Differential and Integral Equations Volume 24, Numbers 5-6 (2011), 541–567

ON THE CRITICAL KDV EQUATION WITH TIME-OSCILLATING NONLINEARITY

X. Carvajal
Instituto de Matemática, UFRJ, Rio de Janeiro, Brazil

M. Panthee [1]
Centro de Matemática, Universidade do Minho, 4710–057, Braga, Portugal

M. Scialom
IMECC-UNICAMP
13083-970, Campinas, São Paulo, Brazil

(Submitted by: Gustavo Ponce)

Abstract. We investigate the initial-value problem (IVP) associated with the equation

$$u_t + \partial_x^3 u + g(\omega t)\partial_x(u^5) = 0,$$

where g is a periodic function. We prove that, for given initial data $\phi \in H^1(\mathbb{R})$, as $|\omega| \to \infty$, the solution u_ω converges to the solution U of the initial-value problem associated with

$$U_t + \partial_x^3 U + m(g)\partial_x(U^5) = 0,$$

with the same initial data, where $m(g)$ is the average of the periodic function g. Moreover, if the solution U is global and satisfies $\|U\|_{L_x^5 L_t^{10}} < \infty$, then we prove that the solution u_ω is also global provided $|\omega|$ is sufficiently large.

1. Introduction

Let us consider the initial-value problem (IVP)

$$u_t + \partial_x^3 u + g(\omega t)\partial_x(u^5) = 0, \quad u(x, t_0) = \phi(x), \tag{1.1}$$

Accepted for publication: October 2010.
AMS Subject Classifications: 35Q35, 35Q53.
[1]X.C. was partially supported by FAPERJ Brazil through grants E-26/111.564/2008 & E-26/110.560/2010 and CNPq Brazil through grant 303849/2008-8, M. P. was partially supported by the Research Center of Mathematics of the University of Minho, Portugal through the FCT Plurianual Funding Program and FAPESP Brazil, and M. S. was partially supported by FAPESP Brazil.

where $x, t, t_0, \omega \in \mathbb{R}$ and $u = u(x,t)$, is a real-valued function and $g \in C(\mathbb{R}, \mathbb{R})$ is a periodic function with period $L > 0$. To simplify the analysis, we translate the initial time t_0 to 0 and consider the following IVP:

$$u_t + \partial_x^3 u + g(\omega(t+t_0))\partial_x(u^5) = 0, \quad u(x,0) = \phi(x). \tag{1.2}$$

Before analyzing the IVP (1.1) with time-oscillating nonlinearity, we discuss some aspects of the critical Korteweg-de Vries (KdV) equation,

$$u_t + \partial_x^3 u + \partial_x(u^5) = 0, \quad u(x,0) = \phi(x), \qquad x, t \in \mathbb{R}. \tag{1.3}$$

In the literature, the equation (1.3) is known as the critical KdV equation because, if one considers the nonlinearity $\partial_x(u^p)$, $p \in \mathbb{Z}$, then for $p < 5$ there exists a global solution for all data in $H^1(\mathbb{R})$, while for $p \geq 5$ the global solutions exist only for small data (i.e., data with small $H^1(\mathbb{R})$-norm). Also, the solitary wave solutions are orbitally stable for $p < 5$ and unstable for $p > 5$; see [3].

Well-posedness issues for the IVP (1.3) have been extensively studied in the literature; see for example [11] and [15], [16], and references therein. A detailed account of the recent well-posedness results can be found in Kenig, Ponce, and Vega [15], where they proved that there exists $\delta > 0$ such that the IVP (1.3) is globally well-posed for any data $\phi \in H^s(\mathbb{R})$, $s \geq 0$ satisfying $\|\phi\|_{L^2_x} < \delta$. They were also able to relax the smallness condition on the given data to obtain a local well-posedness result, but paying the price that the existence time now depends on the shape of the data ϕ and not just on its size. These are the best well-posedness results in the sense that $s = 0$ is the critical exponent given by the scaling argument. However, for data in $H^s(\mathbb{R})$, $s > 0$, they were able to remove the size and shape restriction and got local-well posedness for arbitrary data with life span T of the solution depending on $\|\phi\|_{H^s(\mathbb{R})}$.

We recall that the $L^2_x(\mathbb{R})$ norm and energy are conserved by the flow of (1.3). More precisely,

$$\int_{\mathbb{R}} |u(x,t)|^2 dx = \int_{\mathbb{R}} |\phi(x)|^2 dx \tag{1.4}$$

and

$$E(u(\cdot,t)) := \frac{1}{2}\int_{\mathbb{R}} \big\{(u_x(x,t))^2 - \frac{1}{3}u^6(x,t)\big\} dx = E(\phi) \tag{1.5}$$

are conserved quantities.

These conserved quantities yield an *a priori* estimate if the $\|\phi\|_{L^2_x}$-norm is small enough, which allows one to iterate the local solution to get the global one for data in $H^1(\mathbb{R})$. Note that, for $u \in H^1(\mathbb{R})$, Weinstein [26] proved the following Gagliardo-Nirenberg-type inequality:

$$\frac{1}{3}\|u\|_{L_x^2}^6 \le \Big(\frac{\|u\|_{L_x^2}}{\|Q\|_{L_x^2}}\Big)^4 \|u_x\|_{L_x^2}^2, \tag{1.6}$$

where $Q(x) = \{3c\,\mathrm{sech}^2(2\sqrt{c}\ x)\}^{\frac{1}{4}}$, $c > 0$, is the solitary wave solution of (1.3). So, in view of (1.6), the size restriction on the initial data needed to obtain global solutions for (1.3) in $H^1(\mathbb{R})$ is $\|\phi\|_{L^2(\mathbb{R})} < \|Q\|_{L^2(\mathbb{R})}$. For recent global results for low regularity data, we refer the works of [7] and [8].

Although there are many works that deal with the well-posedness issues for the IVP (1.3) with low regularity initial data, in many practical situations the behavior of the $H^1(\mathbb{R})$ solution holds much importance, e.g., [22] in the blow-up context. Kenig, Ponce, and Vega in [16] gave a detailed account for the concentration of blow-up solutions. Martel and Merle in [20] proved that there exists $\phi \in H^1(\mathbb{R})$ satisfying $\|\phi\|_{L^2(\mathbb{R})} > \|Q\|_{L^2(\mathbb{R})}$ such that the corresponding solution to the IVP (1.3) blows up in finite time. For further results on the existence and dynamics of the blow-up solutions and the stability of the blow-up profile, we refer the readers to the works of Martel and Merle in [20, 21, 22].

Our motivation for considering the critical KdV equation with time-periodic nonlinearity arises from the paper of Abdullaev et al. [1] and Konotop and Pacciani [19], where the authors investigate the effect of a time-oscillating term in factor of the nonlinearity in Bose-Einstein condensates. In [1] the authors investigate solutions which are global for large frequencies, while the authors in [19] study solutions which blow up in finite time. Their results are numerical. Roughly speaking, the physicists claim that the periodic time-dependent term in factor of the nonlinearity would disturb the blow-up solution, either accelerating it or delaying it. Recently, Cazenave and Scialom [5] considered the nonlinear Schrödinger (NLS) equation and got an analytical insight to understand the problem by showing that the solution really depends on the frequency of the oscillating term. They proved that the solution u to the IVP associated to the NLS equation

$$iu_t + \Delta u + \theta(\omega t)|u|^\alpha u = 0, \quad x \in \mathbb{R}^N, \tag{1.7}$$

where $0 < \alpha < \frac{4}{(N-2)^+}$ is an H^1-subcritical exponent and θ is a periodic function, with initial data $\phi \in H^1(\mathbb{R}^N)$ converges as $|\omega| \to \infty$ to the solution U of the limiting equation

$$iU_t + \Delta U + I(\theta)|U|^\alpha U = 0, \quad x \in \mathbb{R}^N, \tag{1.8}$$

with the same initial data, where $I(\theta)$ is the average of θ. Moreover, they also showed that, if the limiting solution U is global and has a certain decay property as $t \to \infty$, then u is also global if $|\omega|$ is sufficiently large.

In this work, we are interested in obtaining similar results for the critical KdV equation. The existence of a blow-up solution in finite time to the IVP (1.3) for some $H^1(\mathbb{R})$-data due to Martel and Merle [20] and the discussion made above strengthen our problem of studying (1.1) with time-oscillating nonlinearity.

Our interest here is to investigate the behavior in $H^1(\mathbb{R})$ of the solution of the IVP (1.1) as $|\omega| \to \infty$. The natural limiting candidate to think of is the solution to the the following IVP:

$$U_t + \partial_x^3 U + m(g)\partial_x(U^5) = 0, \quad U(x,0) = \phi(x), \qquad x,t \in \mathbb{R}, \tag{1.9}$$

where $m(g) := \frac{1}{L}\int_0^L g(t)dt$ is the mean value of g and is a real number. To do so, we need an appropriate well-posedness result for the critical KdV equation in $H^1(\mathbb{R})$. Kenig, Ponce, and Vega in [15] have proved a local well-posedness result for arbitrary data in $H^s(\mathbb{R})$, $s > 0$, with life span of solution depending only on the $H^s(\mathbb{R})$-norm of the initial data. To this end, they used two additional norms, $\|D_t^{s/3}u\|_{L_x^5 L_T^{10}}$ and $\|D_t^{s/3}\partial_x u\|_{L_x^\infty L_T^2}$, which involve time derivatives of the solution. The presence of these norms create extra difficulty in handling the time-oscillating nonlinearity. In our case, it is very important to have an explicit expression that gives the local existence time of the solution. In the literature, we did not find an explicitly written proof of the $H^1(\mathbb{R})$ well-posedness that fulfills our requirement. Therefore, we will provide a new proof for the well-posedness of the IVP (1.3) in $H^1(\mathbb{R})$. Our proof allows us to extend the result to (1.2) and as a consequence to have an estimate of the local existence time.

Actually, we weaken the regularity requirement on the initial data to obtain local well-posedness in $H^s(\mathbb{R})$ without the norms that involve time derivatives. To this end, for $s \in (3/8, 1)$, we use new maximal-function-type estimates and Leibniz's rule for fractional derivatives and prove that the IVP (1.3) (consequently IVPs (1.2) and (1.9)) is locally well-posed in $H^s(\mathbb{R})$.

The only works other than [5] we did find in the literature that address the well-posedness issue for the equations of the KdV family and NLS with explicitly time-dependent nonlinearity were by Nunes [23, 24] and Damergi and Goubet [6]. The authors in [6] deal with the NLS equation in $\mathbb{R}^2$ with nonlinearity $\cos^2(\Omega t)|u|^{p-1}u$ in the critical and supercritical cases. The author in [23] considered the transitional KdV with nonlinearity $f(t)u\partial_x u$,

f a continuous function such that $f' \in L^1_{\text{loc}}(\mathbb{R})$, and proved global well-posedness in $H^s(\mathbb{R})$, $s \geq 1$. The transitional KdV arises in the study of long solitary waves propagating on the thermocline separating two layers of fluids of almost equal densities in which the effect of the change in the depth of the bottom layer, which the wave feels as it approaches the shore, results in the coefficient of the nonlinear term; for details see [18]. In [24], the transitional Benjamin-Ono equation with time-dependent coefficient in the nonlinearity has been considered, and the main result is the global existence of the solution for data in $H^s(\mathbb{R})$, $s \geq \frac{3}{2}$.

Before stating the main results of this work, we define notation that will be used throughout this work.

Notation: We use $\hat{f}$ to denote the Fourier transform of f, which is defined as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

The L^2-based Sobolev space of order s will be denoted by H^s with norm

$$\|f\|_{H^s(\mathbb{R})} = \Big(\int_{\mathbb{R}} (1+\xi^2)^s |\hat{f}(\xi)|^2 \, d\xi \Big)^{1/2}.$$

The Riesz potential of order $-s$ is denoted by $D_x^s = (-\partial_x^2)^{s/2}$. For $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ we define the mixed $L_x^p L_T^q$-norm by

$$\|f\|_{L_x^p L_T^q} = \Big(\int_{\mathbb{R}} \Big(\int_0^T |f(x,t)|^q \, dt \Big)^{p/q} dx \Big)^{1/p},$$

with usual modifications when $p = \infty$. We replace T by t if $[0,T]$ is the whole real line $\mathbb{R}$. We use the notation $f \in H^{\alpha+}$ if $f \in H^{\alpha+\epsilon}$ for $\epsilon > 0$.

We define four more spaces: Z_T^s, X_T, Y_T^s, and Y_T with norms

$$\begin{aligned} \|f\|_{Z_T^s} :=& \|f\|_{L_T^\infty H^s} + \|\partial_x f\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x f\|_{L_x^\infty L_T^2} \\ &+ \|\partial_x f\|_{L_x^{40/3} L_T^{20/7}} + \|D_x^s f\|_{L_T^{20/3} L_x^{20}} + \|f\|_{L_x^8 L_T^\infty}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} \|f\|_{X_T} :=& \|f\|_{L_T^\infty H^1} + \|\partial_x f\|_{L_x^\infty L_T^2} + \|\partial_x^2 f\|_{L_x^\infty L_T^2} \\ &+ \|f\|_{L_x^5 L_T^{10}} + \|\partial_x f\|_{L_x^5 L_T^{10}} + \|\partial_x f\|_{L_x^{20} L_T^{5/2}} + \|f\|_{L_x^4 L_T^\infty}, \end{aligned} \tag{1.11}$$

$$\|f\|_{Y_T^s} := \|D_x^s f\|_{L_x^2 L_T^2} + \|f\|_{L_x^2 L_T^2}, \tag{1.12}$$

and

$$\|f\|_{Y_T} := \|\partial_x f\|_{L_x^2 L_T^2} + \|f\|_{L_x^2 L_T^2}, \tag{1.13}$$

respectively. We replace X_T by X_t or $X_{(T,\infty)}$, if the time integral is taken in the interval $(0,\infty)$ or (T,∞) respectively, and similarly for Z_T^s, Y_T^s, and

Y_T. We would like to note that the space Y_T is obtained from Y_T^s by setting $s = 1$; however, it is not the case for Z_T^s and X_T, and they are independent.

We use the letter C to denote various constants whose exact values are immaterial and which may vary from one line to the next.

First, we state the local well-posedness result for more general IVP:

$$u_t + u_{xxx} + h(t)\partial_x u^5 = 0, \quad u(x,0) = \phi(x), \quad x, t \in \mathbb{R}, \tag{1.14}$$

where $h \in L^\infty$ is a given function of t.

Theorem 1.1. *Suppose $\phi \in H^s(\mathbb{R})$, $s \in (3/8, 1)$. Then there exist $T = T(\|\phi\|_{H^s(\mathbb{R})}, \|h\|_{L_t^\infty}) > 0$ and a unique solution u to the IVP* (1.14) *satisfying*

$$u \in C([0, T]; H^s(\mathbb{R})), \tag{1.15}$$

$$\|\partial_x u\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x u\|_{L_x^\infty L_T^2} < \infty, \tag{1.16}$$

$$\|\partial_x u\|_{L_x^{40/3} L_T^{20/7}} + \|D_x^s u\|_{L_T^{20/3} L_x^{20}} < \infty, \tag{1.17}$$

$$\|u\|_{L_x^8 L_T^\infty} < \infty. \tag{1.18}$$

Moreover, for any $T' \in (0, T)$, there exists a neighborhood $\mathcal{V}$ of ϕ in $H^s(\mathbb{R})$ such that the map $\tilde{\phi} \mapsto \tilde{u}$ from $\mathcal{V}$ into the class defined by (1.15) *to* (1.18) *with T' in place of T is Lipschitz.*

Using Duhamel's principle, we prove Theorem 1.1 by considering the integral equation associated to the IVP (1.14),

$$u(t) = S(t)\phi - \int_0^t S(t - t')h(t')\partial_x(u^5)(t')\, dt', \tag{1.19}$$

where $S(t)$ is the unitary group generated by the operator ∂_x^3 that describes the solution to the linear problem. Our interest is to solve (1.19) using the contraction-mapping principle in appropriate metric spaces.

As particular cases, we have similar well-posedness results for the IVPs (1.2), (1.3), and (1.9) in $H^s(\mathbb{R})$, $s \in (3/8, 1)$, as in Theorem 1.1.

As discussed above, we are interested in proving the convergence of the solution, as $|\omega| \to \infty$, of the IVP (1.2) to that of the IVP (1.9) for given data in $H^1(\mathbb{R})$. For technical reasons (see the proof of Lemma 3.2 below), we use the following local well-posedness results in $H^1(\mathbb{R})$, whose proofs follow by using the space X_T.

Theorem 1.2. *Suppose $\phi \in H^1(\mathbb{R})$. Then there exist*

$$T = T(\|\phi\|_{H^1(\mathbb{R})}, \|h\|_{L_t^\infty}) > 0$$

and a unique solution u to the IVP (1.14) *satisfying*

$$u \in C([0, T]; H^1(\mathbb{R})), \tag{1.20}$$

$$\|\partial_x u\|_{L^\infty_x L^2_T} + \|\partial_x^2 u\|_{L^\infty_x L^2_T} < \infty, \tag{1.21}$$

$$\|u\|_{L^5_x L^{10}_T} + \|\partial_x u\|_{L^5_x L^{10}_T} + \|\partial_x u\|_{L^{20}_x L^{5/2}_T} < \infty, \tag{1.22}$$

$$\|u\|_{L^4_x L^\infty_T} < \infty. \tag{1.23}$$

Moreover, for any $T' \in (0, T)$, there exists a neighborhood $\mathcal{V}$ of ϕ in $H^1(\mathbb{R})$ such that the map $\tilde{\phi} \mapsto \tilde{u}$ from $\mathcal{V}$ into the class defined by (1.20) *to* (1.23) *with T' in place of T is Lipschitz.*

Remark 1.3. As a particular case, we have the similar well-posedness result for the IVP (1.3) ($h(t) = 1$), for given data in $H^1(\mathbb{R})$. Since the average $m(g)$ is a constant, the proof of Theorem 1.2 can also be adapted line by line to obtain the similar well-posedness result for the IVP (1.9). The only difference in this case is that to complete the contraction argument we need to choose $T > 0$ in such a way that $C|m(g)|T^{1/2}\|\phi\|^4_{H^1(\mathbb{R})} < \frac{1}{2}$. So the existence time T depends on $|m(g)|$ and $\|\phi\|_{H^1(\mathbb{R})}$. We also have the following bound:

$$\|U\|_{X_T} \le C\|\phi\|_{H^1(\mathbb{R})}, \quad \forall\ t \in [0, T]. \tag{1.24}$$

Theorem 1.4. *Suppose $\phi \in H^1(\mathbb{R})$. Then there exist*

$$T = T(\|\phi\|_{H^1(\mathbb{R})}, \|g\|_{L^\infty_t}) > 0$$

and a unique solution $u_{\omega,t_0} \in C([0, T]; H^1(\mathbb{R}))$ to the IVP (1.2) *satisfying* (1.21)–(1.23). *Moreover, for any $T' \in (0, T)$, there exists a neighborhood $\mathcal{V}$ of ϕ in $H^1(\mathbb{R})$ such that the map $\tilde{\phi} \mapsto \tilde{u}_{\omega,t_0}$ from $\mathcal{V}$ into the class defined by* (1.20) *to* (1.23) *with T' in place of T is Lipschitz.*

Now, we state the main results of this work.

Theorem 1.5. *Fix $\phi \in H^1(\mathbb{R})$. For given $\omega, t_0 \in \mathbb{R}$, let u_{ω,t_0} be the maximal solution of the IVP* (1.2) *and U be the solution of the limiting IVP* (1.9) *defined on the maximal time of existence $[0, S_{\max})$. Then, for any given $0 < T < S_{\max}$, the solution u_{ω,t_0} exists on $[0, T]$ for all $t_0 \in \mathbb{R}$ and $|\omega|$ large. Moreover, $\|u_{\omega,t_0} - U\|_{X_T} \to 0$, as $|\omega| \to \infty$, uniformly in $t_0 \in \mathbb{R}$. In particular, the convergence holds in $C([0, T]; H^1(\mathbb{R}))$ for all $T \in (0, S_{max})$.*

Theorem 1.6. *Let $\phi \in H^1(\mathbb{R})$ and u_{ω,t_0} be the maximal solution of the IVP* (1.1). *Suppose U is the maximal solution of the IVP* (1.9) *defined on $[0, S_{max})$. If $S_{max} = \infty$ and*

$$\|U\|_{L^5_x L^{10}_t} < \infty, \tag{1.25}$$

then it follows that u_{ω,t_0} *is global for all* $t_0 \in \mathbb{R}$ *if* $|w|$ *is sufficiently large. Moreover,*

$$\|u_{\omega,t_0} - U\|_{X_t} \to 0, \quad \textit{when} \quad |w| \to \infty, \tag{1.26}$$

uniformly in t_0*. In particular, convergence holds in* $L^\infty((0,\infty); H^1(\mathbb{R}))$*.*

Taking into account the result in [20], Theorem 1.6 is very interesting in the sense that when $m(g) = 0$ the solution U to the IVP (1.9) will be global for all initial H^1-data and the solution u_{ω,t_0} to the nonlinear problem (1.2) will be global too, for $|\omega|$ large enough.

Remark 1.7. It is natural to ask if results similar to the ones in Theorem 1.5 and Theorem 1.6 hold for the $H^s(\mathbb{R})$ solution obtained in Theorem 1.1. The convergence result of Lemma 3.2 plays a central role in our analysis. For the norms involved in Theorem 1.1, we could not get such a convergence result. This is one of the reasons that forced us to have different well-posedness results in $H^1(\mathbb{R})$, viz., Theorems 1.2 and 1.4. Note that the indices involved in the norms used in Theorems 1.2 and 1.4 are the admissible triples (see Definition 2.6 below); however, those in Theorem 1.1 are not. As far as we know, there are no blow-up solutions in $H^s(\mathbb{R})$ for $s < 1$, so we did not proceed to have convergence for this range of Sobolev regularity. In this sense, the local well-posedness result of Theorem 1.1 is of independent interest.

Before leaving this section, we discuss the example constructed in [5] in the context of the NLS equation with time-oscillating nonlinearity. The authors in [5] showed that for small frequency $|\omega|$, the solution u_{ω,t_0} blows up in finite time or is global depending on t_0, while for the large frequency $|\omega|$, the solution u_{ω,t_0} is global for all $t_0 \in \mathbb{R}$. The same example can be utilized with small modification in the context of the critical KdV equation. We present it here for the convenience of the readers.

Example 1.8. Let $L > 1$, $0 < \epsilon < \frac{L-1}{2}$, and consider a periodic function g defined by

$$m(g) = 0, \quad \text{and} \quad g(s) = \begin{cases} 1, & |s| \le \epsilon, \\ 0, & 1 \le s \le 1+\epsilon, \end{cases} \tag{1.27}$$

with period L.

Fix $\phi \in H^1(\mathbb{R})$ with $\|\phi\|_{L^2_x} > \|Q\|_{L^2_x}$ such that the solution v of the IVP

$$v_t + v_{xxx} + v^4 \partial_x v = 0, \quad v(x,0) = \phi(x), \tag{1.28}$$

blows up in finite time, say T^*. There exists such a solution $v(x,t)$ of (1.28) with $t \in [0, T^*)$; see [20].

From Theorem 1.5, for this particular ϕ and the periodic function g, we have that the solution u_{ω,t_0} to the IVP (1.2) converges, as $|\omega| \to \infty$, to the solution U of the linear KdV equation with same initial data ϕ. So, in view of Theorem 1.6, u_{ω,t_0} is global as $|\omega| \to \infty$ for all $t_0 \in \mathbb{R}$.

Now we move to analyze the behavior of the solution for $|\omega|$ small. Note that $g(\omega s) = 1$ when $|\omega s| \leq \epsilon$. Therefore, if we consider $|\omega| < \frac{\epsilon}{T^*}$, then we see that the solution v to the IVP (1.28) satisfies (1.2) for $t_0 = 0$ on $[0, T^*)$. By uniqueness, $u_{\omega,0} = v$. Hence the solution $u_{\omega,0}$ of the IVP (1.2) blows-up in finite time, provided $|\omega| < \frac{\epsilon}{T^*}$.

Let $\epsilon = \epsilon(A)$ be as in Corollary 3.4 with $A = \|g\|_{L_t^\infty}$. From the linear estimate (2.7) we have that $S(\cdot)\phi \in L_x^5 L_t^{10}$, so there exists $T > 0$ such that

$$\|S(\cdot)[S(T)\phi]\|_{L_x^5 L_t^{10}} = \|S(\cdot)\phi\|_{L_x^5 L_{(T,\infty)}^{10}} \leq \epsilon. \tag{1.29}$$

For $\omega > 0$, if we consider $t_0 = \frac{1}{\omega}$, we have that $g(\omega(s + t_0)) = 0$ for all $1 \leq \omega(s + t_0) \leq 1 + \epsilon$, i.e., for all $0 \leq s \leq \frac{\epsilon}{\omega}$. Therefore, if we let $\omega > 0$ satisfy $\omega \leq \frac{\epsilon}{T}$ (i.e., $T \leq \frac{\epsilon}{\omega}$), and choose $t_0 = \frac{1}{\omega}$, then $g(\omega(s + t_0)) = 0$ for all $0 \leq s \leq T$. So, with this choice, u_{ω,t_0} solves the linear KdV equation if $0 \leq t \leq T$. Therefore, for $\omega \leq \frac{\epsilon}{T}$, u_{ω,t_0} exists on $[0, T]$ and is given by $S(t)\phi$; in particular, $u_{\omega,t_0}(T) = S(T)\phi$. From (1.29), $\|S(\cdot)u_{\omega,t_0}(T)\|_{L_x^5 L_t^{10}} \leq \epsilon$. Hence, from Corollary 3.4 we conclude that u_{ω,t_0} is global.

This paper is organized as follows. In Section 2, we record some preliminary estimates associated with the linear problem and other relevant results. In Section 3, we give a proof of the local well-posedness result for the critical KdV equation in $H^s(\mathbb{R})$, $s \in (3/8, 1)$, $H^1(\mathbb{R})$ and some other results that will be used in the proof of the main theorems. Finally, the proof of the main results will be given in Section 4.

2. Preliminary estimates

In this section we give some linear estimates associated with the IVP (1.1). These estimates are not new and can be found in the literature. Consequently, we just sketch the idea of the proof and mention the references where they can be found.

Lemma 2.1. *If $u_0 \in L^2(\mathbb{R})$, then*

$$\|\partial_x S(t)u_0\|_{L_x^\infty L_t^2} \leq C\|u_0\|_{L_x^2}. \tag{2.1}$$

If $f \in L^1_x L^2_t$, *then*

$$\left\| \partial_x \int_0^t S(t-t')f(\cdot,t')dt' \right\|_{L^\infty_t L^2_x} \le C\|f\|_{L^1_x L^2_t}, \tag{2.2}$$

$$\left\| \partial_x^2 \int_0^t S(t-t')f(\cdot,t')dt' \right\|_{L^\infty_x L^2_t} \le C\|f\|_{L^1_x L^2_t}. \tag{2.3}$$

Proof. For the proof of the homogeneous smoothing effect (2.1) and the double smoothing effect (2.3), see Theorem 3.5 in [15] (see also Section 4 in [14]). The inequality (2.2) is the dual version of (2.1). □

Now we give the maximal function estimate.

Lemma 2.2. *If* $u_0 \in \dot{H}^{1/4}(\mathbb{R})$, *then*

$$\|S(t)u_0\|_{L^4_x L^\infty_T} \le C\|D_x^{1/4}u_0\|_{L^2(\mathbb{R})}. \tag{2.4}$$

Also, we have

$$\|S(t)u_0\|_{L^\infty_x L^\infty_T} \le C\|u_0\|_{H^{\frac{1}{2}+}(\mathbb{R})}, \tag{2.5}$$

and, for $0 \le \theta \le 1$,

$$\|S(t)u_0\|_{L_x^{\frac{4}{\theta}} L^\infty_T} \le C\|u_0\|_{H^{\frac{2-\theta}{4}+}(\mathbb{R})}. \tag{2.6}$$

Proof. For the proof of the estimate (2.4) we refer to Theorem 3.7 in [15] (see also [13] and [17]). The estimate (2.5) follows from Sobolev embedding. The interpolation between (2.4) and (2.5) yields (2.6). □

In what follows, we state some more estimates that will be used in our analysis.

Lemma 2.3. *If* $u_0 \in L^2(\mathbb{R})$, *then*

$$\|S(t)u_0\|_{L^5_x L^{10}_t} \le C\|u_0\|_{L^2_x}. \tag{2.7}$$

Also we have

$$\|\partial_x S(t)u_0\|_{L^{20}_x L^{5/2}_t} \le C\|D_x^{1/4}u_0\|_{L^2_x}, \tag{2.8}$$

$$\|\partial_x S(t)u_0\|_{L^{40/3}_x L^{20/7}_t} \le C\|D_x^{3/8}u_0\|_{L^2_x}. \tag{2.9}$$

Proof. The proof of the estimates (2.7) and (2.8) can be found in Corollary 3.8 and Proposition 3.17 in [15] respectively. To prove (2.9) we consider the analytic family of operators $T_z u_0 = D_x^{-z/4} D_x^{1-z} S(t)u_0$, with $z \in \mathbb{C},\ 0 \le \Re z \le 1$. Now the estimate (2.9) follows by choosing $z = 3/40$ in the Stein's theorem of analytic interpolation (see [25]) between the smoothing estimate (2.1) and the maximal function estimate (2.4). □

Lemma 2.4. *Let $u_0 \in L_x^2$; then for any $(\theta,\alpha) \in [0,1]\times[0,\frac{1}{2}]$, we have*

$$\|D_x^{\theta\alpha/2} S(t)u_0\|_{L_T^q L_x^p} \le C\|u_0\|_{L_x^2}, \tag{2.10}$$

where $(q,p) = (\frac{6}{\theta(\alpha+1)}, \frac{2}{1-\theta})$.

Proof. See Lemma 2.4 in [12]. □

We state next Leibniz's rule for fractional derivatives, whose proof is also given in [15], Theorem A.8.

Lemma 2.5. *Let $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$, $\alpha_1+\alpha_2 = \alpha$. Let $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \le C\|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}}. \tag{2.11}$$

Moreover, for $\alpha_1 = 0$ the value $q_1 = \infty$ is allowed.

Definition 2.6. *Let $1 \le p, q \le \infty$, $-\frac{1}{4} \le \alpha \le 1$. We say that a triple (p,q,α) is an admissible triple if*

$$\frac{1}{p} + \frac{1}{2q} = \frac{1}{4} \qquad \text{and} \qquad \alpha = \frac{2}{q} - \frac{1}{p}. \tag{2.12}$$

Proposition 2.7. *For any admissible triples (p_j, q_j, α_j), $j = 1, 2$, the following estimate holds:*

$$\left\| D_x^{\alpha_1} \int_0^t S(t-t')f(\cdot,t')dt' \right\|_{L_x^{p_1} L_t^{q_1}} \le C\|D_x^{-\alpha_2} f\|_{L_x^{p_2'} L_t^{q_2'}}, \tag{2.13}$$

where p_2' and q_2' are the conjugate exponents of p_2 and q_2.

Proof. For the proof we refer to Proposition 2.3 in [16]. □

The following results will be used to complete the contraction-mapping argument.

Lemma 2.8. *Let Z_T^s and Y_T^s be the spaces defined earlier and S be the unitary group associated with the operator ∂_x^3; then for $s > 3/8$, we have*

$$\|S(t)u_0\|_{Z_T^s} \le C_0\|u_0\|_{H^s(\mathbb{R})}, \tag{2.14}$$

$$\left\| \int_0^t S(t-t')f(t')dt' \right\|_{Z_T^s} \le CT^{1/2}\|f\|_{Y_T^s}. \tag{2.15}$$

Proof. The estimate (2.14) follows from the linear estimates in Lemmas 2.1, 2.2, 2.3, and Lemma 2.4 with $\alpha = 0$ and $\theta = 9/10$.

Now, we prove the estimate (2.15). Since $S(t)$ is a unitary group in L^2_x, using Hölder's inequality we have

$$\begin{aligned}
&\Big\|\int_0^t S(t-t')f(t')dt'\Big\|_{H^s} \le \Big\|D_x^s\int_0^t S(t-t')f(t')dt'\Big\|_{L^2_x} + \Big\|\int_0^t S(t-t')f(t')dt'\Big\|_{L^2_x}\\
&= \Big\|\int_0^t S(-t')D_x^s f(t')dt'\Big\|_{L^2_x} + \Big\|\int_0^t S(-t')f(t')dt'\Big\|_{L^2_x}\\
&\le \int_0^T \|S(-t')D_x^s f(t')\|_{L^2_x}dt' + \int_0^T \|S(-t')f(t')\|_{L^2_x}dt'\\
&\le CT^{1/2}\big[\|D_x^s f\|_{L^2_xL^2_T} + \|f\|_{L^2_xL^2_T}\big]. \qquad (2.16)
\end{aligned}$$

Using the estimate (2.1), the fact that $S(t)$ is a unitary group in L^2_x, and Hölder's inequality, one can obtain

$$\begin{aligned}
&\Big\|\partial_x\int_0^t S(t-t')f(t')dt'\Big\|_{L^\infty_xL^2_T} \le \int_0^T \|\partial_x S(t)S(-t')f(t')\|_{L^\infty_xL^2_T}dt'\\
&\le C\int_0^T \|S(-t')f(t')\|_{L^2_x}dt' \le CT^{1/2}\|f\|_{L^2_xL^2_T}. \qquad (2.17)
\end{aligned}$$

Similarly,

$$\Big\|D_x^s\partial_x\int_0^t S(t-t')f(t')dt'\Big\|_{L^\infty_xL^2_T} \le CT^{1/2}\|D_x^s f\|_{L^2_xL^2_T}. \qquad (2.18)$$

As in (2.16) and (2.17), the use of the estimate (2.9) yields for $s > 3/8$ that

$$\begin{aligned}
&\Big\|\partial_x\int_0^t S(t-t')f(t')dt'\Big\|_{L^{40/3}_xL^{20/7}_T} \le C\int_0^T \|D_x^{3/8}S(-t')f(t')\|_{L^2_x}dt'\\
&\le C\int_0^T \|D_x^{3/8}f(t')\|_{L^2_x}dt' \le C\int_0^T \|f(t')\|_{H^s(\mathbb{R})}dt' \qquad (2.19)\\
&\le CT^{1/2}\big[\|D_x^s f\|_{L^2_xL^2_T} + \|f\|_{L^2_xL^2_T}\big].
\end{aligned}$$

Now, we use the estimate (2.10) from Lemma 2.4, with $\alpha = 0$ and $\theta = 9/10$, and Hölder's inequality to obtain

$$\begin{aligned}
\Big\|D_x^s\int_0^t S(t-t')f(t')dt'\Big\|_{L^{20/3}_TL^{20}_x} &\le C\int_0^T \|S(-t')D_x^s f(t')\|_{L^2_x}dt' \qquad (2.20)\\
&\le CT^{1/2}\|D_x^s f\|_{L^2_xL^2_T}.
\end{aligned}$$

Finally, using the maximal function estimate (2.6) and the argument as in (2.19) we obtain, for $s > 3/8$,

$$\Big\| \int_0^t S(t-t')f(t')dt' \Big\|_{L_x^8 L_T^\infty} \le C \int_0^T \|S(-t')f(t')\|_{H^{\frac{3}{8}+}(\mathbb{R})} dt' \le CT^{1/2}\big[\|D_x^s f\|_{L_x^2 L_T^2} + \|f\|_{L_x^2 L_T^2}\big]. \tag{2.21}$$

Combining all these estimates we conclude the proof of the lemma. □

Note that it is the estimates (2.19) and (2.21) that give the restriction on s in the local well-posedness result in Theorem 1.1.

Lemma 2.9. *The following estimates hold:*

$$\|\partial_x(u^5)\|_{Y_T^s} \le C\|u\|_{Z_T^s}^5, \tag{2.22}$$

$$\|\partial_x(u^5 - v^5)\|_{Y_T^s} \le F(\|u\|_{Z_T^s}, \|v\|_{Z_T^s})\|u-v\|_{Z_T^s}, \tag{2.23}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally bounded function.

Proof. Using Hölder's inequality, we get

$$\|\partial_x(u^5)\|_{L_x^2 L_T^2} \le C\|u^4\|_{L_x^2 L_T^\infty}\|\partial_x u\|_{L_x^\infty L_T^2} \le C\|u\|_{L_x^8 L_T^\infty}^4 \|\partial_x u\|_{L_x^\infty L_T^2}. \tag{2.24}$$

Similarly, using Leibniz's rule for fractional derivatives (2.11) twice, Hölder's inequality, and the fact that $20 > 20/3$, one obtains

$$\begin{aligned} \|D_x^s\partial_x(u^5)\|_{L_x^2 L_T^2} &\le \|D_x^s(u^4\partial_x u) - u^4 D_x^s \partial_x u - \partial_x u D_x^s(u^4)\|_{L_x^2 L_T^2} \\ &\quad + \|u^4 D_x^s\partial_x u\|_{L_x^2 L_T^2} + \|\partial_x u D_x^s(u^4)\|_{L_x^2 L_T^2} \\ &\le C\big[\|\partial_x u\|_{L_x^{40/3} L_T^{20/7}} \|D_x^s(u^4)\|_{L_x^{40/17} L_T^{20/3}} + \|u^4\|_{L_x^2 L_T^\infty}\|D_x^s\partial_x u\|_{L_x^\infty L_T^2}\big] \\ &\le C\big[\|\partial_x u\|_{L_x^{40/3} L_T^{20/7}} \|u^3\|_{L_x^{8/3} L_T^\infty}\|D_x^s u\|_{L_x^{20} L_T^{20/3}} + \|u\|_{L_x^8 L_T^\infty}^4 \|D_x^s\partial_x u\|_{L_x^\infty L_T^2}\big] \\ &\le C\big[\|\partial_x u\|_{L_x^{40/3} L_T^{20/7}} \|u\|_{L_x^8 L_T^\infty}^3 \|D_x^s u\|_{L_T^{20/3} L_x^{20}} + \|u\|_{L_x^8 L_T^\infty}^4 \|D_x^s\partial_x u\|_{L_x^\infty L_T^2}\big]. \end{aligned} \tag{2.25}$$

In view of the definitions of Z_T^s-norm and Y_T^s-norm, the estimates (2.24) and (2.25) yield the required result (2.22).

To prove (2.23), observe that

$$|\partial_x(u^5 - v^5)| \le C\big[(|u|^4 + |v|^4)|\partial_x(u-v)| + (|\partial_x u| + |\partial_x v|)(|u|^3 + |v|^3)|u-v|\big].$$

Now, with the same argument as in (2.24) and (2.25) we get the desired estimate (2.23). □

With the same ideas as in Lemmas 2.8 and 2.9, we can obtain the corresponding estimates for the spaces X_T and Y_T. We state these results in the following lemma.

Lemma 2.10. *Let X_T and Y_T be the spaces defined earlier and S be the unitary group associated with the operator ∂_x^3; then we have*

$$\|S(t)u_0\|_{X_T} \le C_0\|u_0\|_{H^1(\mathbb{R})}, \tag{2.26}$$

$$\Big\| \int_0^t S(t-t')f(t')dt'\Big\|_{X_T} \le CT^{1/2}\|f\|_{Y_T}. \tag{2.27}$$

$$\|\partial_x(u^5)\|_{Y_T} \le C\|u\|_{X_T}^5. \tag{2.28}$$

$$\|\partial_x(u^5-v^5)\|_{Y_T} \le F(\|u\|_{X_T}, \|v\|_{X_T})\|u-v\|_{X_T}, \tag{2.29}$$

where $F:\mathbb{R}^2\to\mathbb{R}$ is a locally bounded function.

Finally, before leaving this section, we record the following result from [5].

Lemma 2.11. *Let $T>0$, $1\le p<q\le\infty$, and $A,B\ge 0$. If $f\in L^q(0,T)$ satisfies*

$$\|f\|_{L^q_{(0,t)}} \le A + B\|f\|_{L^p_{(0,t)}}, \tag{2.30}$$

for all $t\in(0,T)$, then there exists a constant $K=K(B,p,q,T)$ such that

$$\|f\|_{L^q_{(0,T)}} \le KA. \tag{2.31}$$

3. Proof of the well-posedness results

We start this section by proving the well-posedness results for the IVPs (1.14).

Proof of Theorem 1.1. For $a>0$ and $s\in(3/8,1)$, consider a ball in Z_T^s defined by $\mathcal{B}_a^T=\{u\in C([0,T];Z_T^s):\|u\|_{Z_T^s}<a\}$. Our aim is to show that there exist $a>0$ and $T>0$ such that the application Φ defined by

$$\Phi(u) := S(t)\phi - \int_0^t S(t-t')h(t')\partial_x(u^5)(t')dt' \tag{3.1}$$

maps $\mathcal{B}_a^T$ into $\mathcal{B}_a^T$ and is a contraction.

Using the estimates (2.14), (2.15), and (2.22), we obtain

$$\begin{aligned}\|\Phi\|_{Z_T^s} &\le C_0\|\phi\|_{H^s(\mathbb{R})} + CT^{1/2}\|h\|_{L_t^\infty}\|\partial_x(u^5)\|_{Y_T^s} \\ &\le C_0\|\phi\|_{H^s(\mathbb{R})} + CT^{1/2}\|h\|_{L_t^\infty}\|u\|_{Z_T^s}^5.\end{aligned} \tag{3.2}$$

Hence, for $u \in \mathcal{B}_a^T$,

$$\|\Phi\|_{Z_T^s} \leq C_0\|\phi\|_{H^s(\mathbb{R})} + CT^{1/2}\|h\|_{L_t^\infty} a^5. \tag{3.3}$$

Now, choose $a = 2C_0\|\phi\|_{H^s(\mathbb{R})}$ and T such that $CT^{1/2}\|h\|_{L_t^\infty} a^4 < 1/2$. With these choices we get from (3.3) that

$$\|\Phi\|_{Z_T^s} \leq \frac{a}{2} + \frac{a}{2}.$$

Therefore, Φ maps $\mathcal{B}_a^T$ into $\mathcal{B}_a^T$.

With a similar argument, using (2.23) one can prove that Φ is a contraction. The rest of the proof follows a standard argument.

From the choice of a and T it is clear that the local existence time is given by

$$T \leq C\|\phi\|_{H^s(\mathbb{R})}^{-8}\|h\|_{L_t^\infty}^{-2},$$

and the solution satisfies the following bound: $\|u\|_{Z_T^s} \leq C\|\phi\|_{H^s(\mathbb{R})}$. □

Now we move to providing alternative proofs for the local well-posedness results in $H^1(\mathbb{R})$ that we will use to prove the main results of this work.

Proof of Theorem 1.2. We use the estimates (2.26), (2.27), and (2.28) from Lemma 2.10 to prove that the application Φ defined in (3.1) is a contraction in a ball $\mathcal{B}_a^T = \{u \in C([0,T]; X_T(\mathbb{R})) : \|u\|_{X_T} < a\}$ for some $a > 0$. As in the proof of Theorem 1.1, we can do it by choosing $a = 2C_0\|\phi\|_{H^1(\mathbb{R})}$ and $T > 0$ such that $CT^{1/2}\|h\|_{L_t^\infty} a^4 < 1/2$. The rest of the proof follows a standard argument. □

Proof of Theorem 1.4. As in the proof of Theorem 1.2, this theorem will also be proved by considering the integral equation associated with the IVP (1.2),

$$u(t) = S(t)\phi - \int_0^t S(t-t')g(\omega(t'+t_0))\partial_x(u^5)(t')\,dt', \tag{3.4}$$

and using the contraction-mapping principle.

First of all, notice that the periodic function g is bounded; say $\|g\|_{L_t^\infty} \leq A$, for some positive constant A. Since the norms involved in the space Y_T permit us to take out the $\|g\|_{L_t^\infty}$-norm as a coefficient, the proof of this theorem follows exactly the same argument as in the proof of Theorem 1.2. Moreover, as the initial data ϕ is the same, the choice of the radius a of the ball is exactly the same. However, to complete the contraction-mapping argument, we must select $T > 0$ such that $C\|g\|_{L_t^\infty} T^{1/2} a^4 < \frac{1}{2}$, which implies that the existence T is given by

$$T = T(\|g\|_{L_t^\infty}, \|\phi\|_{H^1(\mathbb{R})}) = \frac{C}{\|g\|_{L_t^\infty}^2 \|\phi\|_{H^1(\mathbb{R})}^8}. \tag{3.5}$$

Furthermore, from the proof, one can obtain

$$\|u\|_{X_T} \leq C\|\phi\|_{H^1(\mathbb{R})}. \tag{3.6}$$

The following lemmas will be used in the proof of the main results.

Lemma 3.1. *Let X_T and Y_T be spaces as defined in* (1.11) *and* (1.13)*. Let $f \in Y_T$; then we have the following convergence:*

$$\int_0^t g(\omega(t'+t_0))S(t-t')f(t')dt' \to m(g)\int_0^t S(t-t')f(t')dt', \tag{3.7}$$

whenever $|\omega| \to \infty$, in the X_T-norm.

Proof. Using the linear estimates and the fact that g is bounded, we have

$$\Big\| \int_0^t g(\omega(t'+t_0))S(t-t')f(t')dt' \Big\|_{X_T} \leq CT^{1/2}A\|f\|_{Y_T}. \tag{3.8}$$

So, by a density argument, it is enough to prove (3.7) considering $f \in C_c^1(\mathbb{R}; \mathcal{S}(\mathbb{R}))$. Define the function $\psi(t) := g(t) - m(g)$ and $\Psi(t) := \int_0^t \psi(t')dt'$; then

$$\frac{d}{dt'}\Psi(\omega(t'+t_0)) = \omega\psi(\omega(t'+t_0)).$$

Integrating by parts, we obtain

$$\begin{aligned}\int_0^t \psi(\omega(t'+t_0))S(t-t')f(t')dt' &= \frac{1}{\omega}\Psi(\omega(t+t_0))f(t) - \frac{1}{\omega}\Psi(\omega t_0)S(t)f(0)\\ &- \frac{1}{\omega}\int_0^t \Psi(\omega(t'+t_0))S(t-t')[f_t(t') + \partial_x^3 f(t')]dt'. \end{aligned}\tag{3.9}$$

Using the triangle inequality, linear estimates, and Lemma 2.8 we get

$$\begin{aligned}&\Big\| \int_0^t \psi(\omega(t'+t_0))S(t-t')f(t')dt' \Big\|_{X_T}\\ &\leq \frac{C}{\omega}\|\Psi\|_{L^\infty}\big[\|f\|_{X_T} + \|f(0)\|_{H^1} + T^{1/2}\|f_t + \partial_x^3 f\|_{Y_T}\big].\end{aligned}\tag{3.10}$$

Finally, letting $|\omega| \to \infty$, we obtain the desired result. □

Lemma 3.2. *Let the initial data* $\phi \in H^1(\mathbb{R})$. *Let* u_{ω,t_0} *be the maximal solution of the IVP* (1.1). *Suppose* U *is the maximal solution of the IVP* (1.9) *defined in* $[0, S_{max})$. *Let* $0 < T < S_{max}$ *and let* u_{ω,t_0} *exist in* $[0,T]$ *for* $|\omega|$ *large, and suppose that*

$$\limsup_{|\omega|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L_T^\infty H^1(\mathbb{R})} < \infty, \tag{3.11}$$

and

$$\limsup_{|\omega|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L_x^4 L_T^\infty} < \infty. \tag{3.12}$$

Then, for all $t \in [0,T]$,

$$\sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0} - U\|_{X_T} \to 0, \quad \text{as } |\omega| \to \infty. \tag{3.13}$$

In particular, $u_{\omega,t_0} \to U$ *as* $|\omega| \to \infty$, *in* $H^1(\mathbb{R})$.

Proof. Since u_{ω,t_0} and U have the same initial data ϕ, from Duhamel's formula we have

$$\begin{aligned} u_{\omega,t_0} - U &= \int_0^t g(\omega(t'+t_0))S(t-t')\partial_x(u_{\omega,t_0}^5)dt' - m(g)\int_0^t S(t-t')\partial_x(U^5)dt' \\ &= \int_0^t g(\omega(t'+t_0))S(t-t')\partial_x(u_{\omega,t_0}^5 - U^5)dt' \qquad (3.14) \\ &\quad + \int_0^t [g(\omega(t'+t_0)) - m(g)]S(t-t')\partial_x(U^5)dt' =: I_1 + I_2. \end{aligned}$$

We note that

$$|u^5 - v^5| \le C(|u|^4 + |v|^4)|u - v| \tag{3.15}$$

and

$$|\partial_x(u^5 - v^5)| \le C\big[(|u|^4 + |v|^4)|\partial_x(u-v)| + (|\partial_x u| + |\partial_x v|)(|u|^3 + |v|^3)|u - v|\big]. \tag{3.16}$$

Let $\|g\|_{L_T^\infty} \le A$. Use of (2.2), (3.15), Hölder's inequality, and the assumptions (3.11) and (3.12) yield

$$\begin{aligned} &\|I_1\|_{L_T^\infty L_x^2} \le C\|g\|_{L_T^\infty}\|u_{\omega,t_0}^5 - U^5\|_{L_x^1 L_T^2} \qquad (3.17) \\ &\le CA\|u_{\omega,t_0}^4(u_{\omega,t_0} - U)\|_{L_x^1 L_T^2} + \|U^4(u_{\omega,t_0} - U)\|_{L_x^1 L_T^2} \\ &\le CA\|u_{\omega,t_0}^4\|_{L_x^2 L_T^\infty}\|u_{\omega,t_0} - U\|_{L_x^2 L_T^2} + \|U^4\|_{L_x^2 L_T^\infty}\|u_{\omega,t_0} - U\|_{L_x^2 L_T^2} \\ &\le CA\Big[\|u_{\omega,t_0}^2\|_{L_x^\infty L_T^\infty}\|u_{\omega,t_0}^2\|_{L_x^2 L_T^\infty} + \|U^2\|_{L_x^\infty L_T^\infty}\|U^2\|_{L_x^2 L_T^\infty}\Big]\|u_{\omega,t_0} - U\|_{L_T^2 L_x^2} \\ &\le CA\Big[\|u_{\omega,t_0}\|_{L_T^\infty H^1(\mathbb{R})}^2\|u_{\omega,t_0}\|_{L_x^4 L_T^\infty}^2 + \|U\|_{L_T^\infty H^1(\mathbb{R})}^2\|U\|_{L_x^4 L_T^\infty}^2\Big]\|u_{\omega,t_0} - U\|_{L_T^2 L_x^2} \end{aligned}$$

$\leq CA\|u_{\omega,t_0} - U\|_{L^2_T L^2_x}.$

Again, using (2.2) and (3.16), one can obtain

$$\|\partial_x I_1\|_{L^\infty_T L^2_x} \leq CA\|\partial_x(u^5_{\omega,t_0} - U^5)\|_{L^1_x L^2_T} \tag{3.18}$$
$$\leq CA\Big[\|(|u_{\omega,t_0}|^4 + |U|^4)\partial_x(u_{\omega,t_0} - U)\|_{L^1_x L^2_T}$$
$$+ \|(|\partial_x u_{\omega,t_0}| + |\partial_x U|)(|u_{\omega,t_0}|^3 + |U|^3)(u_{\omega,t_0} - U)\|_{L^1_x L^2_T}\Big] =: CA[J_1 + J_2].$$

With the same argument as in (3.17),

$$J_1 \leq C\|\partial_x(u_{\omega,t_0} - U)\|_{L^2_T L^2_x}. \tag{3.19}$$

Now we move to estimate the first term, $\|u^3_{\omega,t_0}\partial_x u_{\omega,t_0}(u_{\omega,t_0} - U)\|_{L^1_x L^2_T}$ in J_2; the estimates for the other terms are similar. We have

$$\|u^3_{\omega,t_0}\partial_x u_{\omega,t_0}(u_{\omega,t_0} - U)\|_{L^1_x L^2_T} \tag{3.20}$$
$$\leq C\|u^2_{\omega,t_0}\|_{L^2_x L^\infty_T}\|u_{\omega,t_0}\partial_x u_{\omega,t_0}(u_{\omega,t_0} - U)\|_{L^2_x L^2_T}$$
$$\leq C\|u_{\omega,t_0}\|^2_{L^4_x L^\infty_T}\|u_{\omega,t_0}\partial_x u_{\omega,t_0}(u_{\omega,t_0} - U)\|_{L^2_T L^2_x}$$
$$\leq C\|u_{\omega,t_0}\|^2_{L^4_x L^\infty_T}\|u_{\omega,t_0}\|_{L^\infty_T L^\infty_x}\|\partial_x u_{\omega,t_0}\|_{L^\infty_T L^2_x}\|(u_{\omega,t_0} - U)\|_{L^2_T L^\infty_x}$$
$$\leq C\|u_{\omega,t_0}\|^2_{L^4_x L^\infty_T}\|u_{\omega,t_0}\|^2_{L^\infty_T H^1(\mathbb{R})}\|(u_{\omega,t_0} - U)\|_{L^2_T H^1(\mathbb{R})}$$
$$\leq C\|(u_{\omega,t_0} - U)\|_{L^2_T H^1(\mathbb{R})}.$$

Inserting (3.19) and (3.20) in (3.18), we get

$$\|\partial_x I_1\|_{L^\infty_T L^2_x} \leq CA\|(u_{\omega,t_0} - U)\|_{L^2_T H^1(\mathbb{R})}. \tag{3.21}$$

Combining (3.17) and (3.21), we obtain

$$\|I_1\|_{L^\infty_T H^1(\mathbb{R})} \leq CA\|(u_{\omega,t_0} - U)\|_{L^2_T H^1(\mathbb{R})}. \tag{3.22}$$

From Lemma 3.1, we have

$$\|I_2\|_{L^\infty_T H^1(\mathbb{R})} \leq C_\omega \to 0, \quad \text{as } |\omega| \to \infty. \tag{3.23}$$

Therefore, we have

$$\|u_{\omega,t_0} - U\|_{L^\infty_T H^1(\mathbb{R})} \leq CA\|(u_{\omega,t_0} - U)\|_{L^2_T H^1(\mathbb{R})} + C_\omega. \tag{3.24}$$

Applying Lemma 2.11 in (3.24), we get

$$\|u_{\omega,t_0} - U\|_{L^\infty_T H^1(\mathbb{R})} \leq KC_\omega \to 0, \qquad as\ |\omega| \to \infty. \tag{3.25}$$

From (3.24) and (3.25), it is easy to conclude that

$$\|(u_{\omega,t_0} - U)\|_{L^2_T H^1(\mathbb{R})} \to 0, \quad as\ |\omega| \to \infty. \tag{3.26}$$

Now, we move to estimate the other norms involved in the definition of X_T. Let

$$\begin{aligned}\mathfrak{L}_1 &:= \|\partial_x(u_{\omega,t_0} - U)\|_{L^\infty_x L^2_T} + \|\partial_x^2(u_{\omega,t_0} - U)\|_{L^\infty_x L^2_T} \\ &\quad + \|u_{\omega,t_0} - U\|_{L^5_x L^{10}_T} + \|D_x(u_{\omega,t_0} - U)\|_{L^5_x L^{10}_T} \\ \mathfrak{L}_2 &:= \|\partial_x(u_{\omega,t_0} - U)\|_{L^{20}_x L^{5/2}_T} + \|u_{\omega,t_0} - U\|_{L^4_x L^\infty_T}.\end{aligned}$$

Use of (2.2), (2.3), the estimate (2.13) from Proposition 2.7 with admissible triples $(p_1, q_1, \alpha_1) = (5, 10, 0)$, and $(p_2, q_2, \alpha_2) = (\infty, 2, 1)$ in (3.14) yields

$$\mathfrak{L}_1 \le CA\|\partial_x(u^5_{\omega,t_0} - U^5)\|_{L^1_x L^2_T} + CA\|u^5_{\omega,t_0} - U^5\|_{L^1_x L^2_T} + \|I_2\|_{X_T}. \tag{3.27}$$

Therefore, with the same argument as in (3.17)–(3.21), we can obtain

$$\mathfrak{L}_1 \le CA\|u_{\omega,t_0} - U\|_{L^2_T H^1} + C_\omega. \tag{3.28}$$

Hence, using Lemma 3.2 and (3.26) we get from (3.28) that

$$\mathfrak{L}_1 \overset{|\omega|\to\infty}{\to} 0. \tag{3.29}$$

Finally, to estimate $\mathfrak{L}_2$ we use Proposition 2.7 with admissible triples $(p_1, q_1, \alpha_1) = (20, 5/2, 3/4)$ and $(p_2, q_2, \alpha_2) = (20/3, 5, 1/4)$, to get

$$\left\|\partial_x \int_0^t S(t-t')f(\cdot,t')dt'\right\|_{L^{20}_x L^{5/2}_T} \le C\|f\|_{L^{20/17}_x L^{5/4}_T}, \tag{3.30}$$

and with admissible triples $(p_1, q_1, \alpha_1) = (4, \infty, -1/4)$ and $(p_2, q_2, \alpha_2) = (20/3, 5, 1/4)$, to get

$$\left\|\int_0^t S(t-t')f(\cdot,t')dt'\right\|_{L^4_x L^\infty_T} \le C\|f\|_{L^{20/17}_x L^{5/4}_T}. \tag{3.31}$$

Using (3.30), (3.31), and the definition of X_T, we get from (3.14) that

$$\mathfrak{L}_2 \le CA\|\partial_x(u^5_{\omega,t_0} - U^5)\|_{L^{20/17}_x L^{5/4}_T} + \|I_2\|_{X_T}. \tag{3.32}$$

Using (3.16), we can obtain

$$\begin{aligned}\|\partial_x(u^5_{\omega,t_0} - U^5)\|_{L^{20/17}_x L^{5/4}_T} &\le C\Big[\|(|u_{\omega,t_0}|^4 + |U|^4)\partial_x(u_{\omega,t_0} - U)\|_{L^{20/17}_x L^{5/4}_T} \\ &\quad + \|(|\partial_x u_{\omega,t_0}| + |\partial_x U|)(|u_{\omega,t_0}|^3 + |U|^3)(u_{\omega,t_0} - U)\|_{L^{20/17}_x L^{5/4}_T}\Big] \\ &=: C[\tilde{J}_1 + \tilde{J}_2].\end{aligned} \tag{3.33}$$

Hölder's inequality, the fact that $20/13 > 10/7$, Sobolev immersion, and the assumption (3.11) imply that

$$\begin{aligned}\tilde{J}_1 &\le C\|\partial_x(u_{\omega,t_0}-U)\|_{L^5_xL^{10}_T}\{\|u^4_{\omega,t_0}\|_{L^{20/13}_xL^{10/7}_T}+\|U^4\|_{L^{20/13}_xL^{10/7}_T}\}\\ &\le C\|\partial_x(u_{\omega,t_0}-U)\|_{L^5_xL^{10}_T}\{\|u^4_{\omega,t_0}\|_{L^{10/7}_TL^{20/13}_x}+\|U^4\|_{L^{10/7}_TL^{20/13}_x}\}\\ &\le C\|\partial_x(u_{\omega,t_0}-U)\|_{L^5_xL^{10}_T}\,T^{7/10}\{\|u_{\omega,t_0}\|^4_{L^\infty_TH^1}+\|U\|^4_{L^\infty_TH^1}\}\\ &\le C\,T^{7/10}\|\partial_x(u_{\omega,t_0}-U)\|_{L^5_xL^{10}_T}.\end{aligned} \tag{3.34}$$

As in (3.18), we give details in estimating the first term, $\|u^3_{\omega,t_0}\partial_xu_{\omega,t_0}(u_{\omega,t_0}-U)\|_{L^{20/17}_xL^{5/4}_T}$ in $\tilde{J}_2$; the estimates for the other terms are similar. Here too, Hölder's inequality, the fact that $20/3 > 5$, Sobolev immersion, and the assumption (3.11) yield

$$\begin{aligned}&\|u^3_{\omega,t_0}\partial_xu_{\omega,t_0}(u_{\omega,t_0}-U)\|_{L^{20/17}_xL^{5/4}_T}\\ &\le C\|u^3_{\omega,t_0}\|_{L^{20/3}_xL^5_T}\|\partial_xu_{\omega,t_0}\|_{L^2_xL^2_T}\|u_{\omega,t_0}-U\|_{L^5_xL^{10}_T}\\ &\le C\|u^3_{\omega,t_0}\|_{L^5_TL^{20/3}_x}\|\partial_xu_{\omega,t_0}\|_{L^2_TL^2_x}\|u_{\omega,t_0}-U\|_{L^5_xL^{10}_T}\\ &\le C\,T^{7/10}\|u_{\omega,t_0}\|^4_{L^\infty_TH^1}\|u_{\omega,t_0}-U\|_{L^5_xL^{10}_T}\\ &\le C\,T^{7/10}\|u_{\omega,t_0}-U\|_{L^5_xL^{10}_T}.\end{aligned} \tag{3.35}$$

In view of (3.33), (3.34), and (3.35), we get from (3.32) that

$$\mathfrak{L}_2 \le CA\,T^{7/10}\{\|\partial_x(u_{\omega,t_0}-U)\|_{L^5_xL^{10}_T}+\|u_{\omega,t_0}-U\|_{L^5_xL^{10}_T}\}+C_\omega. \tag{3.36}$$

Therefore, Lemma 3.2 and (3.29) imply

$$\mathfrak{L}_2 \overset{|\omega|\to\infty}{\to} 0. \tag{3.37}$$

Now, the proof of the lemma follows by combining (3.25), (3.29), and (3.37). □

In what follows, we consider the critical KdV equation (1.14) with a more general time-dependent coefficient on the nonlinearity.

Proposition 3.3. *Given any $A > 0$, there exist $\epsilon = \epsilon(A)$ and $B > 0$ such that if $\|h\|_{L^\infty} \le A$ and if $\phi \in H^1(\mathbb{R})$ satisfies*

$$\|S(t)\phi\|_{L^5_xL^{10}_t} \le \epsilon, \tag{3.38}$$

then the corresponding solution u of (1.14) is global and satisfies

$$\|u\|_{L^5_xL^{10}_t} \le 2\,\|S(t)\phi\|_{L^5_xL^{10}_t}, \tag{3.39}$$

$$\|u\|_{X_t} \le B\|\phi\|_{H^1(\mathbb{R})}. \tag{3.40}$$

Conversely, if the solution u of (1.14) *is global and satisfies*

$$\|u\|_{L^5_x L^{10}_t} \le \epsilon, \tag{3.41}$$

then

$$\|S(t)\phi\|_{L^5_x L^{10}_t} \le 2\|u\|_{L^5_x L^{10}_t}. \tag{3.42}$$

Proof. Since $\|h\|_{L^\infty_t} \le A$, as in Theorem 1.4 we can prove local well-posedness for the IVP (1.14) in $H^1(\mathbb{R})$ with time of existence

$$T = T(\|\phi\|_{H^1(\mathbb{R})}, \|h\|_{L^\infty}).$$

Let $u \in C([0, T_{max}); H^1(\mathbb{R}))$ be the maximal solution of the IVP (1.14). For $0 \le t < T_{max}$, we have that

$$u(t) = S(t)\phi + w(t), \tag{3.43}$$

where

$$w(t) = -\int_0^t S(t-t')h(t')\partial_x(u^5)(t')\,dt'.$$

Using (2.13) from Proposition 2.7 for admissible triples $(5, 10, 0)$ and $(\infty,2,1)$, we obtain

$$\|w\|_{L^5_x L^{10}_T} \le CA\|u^5\|_{L^1_x L^2_T} = CA\|u\|^5_{L^5_x L^{10}_T}. \tag{3.44}$$

From (3.43) and (3.44) it follows that

$$|\,\|u\|_{L^5_x L^{10}_T} - \|S(t)\phi\|_{L^5_x L^{10}_T}| \le CA\|u\|^5_{L^5_x L^{10}_T}. \tag{3.45}$$

Thus, for all $T \in (0, T_{max})$ one has

$$\|u\|_{L^5_x L^{10}_T} \le \epsilon + CA\|u\|^5_{L^5_x L^{10}_T}. \tag{3.46}$$

Choose $\epsilon = \epsilon(A)$ such that

$$CA(2\epsilon)^4 < 1/2, \tag{3.47}$$

and suppose that the estimate (3.38) holds. As the norm is continuous on T and vanishes at $T = 0$, using a continuity argument, the estimate (3.46) and the choice of ϵ in (3.47) imply that

$$\|u\|_{L^5_x L^{10}_{T_{max}}} \le 2\epsilon. \tag{3.48}$$

Moreover, from (3.45),

$$\begin{aligned} \|u\|_{L^5_x L^{10}_{T_{max}}} &\le \|S(t)\phi\|_{L^5_x L^{10}_{T_{max}}} + cA\|u\|^5_{L^5_x L^{10}_{T_{max}}} \\ &\le \|S(t)\phi\|_{L^5_x L^{10}_{T_{max}}} + CA(2\epsilon)^4\|u\|_{L^5_x L^{10}_{T_{max}}}. \end{aligned} \tag{3.49}$$

Therefore, with the choice of ϵ satisfying (3.47), the estimate (3.49) yields

$$\|u\|_{L^5_xL^{10}_{T_{max}}} \leq 2\|S(t)\phi\|_{L^5_xL^{10}_{T_{max}}}. \tag{3.50}$$

In what follows, we will show that $T_{max} = \infty$. The inequalities (2.2), (2.3), (2.13) with admissible triples $(5, 10, 0)$ and $(\infty, 2, 1)$, and Hölder's inequality imply

$$\begin{aligned} &\|w\|_{L^\infty_T H^1} + \|\partial_x w\|_{L^\infty_x L^2_T} + \|\partial_x^2 w\|_{L^\infty_x L^2_T} + \|w\|_{L^5_xL^{10}_T} + \|\partial_x f\|_{L^5_xL^{10}_T} \\ &\leq CA\|u\|^4_{L^5_xL^{10}_T}\|u\|_{X_T}. \end{aligned} \tag{3.51}$$

Now using (3.30), (3.31), and Hölder's inequality, we have

$$\begin{aligned} &\|\partial_x w\|_{L^{20}_x L^{5/2}_T} + \|w\|_{L^4_x L^\infty_T} \leq CA\|\partial_x u^5\|_{L^{20/17}_x L^{5/4}_T} \\ &\leq CA\|u^4\|_{L^{5/4}_x L^{5/2}_T}\|\partial_x u\|_{L^{20}_x L^{5/2}_T} \leq CA\|u\|^4_{L^5_xL^{10}_T}\|\partial_x u\|_{L^{20}_x L^{5/2}_T}. \end{aligned} \tag{3.52}$$

Combining (3.51) and (3.52), we obtain

$$\|w\|_{X_T} \leq CA\|u\|^4_{L^5_xL^{10}_T}\|u\|_{X_T}. \tag{3.53}$$

This estimate with (3.47) and (3.48) gives

$$\|w\|_{X_T} \leq CA(2\epsilon)^4\|u\|_{X_T} < \frac{1}{2}\|u\|_{X_T}. \tag{3.54}$$

Using (3.43) we obtain

$$\|u\|_{X_T} \leq \|S(t)\phi\|_{X_T} + \|w\|_{X_T} \leq C\|\phi\|_{H^1(\mathbb{R})} + \frac{1}{2}\|u\|_{X_T}, \tag{3.55}$$

for all $T \in (0, T_{max})$. Therefore, we have

$$\|u\|_{X_{T_{max}}} \leq 2\,C\|\phi\|_{H^1(\mathbb{R})}. \tag{3.56}$$

Hence, from the definition of $\|u\|_{X_{T_{max}}}$, we have that

$$\|u\|_{L^\infty_{T_{max}}H^1(\mathbb{R})} \leq C\|u(0)\|_{H^1(\mathbb{R})}. \tag{3.57}$$

Now, combining the local existence from Theorem 1.4 and the estimate (3.57), the blow-up alternative implies that $T_{max} = \infty$. Finally, the estimates (3.50) and (3.56) yield (3.39) and (3.40) respectively with $B = 2C$.

Conversely, let $T_{max} = \infty$ and (3.41) hold. With an argument similar to (3.45), we can get

$$|\,\|u\|_{L^5_xL^{10}_t} - \|S(t)\phi\|_{L^5_xL^{10}_t}| \leq CA\|u\|^5_{L^5_xL^{10}_t}. \tag{3.58}$$

Thus, from (3.58) in view of (3.41) and (3.47), one has

$$\|S(t)\phi\|_{L^5_xL^{10}_t} \leq \|u\|_{L^5_xL^{10}_t} + CA\epsilon^4\|u\|_{L^5_xL^{10}_t} \leq 2\|u\|_{L^5_xL^{10}_t}. \tag{3.59}$$

Corollary 3.4. *Let $h \in L^\infty(\mathbb{R})$ satisfy $\|h\|_{L^\infty} \le A$ and ϵ and B be as in Proposition* 3.3. *Given $\phi \in H^1(\mathbb{R})$, let u be the solution of the IVP* (1.14) *defined on the maximal interval $[0, T_{max})$. If there exists $T \in (0, T_{max})$ such that $\|S(t)u(T)\|_{L^5_x L^{10}_t} \le \epsilon$, then the solution u is global. Moreover,*

$$\|u\|_{L^5_x L^{10}_{(T,\infty)}} \le 2\epsilon \quad \textit{and} \quad \|u\|_{X_{(T,\infty)}} \le B\|u(T)\|_{H^1(\mathbb{R})}.$$

Proof. The proof follows by using a standard extension argument. For details we refer to the proof of Corollary 2.4 in [5]. □

4. Proof of the main results

This section is devoted to providing the proofs of the main results of this work. Lemma 3.2 and the local existence Theorem 1.4 are the main tools used in the proof of Theorem 1.5, while Proposition 3.3 and Theorem 1.5 are fundamental in proving Theorem 1.6. Once we have results from Lemma 3.2 and Proposition 3.3, the idea used to complete the proofs of the main theorems is similar to that in [5].

Proof of Theorem 1.5. Let $A = \|g\|_{L^\infty}$ and $T \in (0, S_{\max})$ be fixed and set

$$M_0 = 2 \sup_{t\in[0,T]} \|U(t)\|_{H^1(\mathbb{R})}. \tag{4.1}$$

In particular, for $t = 0$, (4.1) gives $\|\phi\|_{H^1(\mathbb{R})} \le M_0/2$. From Theorem 1.4, we have that for all $\omega, t_0 \in \mathbb{R}$, u_{ω,t_0} exists on $[0, \delta]$. Using (3.5) we have that the existence time δ is given by

$$\delta = \frac{C}{A^2 M_0^8}. \tag{4.2}$$

Moreover, from (3.6)

$$\limsup_{|w|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L^\infty_\delta H^1(\mathbb{R})} \le C\|\phi\|_{H^1(\mathbb{R})} \tag{4.3}$$

and

$$\limsup_{|w|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L^4_x L^\infty_\delta H^1(\mathbb{R})} \le C\|\phi\|_{H^1(\mathbb{R})}. \tag{4.4}$$

From Lemma 3.2, we have that $\sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0} - U\|_{X_T} \overset{|w|\to\infty}{\to} 0$; in particular,

$$\sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}(\delta) - U(\delta)\|_{H^1(\mathbb{R})} \overset{|w|\to\infty}{\to} 0. \tag{4.5}$$

Combining (4.1) and (4.5), for $|w|$ sufficiently large we deduce that

$$\sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}(\delta)\|_{H^1(\mathbb{R})} \leq M_0. \tag{4.6}$$

We suppose $\delta \leq T$; otherwise we are done. Using Theorem 1.4 we can extend the solution u_{ω,t_0} (as in the proof of Corollary 3.4) on the interval $[0, 2\delta]$, with $\|\tilde{u}_{\omega,t_0}\|_{L_t^\infty(0,\delta)H^1(\mathbb{R})} \leq C\|\tilde{u}_{\omega,t_0}(0)\|_{H^1(\mathbb{R})}$, where $\tilde{u}_{\omega,t_0}(t) = u_{\omega,t_0}(t+\delta)$; i.e., $\|u_{\omega,t_0}\|_{L_t^\infty(\delta,2\delta)H^1(\mathbb{R})} \leq C\|u_{\omega,t_0}(\delta)\|_{H^1(\mathbb{R})} \leq C^2\|\phi\|_{H^1(\mathbb{R})}$. Therefore, (4.3) gives

$$\limsup_{|w|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L_t^\infty(0,2\delta)H^1(\mathbb{R})} \leq C(1+C)\|\phi\|_{H^1(\mathbb{R})}. \tag{4.7}$$

Similarly, from (4.4),

$$\limsup_{|w|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L_x^4 L_{2\delta}^\infty H^1(\mathbb{R})} \leq C(1+C)\|\phi\|_{H^1(\mathbb{R})}. \tag{4.8}$$

So, we can again apply Lemma 3.2. Iterating this argument a finite number of times with the same time of existence in each iteration, we see that

$$\limsup_{|w|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L_T^\infty H^1(\mathbb{R})} \leq C\|\phi\|_{H^1(\mathbb{R})}$$

and

$$\limsup_{|w|\to\infty} \sup_{t_0\in\mathbb{R}} \|u_{\omega,t_0}\|_{L_x^4 L_T^\infty} \leq C\|\phi\|_{H^1(\mathbb{R})}.$$

The result is therefore a consequence of Lemma 3.2. □

Proof of Theorem 1.6. Let $\epsilon \in (0, \epsilon(A))$, where $\epsilon(A)$ is as in Proposition 3.3. If T is sufficiently large, from (1.25), we have that

$$\|U\|_{L_x^5 L_{(T,\infty)}^{10}} \leq \frac{\epsilon}{4}. \tag{4.9}$$

Applying Proposition 3.3 to the global solution $\tilde{U}(t) = U(t+T)$, the inequality (3.42) gives

$$\|S(t)U(T)\|_{L_x^5 L_t^{10}} = \|S(t)\tilde{U}(0)\|_{L_x^5 L_t^{10}} \leq 2\|\tilde{U}\|_{L_x^5 L_t^{10}} = 2\|U\|_{L_x^5 L_{(T,\infty)}^{10}} \leq \frac{\epsilon}{2}. \tag{4.10}$$

From this inequality and Corollary 3.4 we get

$$\|U\|_{X_{(T;\infty)}} \leq B\|U(T)\|_{H^1(\mathbb{R})}. \tag{4.11}$$

From Theorem 1.5 it follows that

$$\sup_{t_0\in\mathbb{R}} \sup_{0\leq t\leq T} \|u_{\omega,t_0}(t) - U(t)\|_{H^1(\mathbb{R})} \to 0, \quad \text{as} \quad |\omega| \to \infty. \tag{4.12}$$

Thus, if $|w|$ is sufficiently large, the triangle inequality along with (4.12) gives

$$\begin{aligned}\|S(t)u_{\omega,t_0}(T)\|_{L_x^5L_t^{10}} &\leq \|S(t)u_{\omega,t_0}(T)-S(t)U(T)\|_{L_x^5L_t^{10}}+\|S(t)U(T)\|_{L_x^5L_t^{10}}\\ &\leq \|u_{\omega,t_0}(T)-U(T)\|_{L_x^2}+\frac{\epsilon}{2}\leq \epsilon. \end{aligned} \tag{4.13}$$

Therefore, Corollary 3.4 implies that u_{ω,t_0} is global. Moreover,

$$\sup_{t_0\in\mathbb{R}}\|u_{\omega,t_0}\|_{L_x^5L_{(T,\infty)}^{10}}\leq 2\epsilon \quad \text{and} \quad \|u_{\omega,t_0}\|_{X_{(T,\infty)}}\leq B\|u_{\omega,t_0}(T)\|_{H^1(\mathbb{R})}, \tag{4.14}$$

for $|w|$ sufficiently large.

Let $M_0=\sup_{0\leq t\leq T}\|U(t)\|_{H^1(\mathbb{R})}$, as in (4.1). Now, we move to prove (1.26). The inequalities (4.12) and (4.14) show that there exists $L>0$ such that

$$\sup_{|w|\geq L}\sup_{t_0\in\mathbb{R}}\sup_{t\geq 0}\|u_{\omega,t_0}(t)\|_{H^1(\mathbb{R})}\leq (1+M_0)+B\|u_{\omega,t_0}(T)\|_{H^1(\mathbb{R})}=M_1<\infty. \tag{4.15}$$

In what follows, we prove that $u_{\omega,t_0}\to U$ in the $\|\cdot\|_{X_t}$-norm, when $|\omega|\to\infty$.

Using Duhamel's formulas for u_{ω,t_0} and U we have

$$\begin{aligned} u_{\omega,t_0}(T+t)-U(T+t) &= S(t)(u_{\omega,t_0}(T)-U(T)) \\ &\quad -\int_0^t S(t-t')g(\omega(T+t'+t_0))\partial_x(u_{\omega,t_0}^5)(T+t')dt' \\ &\quad +m(g)\int_0^t S(t-t')\partial_x(U^5)(T+t')dt' \\ &=: I_1+I_2+I_3. \end{aligned} \tag{4.16}$$

Using properties of the unitary group $S(t)$ we have by (4.12) that

$$\|I_1\|_{X_t}=\|S(t)(u_{\omega,t_0}(T)-U(T))\|_{X_t}\leq C\|u_{\omega,t_0}(T)-U(T)\|_{H^1(\mathbb{R})}\overset{|\omega|\to\infty}{\to}0. \tag{4.17}$$

With the same argument as in (3.53), we have

$$\|I_2\|_{X_t}\leq CA\|u_{\omega,t_0}\|^4_{L_x^5L_{(T,\infty)}^{10}}\|u_{\omega,t_0}\|_{X_{(T,\infty)}}. \tag{4.18}$$

From (4.18), with the use of (4.14) and (4.15), we have

$$\|I_2\|_{X_t}\leq CA(2\epsilon)^4BM_1. \tag{4.19}$$

As in I_2, using (4.9) and (4.11), we get

$$\|I_3\|_{X_t}\leq CA\|U\|^4_{L_x^5L_{(T,\infty)}^{10}}\|U\|_{X_{(T,\infty)}}\leq CA\Big(\frac{\epsilon}{4}\Big)^4BM_0. \tag{4.20}$$

Now given $\beta > 0$, we choose $\epsilon > 0$ sufficiently small (T sufficiently large) such that $CA(2\epsilon)^4\big[BM_0 + BM_1\big] < \beta/3$ and $|\omega|$ is sufficiently large, so that (4.16), (4.17), (4.19), and (4.20) imply

$$\begin{aligned}\|u_{\omega,t_0}(t) - U(t)\|_{X_{(T,\infty)}} &= \|u_{\omega,t_0}(T+t) - U(T+t)\|_{X_t} \\ &\leq \|I_1\|_{X_t} + \|I_2\|_{X_t} + \|I_3\|_{X_t} < \beta.\end{aligned} \tag{4.21}$$

On the other hand, from Theorem 1.5, we have

$$\|u_{\omega,t_0}(t) - U(t)\|_{X_{(0,T)}} = \|u_{\omega,t_0}(t) - U(t)\|_{X_T} \overset{|\omega|\to\infty}{\to} 0. \tag{4.22}$$

Therefore, from (4.21) and (4.22), we can conclude the proof of the theorem.

Acknowledgments. The authors are grateful to Gustavo Ponce, Felipe Linares and the anonymous referee for valuable comments and suggestions that helped to improve the manuscript.

References

[1] F.K. Abdullaev, J.G. Caputo, R.A. Kraenkel, and B.A. Malomed, *Controlling collapse in Bose-Einstein condensates by temporal modulation of the scattering length*, Phys. Rev. A, 67 012605 (2003).

[2] J. Angulo, J. Bona, F. Linares, and M. Scialom, *Scaling, stability and singularities for nonlinear dispersive wave equations: the critical case*, Nonlinearity, 15 (2002), 759–786.

[3] J.L. Bona, P. Souganidis, and W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type equation*, Proc. Roy. Soc. London Ser A, 411 (1987), 395–412.

[4] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notices, 5 (1998), 253–283.

[5] T. Cazenave and M. Scialom, *A Schrödinger equation with time-oscillating nonlinearity*, Revista Matemática Complutense, 23 (2010), 321–339.

[6] I. Damergi and O. Goubet, *Blow-up solutions to the nonlinear Schrödinger equation with oscillating nonlinearities*, J. Math. Anal. Appl., 352 (2009), 336–344.

[7] L.G. Farah, *Global rough solutions to the critical generalized KdV equation*, J. Diff. Equations 249 (2010), 1968–1985.

[8] G. Fonseca, F. Linares, and G. Ponce, *Global existence of the critical generalized KdV equation*, Proc. Amer. Math. Soc., 131 (2003), 1847–1855.

[9] A. Friedman, "Partial Differential Equations," Holt, Rinehart and Winston, Inc., 1969.

[10] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal., 74 (1987), 160–197.

[11] T. Kato, *On the Cauchy problem for the (generalized), Korteweg-de Vries equation*, Advances in Mathematics Supplementary Studies, Studies in Appl. Math., 8 (1983), 93–128.

[12] C.E. Kenig, G. Ponce, and L. Vega, *On the (generalized), Korteweg-de Vries equation*, Duke Math. J., 59 (1989), 585–610.

[13] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc., 4 (1991), 323–347.

[14] C.E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana University Math. J., 40 (1), (1991), 33–69.

[15] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math., 46 (1993), 527–620.

[16] C.E. Kenig, G. Ponce, and L. Vega, *On the concentration of blow up solutions for the generalized KdV equation critical in* L^2, "Nonlinear Wave Equations" (Providence, RI, 1998), 131–156, Contemp. Math., 263, Amer. Math. Soc., Providence, RI, 2000.

[17] C.E. Kenig and A. Ruiz, *A strong type (2,2), estimate for a maximal operator associated to the Schrödinger equation*, Trans. Amer. Math. Soc., 230 (1983), 239–246.

[18] C.J. Knickerbocker and A.C. Newell, *Internal solitary waves near a turning point*, Phys. Lett., 75 A (1980), 326–330.

[19] V.V. Konotop and P. Pacciani, *Collapse of solutions of the nonlinear Schrödinger equation with a time dependent nonlinearity: application to the Bose-Einstein condensates*, Phys. Rev. Lett., 94 (2005), 240405 .

[20] Y. Martel and F. Merle, *Blow up in finite time and dynamics of blow up solutions for the critical generalized KdV equation*, J. Amer. Math. Soc., 15 (2002), 617–664.

[21] Y. Martel and F. Merle, *Stability of blowup profile and lower bounds on blowup rate for the critical generalized KdV equation*, Ann. of Math., 155 (2002), 235–280.

[22] F. Merle, *Existence of blow-up solutions in the energy space for the critical generalized KdV equation*, J. Amer. Math. Soc., 14 (2001), 555–578.

[23] W.V.L. Nunes, *Global well-posedness for the transitional Korteweg-de Vries equation* Appl. Math. Lett., 11 (1998), 15–20.

[24] W.V.L. Nunes, *On the well-posedness and scattering for the transitional Benjamin-Ono equation*, Mat. Contemp., 3 (1992), 127–148.

[25] E. Stein and G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces," Princeton University Press, 1971.

[26] M. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., 39 (1986), 51–67.