+\nu_j+k}\left\|\partial_t^k u_j(t,\cdot)\right\|_{M_{q,1}^{m+1-k}} \quad (1 \leq j \leq N),$$

$$\|g_j\|_{\bar{Y}_j} = \sum_{k=0}^{m+1} \sup_{t\geq 0, p\leq q\leq\infty} (1+t)^{-n/2q+\nu_j+k+1} \left\| \partial_t^k g_j(t,\cdot) \right\|_{M_{q,1}^{m+1-k}} \quad (1 \leq j \leq N),$$

respectively. Let $\bar{X} = \prod_{j=1}^N \bar{X}_j$ and $\bar{Y} = \prod_{j=1}^N \bar{Y}_j$ be Banach spaces equipped with the norms

$$\|(u_j)_{1\leq j\leq N}\|_{\bar{X}} = \sum_{j=1}^N \|u_j\|_{\bar{X}_j}, \quad \|(f_j)_{1\leq j\leq N}\|_{\bar{Y}} = \sum_{j=1}^N \|f_j\|_{\bar{Y}_j},$$

respectively.

From Lemmas 2.1 and 2.3, one obtains the next lemma.

Lemma 4.1. *Let m be a positive integer, and let $p, q \in [1,\infty]$.*

(1) *Let $\bar{M}_{p,q}^m$ be a function space defined by*

$$\bar{M}_{p,q}^m = \{u \in M_{p,q} : \partial_x^\alpha u \in M_{p,q} \quad \forall|\alpha| \leq m\}$$

equipped with the norm

$$\|u\|_{\bar{M}_{p,q}^m} = \sum_{|\alpha|\leq m} \|\partial_x^\alpha u\|_{M_{p,q}}.$$

Then, $M_{p,q}^m = \bar{M}_{p,q}^m$ with equivalent norm.

(2) *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1/p = \theta/p_1 + (1-\theta)/p_2$ with $0 \leq \theta \leq 1$, then*

$$\|u\|_{M_{p,q}^m} \leq \|u\|_{M_{p_1,q}^m}^\theta \|u\|_{M_{p_2,q}^m}^{1-\theta} \quad \forall u \in M_{p_1,q}^m.$$

Proof. (1) Let us assume that $u \in M_{p,q}^m$. Then, Lemma 2.1(3) shows that

$$(1-\Delta)^{m/2} u \in M_{p,q}, \quad \|u\|_{M_{p,q}^m} \sim \left\|(1-\Delta)^{m/2} u\right\|_{M_{p,q}}.$$

Introduce a function $\mu_\alpha := \xi^\alpha (1+|\xi|^2)^{-m/2}$ when $|\alpha| \leq m$, then Lemma 2.3 shows that

$$\partial_x^\alpha u = c_\alpha \mathcal{F}^{-1} \mu_\alpha(\xi) \mathcal{F}\left((1-\Delta)^{m/2} u\right) \in M_{p,q}, \quad \forall|\alpha| \leq m,$$

and

$$\|\partial_x^\alpha u\|_{M_{p,q}} \leq C \left\|(1-\Delta)^{m/2} u\right\|_{M_{p,q}} \leq C \|u\|_{M_{p,q}^m}, \quad \forall|\alpha| \leq m.$$

Hence, one obtains

$$M_{p,q}^m \subseteq \bar{M}_{p,q}^m, \quad \|u\|_{\bar{M}_{p,q}^m} \leq C \|u\|_{M_{p,q}^m}.$$

Let us assume that $u \in \bar{M}_{p,q}^m$. Let $\bar{\omega}_1, \cdots, \bar{\omega}_n$ be smooth function on $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ such that $\sum_{i=1}^n \bar{\omega}_i(\xi) \equiv 1$, and $|\xi_i| \geq a > 0$ when $\bar{\omega}_i(\xi) \neq 0$

$(i = 1, \cdots, n)$, and let ω_0 be a smooth radial function such that $\omega_0(\xi) = 1$ when $|\xi| \le 1$, and $\omega_0(\xi) = 0$ when $|\xi| \ge 2$. Let us introduce functions $\omega_1, \cdots, \omega_n$ by

$$\omega_j(\xi) = \begin{cases} (1 - \omega_0(\xi))\, \omega_j\left(\frac{\xi}{|\xi|}\right), & \xi \neq 0, \\ 0, & \xi = 0, \end{cases}$$

then it follows that $\sum_{j=0}^{n} \omega_j(\xi) \equiv 1$ and

$$\left|\partial_\xi^\alpha \bar{\mu}_0(\xi)\right| + \sum_{j=1}^{n} \left|\partial_\xi^\alpha \bar{\mu}_j(\xi)\right| \le C_{\alpha,m} \quad \forall \alpha,$$

where $\bar{\mu}_0(\xi) = (1 + |\xi|^2)^{m/2}\omega_0(\xi)$, $\bar{\mu}_j(\xi) = (1 + |\xi|^2)^{m/2}/\xi_j^m \omega_j(\xi)$ $(j = 1, \cdots, n)$. Since

$$(1 - \Delta)^{m/2} u = \mathcal{F}^{-1}\left(\bar{\mu}_0(\xi)\hat{u}\right) + \sum_{j=1}^{n} c_{j,m} \mathcal{F}^{-1} \bar{\mu}_j(\xi) \mathcal{F}\left(\partial_{x_j}^m u\right),$$

Lemmas 2.1 and 2.3 show that $\bar{M}_{p,q}^m \subseteq M_{p,q}^m$, $\|u\|_{M_{p,q}^m} \le C \|u\|_{\bar{M}_{p,q}^m}$.

(2) Since

$$\|\Box_\gamma \partial_x^\alpha u\|_{L^p} \le \|\Box_\gamma \partial_x^\alpha u\|_{L^{p_1}}^\theta \|\Box_\gamma \partial_x^\alpha u\|_{L^{p_2}}^{1-\theta}, \quad \forall \alpha \in \mathbb{Z}_+^n, \forall \gamma \in \mathbb{Z}^n,$$

one obtains the desired estimates from Hölder's inequality and Lemma 4.1 (1). □

From Lemma 2.4 and 4.1 one obtains the next lemma.

Lemma 4.2. *Let m_1 be a non-negative integer, and let $p, p_1, \cdots, p_k \in [1, \infty]$ satisfy $1/p_1 + \cdots + 1/p_k = 1/p$. Assume that $f_j \in M_{p_j,1}^{m_1}$ for $j = 1, \cdots, k$, then it follows that $\prod_{j=1}^k f_j \in M_{p,1}^{m_1}$, and*

$$\left\| \prod_{j=1}^{k} f_j \right\|_{M_{p,1}^{m_1}} \le C \prod_{j=1}^{k} \|f_j\|_{M_{p_j,1}^{m_1}}.$$

Lemma 4.3. *Let $u, v \in \bar{X}$. Then $F(u), F(v) \in \bar{Y}$, and they satisfy*

$$\|F_j(u) - F_j(v)\|_{\bar{Y}_j} \le C \left(\|u\|_{\bar{X}} + \|v\|_{\bar{X}}\right)^{\sigma_j - 1} \|u - v\|_{\bar{X}} \quad (1 \le j \le N).$$

Proof. Let $q \in [p, \infty]$ and $j \in \{1, \cdots, N\}$ be fixed and assume that $\sigma_{j,k} \neq 0$ $(1 \le k \le N)$. Let $i_1, \cdots, i_N$ be non-negative integers such that $i := i_1 + \cdots + i_n \le m + 1$. Leibnitz' rule and Lemma 4.2 show that

$$\left\|\partial_t^{i_l}\left(w_l^{\sigma_{j,l}}\right)\right\|_{M_{\infty,1}^{m+1-i}} \le C\,(1 + t)^{-\sigma_{j,l}\nu_l - i} \|w_l\|_{\bar{X}_l}^{\sigma_{j,l}} \quad \forall w_l \in \bar{X}_l, l = 1, \cdots, N,$$

$$\left\|\partial_t^{i_k}\left(u_k^{\sigma_{j,k}}-v_k^{\sigma_{j,k}}\right)\right\|_{M_{q,1}^{m+1-i}}$$
$$\leq C\,(1+t)^{n/2q-\sigma_{j,k}\nu_k-I_k}\left(\|u_k\|_{\bar{X}_k}+\|v_k\|_{\bar{X}_k}\right)^{\sigma_{j,k}-1}\|u_k-v_k\|_{\bar{X}X_k}.$$

Hence it follows that

$$\left\|\partial_t^{i_1}\left(u_1^{\sigma_{j,1}}\right)\cdots\partial_t^{i_k}\left(u_k^{\sigma_{j,k}}-v_k^{\sigma_{j,k}}\right)\cdots\partial_t^{i_N}\left(v_N^{\sigma_{j,N}}\right)\right\|_{M_{q,1}^{m+1-i}}$$
$$\leq C\left\|\partial_t^{i_1}(u_1^{\sigma_{j,1}})\right\|_{M_{\infty,1}^{m+1-i}}\cdots\left\|\partial_t^{i_k}(u_k^{\sigma_{j,k}}-v_k^{\sigma_{j,k}})\right\|_{M_{q,1}^{m+1-i}}\cdots\left\|\partial_t^{i_N}(v_N^{\sigma_{j,N}})\right\|_{M_{\infty,1}^{m+1-i}}$$
$$\leq C(1+t)^{n/2q-\nu_j-1-i}\left(\|u\|_{\bar{X}}+\|v\|_{\bar{X}}\right)^{\sigma_j-1}\|u_k-v_k\|_{\bar{X}_k}; \tag{4.1}$$

in the above we used the equality $\sum_{k=1}^{N}\sigma_{j,k}\nu_k=\nu_j+1$. Let $i\in[0,m+1]$ be an integer. Then, Leibnitz' rule shows that $\partial_t^i\left(F(u)-F(v)\right)$ is a linear combination of the terms

$$\partial_t^{i_1}\left(u_1^{\sigma_{j,1}}\right)\cdots\partial_t^{i_k}\left(u_k^{\sigma_{j,k}}-v_k^{\sigma_{j,k}}\right)\cdots\partial_t^{i_N}\left(v_N^{\sigma_{j,N}}\right),\quad i=i_1+\cdots+i_N, 1\leq k\leq N.$$

Hence, (4.1) shows that

$$\left\|\partial_t^i\left(F_j(u(t,\cdot))-F_j(v(t,\cdot))\right)\right\|_{M_{q,1}^{m+1-i}}$$
$$\leq C(1+t)^{n/2q-\nu_j-1-i}\left(\|u\|_{\bar{X}}+\|v\|_{\bar{X}}\right)^{\sigma_j-1}\|u-v\|_{\bar{X}} \tag{4.2}$$

for any $q\in[p,\infty]$ and $i=0,1,\cdots,m+1$. From (4.2) one sees that $F(u)\in\bar{X}$ and obtains the desired estimate

$$\|(F(u)-F(v))\|_{\bar{X}_j}\leq C\left(\|u\|_{\bar{X}}+\|v\|_{\bar{X}}\right)^{\sigma_j-1}\|u-v\|_{\bar{X}}. \qquad \square$$

From Lemma 2.8 one obtains the next lemma.

Lemma 4.4. *Let $\varphi=(\varphi_1,\cdots,\varphi_N)\in M_{p,1}^{m+1,1}\times\cdots\times M_{p,1}^{m+1,1}$, and $\psi=(\psi_1,\cdots,\psi_N)\in M_{p,1}^{m}\times\cdots\times M_{p,1}^{m}$, then $S_0(\cdot)\varphi, S_1(\cdot)\psi\in X$, and they satisfy*

$$\|S_0(\cdot)\varphi\|_{\bar{X}}+\|S_1(\cdot)\psi\|_{\bar{X}}\leq\sum_{j=1}^{N}\left(\|\varphi_j\|_{M_{p,1}^{m+1}}+\|\psi_j\|_{M_{p,1}^{m}}\right).$$

Lemma 4.5. *Let G be an operator defined by*

$$Gf=\int_0^t S_1(t-\tau)f(\tau,\cdot)\,d\tau,\quad f\in\bigcup_{j=1}^{N}\bar{Y}_j.$$

When $f\in\bar{Y}_j$ $(j=1,\cdots,N)$, then $Gf\in\bar{X}_j$, and it satisfies

$$\|Gf\|_{\bar{X}_j}\leq C\,\|f\|_{\bar{Y}_j},\quad j=1,\cdots,N.$$

Proof. From Lemma 2.8 one sees that $Gf \in \bigcap_{l=0}^{m+1} C^l([0,\infty); M_{p,1}^{m+1-l})$, and moreover, one obtains the estimates

$$\left\|\partial_t^l S_1(t-\tau)f(\tau,\cdot)\right\|_{M_{q,1}^{m+1-l}} \le C(1+t-\tau)^{-(n/2)(1/p-1/q)-l}\|f(\tau,\cdot)\|_{M_{p,1}^m}$$

$$\le C(1+t-\tau)^{-(n/2)(1/p-1/q)-l}(1+\tau)^{n/2p-\nu_j-1}\|f\|_{\bar{Y}_j}, \tag{4.3}$$

$$\left\|S_1(\tau)\partial_t^l f(t-\tau,\cdot)\right\|_{M_{q,1}^{m+1-l}} \le C\left\|\partial_t^k f(t-\tau,\cdot)\right\|_{M_{q,1}^{m-l}} \tag{4.4}$$

$$\le C(1+t-\tau)^{n/2q-\nu_j-1-l}\|f\|_{\bar{Y}_j}, \tag{4.5}$$

for any $0 \le \tau \le t$, $q \in [p,\infty]$ and $l \in \{0,1,\cdots,m+1\}$. Furthermore one obtains

$$\left\|\partial_t^{l_1} S_1(t/2)\partial_t^{l_2} f(t/2,\cdot)\right\|_{M_{q,1}^{m+1-l_1-l_2}} \tag{4.6}$$

$$\le C(1+t)^{-(n/2)(1/p-1/q)-l_1}\left\|\partial_t^{l_2} f(t/2,\cdot)\right\|_{M_{p,1}^{m-l_2}} \tag{4.7}$$

$$\le C(1+t)^{n/2q-\nu_j-l_1-l_2-1}\|f\|_{\bar{Y}_j},$$

for any $t \ge 0$, $q \in [p,\infty]$ and $l_1, l_2 \in \{0,1.\cdots,m+1\}$ such that $l_1+l_2 \le m+1$.

Since $n/2p - \nu_j - 1 > -1$, Lemma 2.9 and (4.3)–(4.5) show

$$\|(Gf)(t,\cdot)\|_{M_{p,1}^{m+1}} \le C\int_0^t (1+\tau)^{n/2p-\nu_j-1}\|f\|_{\bar{Y}_j}\,d\tau \le C(1+t)^{n/2p-\nu_j}\|f\|_{\bar{Y}_j}, \tag{4.8}$$

and

$$\|(Gf)(t,\cdot)\|_{M_{\infty,1}^{m+1}} \tag{4.9}$$

$$\le C\Big(\int_0^{t/2}(1+t-\tau)^{-n/2p}(1+\tau)^{n/2p-\nu_j-1}d\tau + C\int_{t/2}^t (1+\tau)^{-\nu_j-1}d\tau\Big)\|f\|_{\bar{Y}_j}$$

$$\le C(1+t)^{n/2p-\nu_j}\|f\|_{\bar{Y}_j}.$$

From Lemma 4.1 and (4.8)–(4.9) one obtains

$$\|(Gf)(t,\cdot)\|_{M_{q,1}^{m+1}} \le \|(Gf)(t,\cdot)\|_{M_{p,1}^{m+1}}^{p/q}\|(Gf)(t,\cdot)\|_{M_{\infty,1}^{m+1}}^{1-p/q}$$

$$\le C(1+t)^{n/2q-\nu_j}\|f\|_{\bar{Y}_j} \tag{4.10}$$

for any $t \ge 0$, $q \in [p,\infty]$. Since

$$Gf(t,\cdot) = \int_0^{t/2} S_1(t-\tau)f(\tau,\cdot)\,d\tau + \int_0^{t/2} S_1(\tau)f(t-\tau,\cdot)\,d\tau,$$

easy calculation shows that

$$\begin{aligned}\partial_t^l Gf(t,\cdot) &= \int_0^{t/2} \partial_t^l S_1(t-\tau) f(\tau,\cdot)\, d\tau + \int_0^{t/2} S_1(\tau)\partial_t^l f(t-\tau,\cdot)\, d\tau \\ &+ \sum_{0\le l_1\le l-1, 0\le l_2\le l-1, l_1+l_2=l-1} c\partial_t^{l_1} S_1(t/2)\partial_t^{l_2} f(t/2,\cdot)\end{aligned}$$

when $l \in [1, m+1]$ is an integer. Hence, Lemma 2.9 and (4.3)–(4.6) show

$$\begin{aligned}&\left\| \int_0^{t/2} \partial_t^k S_1(t-\tau) f(\tau,\cdot)\, d\tau \right\|_{M_{q,1}^{m+1-k}} \\ &\le C \int_0^{t/2} (1+t-\tau)^{-(n/2)(1/p-1/q)-k} (1+\tau)^{n/2p-\nu_j-1}\, d\tau \, \|f\|_{\bar{Y}_j}, \\ &\le C(1+t)^{n/2q-\nu_j-k} \|f\|_{\bar{Y}_j},\end{aligned}$$

$$\begin{aligned}&\left\| \int_0^{t/2} S_1(\tau)\partial_t^l f(t,\cdot)\, d\tau \right\|_{M_{q,1}^{m+1-l}} \le C \int_0^{t/2} (1+t-\tau)^{n/2q-l-\nu_j-1} \|f\|_{\bar{Y}_j} \\ &\le C(1+t)^{n/2q-\nu_j-l} \|f\|_{\bar{Y}_j},\end{aligned}$$

and

$$\begin{aligned}&\sum_{0\le l_1\le l-1, 0\le l_2\le l-1, l_1+l_2=l-1} \left\| \partial_t^{l_1} S_1(t/2)\partial_t^{l_2} f(t/2,\cdot) \right\|_{M_{q,1}^{m+1-l_1-l_2}} \\ &\le C(1+t)^{n/2q-\nu_j-l} \|f\|_{\bar{Y}_j}\end{aligned}$$

for any $q \in [p, \infty]$ and $l = 1, \cdots, m+1$. From the above estimates one obtains

$$\left\| \partial_t^l Gf(t) \right\|_{M_{q,1}^{m+1-l}} \le C(1+t)^{n/2q-\nu_j-l} \|f\|_{\bar{Y}_j} \quad \forall t \ge 0 \tag{4.11}$$

for any $q \in [p, \infty]$ and $l \in \{1, \cdots, m+1\}$. (4.10)–(4.11) gives the desired estimate

$$\|Gf\|_{\bar{X}_j} \le C \|f\|_{\bar{Y}_j}. \qquad \square$$

Proof of Theorem 2. For $\eta > 0$, Lemma 4.4 shows that

$$\|S_0(t)\varphi + S_1(t)(\varphi + \psi)\|_X \le C\epsilon \le C\epsilon_0 \le \frac{\eta}{2},$$

provided that $0 < C\epsilon < \eta/2$. Define a mapping Φ on $\bar{X}$ by $\Phi(u) = v$ for $u \in \bar{X}$, where

$$v_j(t,\cdot) = S_0(t)\varphi_j + S_1(t)\left(\frac{\varphi_j}{2} + \psi_j\right) + \int_0^t S_1(t-\tau)F_j(u(\tau,\cdot))\,d\tau, \ \ (1 \le j \le N).$$

For $\eta > 0$, let us define a closed subset $\bar{X}(\eta)$ of $\bar{X}$ by $\bar{X}(\eta) = \{u \in \bar{X} : \|u\|_{\bar{X}} \le \eta\}$. From Lemmas 4.3 and 4.5 one sees that there exist positive constants ϵ_0 and $\eta_0 \le 1$ such that Φ is a contraction map on $\bar{X}(\eta_0)$ when

$$\sum_{j=1}^N \left(\|\varphi_j\|_{M^{m+1}_{p,1}} + \|\psi_j\|_{M^m_{p,1}}\right) \le \epsilon_0.$$

From the contraction mapping theorem one sees that there exists a unique fixed point $u \in \bar{X}(\eta)$ of Φ : $u = \{u_j\}_{j=1,\cdots,N} \in \bar{X}(\eta)$ that satisfies

$$u_j(t,\cdot) = S_0(t)\varphi_j + S_1(t)\left(\frac{\varphi_j}{2} + \psi_j\right) + \int_0^t S_1(t-\tau)F_j(u(\tau,\cdot))\,d\tau, \ \ (1 \le j \le N).$$

Therefore, $u \in \bar{X}$ is the desired solution. Moreover, from Lemmas 4.3–4.5 one obtains

$$\|u\|_{\bar{X}} \le C\sum_{j=1}^N \left(\|\varphi_j\|_{M^{m+1}_{p,1}} + \|\psi_j\|_{M^m_{p,1}}\right) + C\,\|u\|^{\sigma}_{\bar{X}}, \tag{4.12}$$

where $\sigma = \min(\sigma_1, \cdots, \sigma_N)$. When ϵ_0 is sufficiently small one may choose η_0 so that $C\eta^{\sigma} \le 1/2$, hence, (4.12) shows that

$$\|u\|_{\bar{X}} \le C\sum_{j=1}^N \left(\|\varphi_j\|_{M^{m+1}_{p,1}} + \|\psi_j\|_{M^m_{p,1}}\right). \tag{4.13}$$

Therefore, one claims that

$$\left\|\partial_t^k u_j\right\|_{M^{m+1-k}_{q,1}} \le C(1+t)^{n/2q-\nu_j-k}\sum_{j=1}^N \left(\|\varphi_j\|_{M^{m+1}_{p,1}} + \|\psi_j\|_{M^m_{p,1}}\right)$$

for any $t \ge 0$, $q \in [p,\infty]$, $j \in \{1,\cdots,N\}$ and $k \in \{0,1,\cdots,m+1\}$.

We omit the proof of uniqueness of the solution because it is rather easy.

□

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