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BLOW-UP IN SEVERAL POINTS FOR THE NONLINEAR SCHRÖDINGER EQUATION ON A BOUNDED DOMAIN

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Abstract. Given p points in a bounded domain of $\mathbb{R}^d$, with $d = 2, 3$, we show the existence of solutions of the L^2-critical focusing nonlinear Schrödinger equation blowing up exactly at these points.

1. Introduction

We consider the L^2-critical focusing nonlinear Schrödinger equation posed on a bounded and regular domain Ω of $\mathbb{R}^d$ (with $d = 2, 3$):

$$i\partial_t u + \Delta u = -|u|^{4/d}u, \quad (t, x) \in [0, T) \times \Omega. \tag{1.1}$$

We add initial data and the Dirichlet boundary condition:

$$\begin{cases} u(t, x) = 0, & (t, x) \in [0, T) \times \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Thanks to the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, one can show that the equation (1.1) is locally well-posed in the space $H^2(\Omega) \cap H^1_0(\Omega)$: for every initial data $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, there exists a time $T \in (0, +\infty]$ and a unique function $u \in C([0, T), H^2(\Omega) \cap H^1_0(\Omega))$ a solution of (1.1) with initial data u_0. If u is a solution of (1.1), the energy and the mass are conserved: for every $t \in [0, T)$,

$$\begin{aligned} E(t) &:= \frac{1}{2}\|\nabla u(t)\|^2_{L^2} - \frac{d}{4+2d}\|u(t)\|^{\frac{4+2d}{d}}_{L^{\frac{4+2d}{d}}} = E(0), \\ M(t) &:= \|u(t)\|_{L^2} = M(0). \end{aligned}$$

Moreover, we have the following blow-up criteria:

$$\text{if} \quad T < +\infty, \quad \text{then} \quad \|u(t)\|_{H^2} \to +\infty, \quad \text{when} \quad t \to T.$$

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Note that if $d = 2$, the equation (1.1) is also well posed in $H_0^1(\Omega)$ and even globally well posed if the L^2-norm of the initial data is smaller than $\|Q\|_{L^2(\mathbb{R}^2)}$ (see [1, 3, 13]).

Blow-up solutions for the equation (1.1) have been extensively studied in $\mathbb{R}^n$, and we expect that some results remain valid in bounded domains or more generally in the setting of flat geometries. Among papers concerning the study of the nonlinear Schrödinger equation on a domain, we can mention [2, 6, 7, 11].

In [9], Merle shows that if Ω is the whole space $\mathbb{R}^d$ (without restriction on d) then given p points in $\mathbb{R}^d$, there exists a solution of the focusing nonlinear Schrödinger equation with L^2-critical nonlinearity that blows up at the p points. The aim of this paper is to show that this result is still true if $\mathbb{R}^d$ is replaced by a bounded and regular domain of $\mathbb{R}^d$ with $d = 2, 3$. We can not adapt the construction of Merle because the proof crucially uses the dispersion estimate

$$\exists C > 0,\ \forall t > 0,\ \forall v \in L^1(\mathbb{R}^d),\ \|e^{it\Delta} v\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{t^{d/2}} \|v\|_{L^1(\mathbb{R}^d)},$$

which turns out to be false if we replace $\mathbb{R}^d$ by a bounded domain. To prove our result, we use the perturbation method introduced in [10] and used in [4] to treat the case of a point in dimension 2. Because of lack of regularity of the nonlinearity, the case of the dimension 3 requires a change in the choice of the weighted space where we show the property of contraction.

In the following, we denote by Q the unique ([8, 14]) radially symmetric and strictly positive solution of

$$-\Delta Q + Q = |Q|^{4/d} Q$$

and satisfying the exponential decay at infinity:

$$\forall \alpha \in \mathbb{N}^n,\ \exists C_\alpha > 0,\ \exists D_\alpha > 0,\ \forall x \in \mathbb{R}^d,\ |\partial^\alpha Q(x)| \leq C_\alpha e^{-D_\alpha |x|}.$$

2. Statement and proof of the result

Now we state the theorem of the paper.

Theorem 2.1. *Let Ω be a bounded and regular domain of $\mathbb{R}^d$ (with $d = 2, 3$) and $x_1, \ldots, x_p$ be p distinct points in Ω. Let $\varphi_1, \ldots, \varphi_p \in C_0^\infty(\Omega)$ with disjoint supports and such that $0 \leq \varphi_k \leq 1$ and $\varphi_k = 1$ near x_k. Then we have the following:*

(1) *There exists* $\lambda_0 > 0$ *such that for every* $\lambda \geq \lambda_0$, *there exists a time* $T_\lambda > 0$ *and* $u_\lambda \in C([0, T_\lambda), H^2(\Omega) \cap H_0^1(\Omega))$ *such that the function* h_λ *defined by*

$$h_\lambda(t,x) = \frac{1}{\lambda^{d/2}(T_\lambda - t)^{d/2}} \sum_{k=1}^{p} e^{\frac{i(4-\lambda^2|x-x_k|^2)}{4\lambda^2(T_\lambda - t)}} \varphi_k(x) Q\Big(\frac{x - x_k}{\lambda(T_\lambda - t)}\Big) + u_\lambda(t,x)$$

is a solution of (1.1). *Moreover,*

$$\exists\ \gamma > 0,\ C > 0,\ \forall \lambda \geq \lambda_0,\ \forall t \in [0, T_\lambda),\ \|u_\lambda(t)\|_{H^2(\Omega)} \leq C e^{-\frac{\gamma}{\lambda(T_\lambda - t)}}.$$

(2) *For* $\lambda \geq \lambda_0$, *the solution* h_λ *blows up in* H^1 *at time* $t = T_\lambda$ *at the points* $x_1, \dots, x_p$ *with speed* $(T_\lambda - t)^{-1}$. *More precisely,* h_λ *satisfies*

(*i*) *for* $R > 0$ *small enough, for every* k,

$$\|h_\lambda(t)\|_{L^2(\overline{B}(x_k, R))} \xrightarrow[t \to T_\lambda]{} \|Q\|_{L^2(\mathbb{R}^d)},$$

(*ii*) *for all* $t \in [0, T_\lambda)$, $\|h_\lambda(t)\|_{L^2(\Omega)} = \sqrt{p}\|Q\|_{L^2(\mathbb{R}^d)}$,

(*iii*) $|h_\lambda(t)|^2 \xrightarrow[t \to T_\lambda]{} \|Q\|^2_{L^2(\mathbb{R}^d)} \sum_{k=1}^{p} \delta_{x_k}$ *in the sense of measures,*

(*iv*) $\|\nabla h_\lambda(t)\|_{L^2(\Omega)} \underset{t \to T_\lambda}{\sim} \dfrac{\sqrt{p}\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\lambda(T_\lambda - t)}$.

Scheme of the proof. If $\Omega = \mathbb{R}^d$, we know an explicit solution u_i of (1.1) which blows up at x_i. Next we consider the function $u = \varphi_1 u_1 + \cdots + \varphi_p u_p$ where φ_k is a cut-off function near x_k. Therefore, u is a function which vanishes on the boundary and has the same behavior as u_k near x_k because of the cut-off functions. Thus, u blows up at the points $x_1, \dots, x_p$. However, u is not a solution of (1.1), but we shall show the existence of a rest r such that $u + r$ is a solution of (1.1) and keeps the behavior of u when t tends to the blow-up time. For this, we will impose the condition that $r(t)$ tends to 0 at the blow-up time. To prove the existence of the rest r, we perform a fixed-point argument in a suitable weighted space.

Proof. For $T > 0$ and $\lambda > 0$, we introduce

$$\begin{cases} r_\lambda^k(t,x) = \dfrac{1}{\lambda^{d/2}(T - t)^{d/2}} e^{\frac{i(4-\lambda^2|x-x_k|^2)}{4\lambda^2(T-t)}} Q\Big(\dfrac{x - x_k}{\lambda(T - t)}\Big), \\ r_\lambda(t,x) = \displaystyle\sum_{k=1}^{p} \varphi_k(x) r_\lambda^k(t,x). \end{cases}$$

For every k, r_λ^k is a solution of (1.1) on $\mathbb{R}^d$. This solution blows up in $H^1(\mathbb{R}^d)$ at time T and at x_k. We seek a condition on u_λ for which the function $h_\lambda := r_\lambda + u_\lambda$ is a solution of (1.1) on Ω. We have the equality

$$i\partial_t h_\lambda + \Delta h_\lambda = \sum_{k=1}^{p} \Big(r_\lambda^k \ \Delta\varphi_k + 2\nabla\varphi_k \cdot \nabla r_\lambda^k - \varphi_k |r_\lambda^k|^{4/d} r_\lambda^k \Big) + (i\partial_t + \Delta)u_\lambda.$$

Thus, h_λ is a solution of (1.1) if and only if u_λ satisfies

$$\begin{aligned}(i\partial_t + \Delta)u_\lambda &= -|r_\lambda + u_\lambda|^{4/d}(r_\lambda + u_\lambda)\\ &\quad - \sum_{k=1}^{p} \Big(r_\lambda^k \Delta\varphi_k + 2\nabla\varphi_k \cdot \nabla r_\lambda^k - \varphi_k |u_\lambda^k|^{4/d} u_\lambda^k \Big) = S_0 + S(u_\lambda),\end{aligned}$$

where we denote

$$S_0 = -\Big|\sum_{k=1}^{p} \varphi_k r_\lambda^k\Big|^{\frac{4}{d}} \Big(\sum_{k=1}^{p} \varphi_k r_\lambda^k\Big) - \sum_{k=1}^{p} \Big(r_\lambda^k \ \Delta\varphi_k + 2\nabla\varphi_k \cdot \nabla r_\lambda^k - \varphi_k |r_\lambda^k|^{4/d} r_\lambda^k \Big),$$

$$S(u) = \Big|\sum_{k=1}^{p} \varphi_k r_\lambda^k\Big|^{4/d} \Big(\sum_{k=1}^{p} \varphi_k r_\lambda^k\Big) - \Big|u + \sum_{k=1}^{p} \varphi_k r_\lambda^k\Big|^{4/d} \Big(u + \sum_{k=1}^{p} \varphi_k r_\lambda^k\Big).$$

To not perturb the behavior of r_λ at the blow-up time, we impose the condition that $u(t)$ tends to 0 when t tends to T; this leads to considering the integral formulation

$$u(t) = i\int_t^T e^{i(t-s)\Delta} \left(S_0(s) + S(u)(s)\right) ds. \tag{2.1}$$

We introduce

$$I_0(t) = i\int_t^T e^{i(t-s)\Delta} S_0(s) ds, \quad I(u)(t) = i\int_t^T e^{i(t-s)\Delta} S(u(s)) ds.$$

To estimate the terms I_0 and I we begin with the following lemma.

Lemma 1. *There exists a constant $C > 0$ such that for every $u, v \in H^2(\Omega)$,*

(i) $\||u|^{4/d}u - |v|^{4/d}v\|_{L^2(\Omega)} \le C\|u - v\|_{L^2(\Omega)} \left(\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}\right)^{4/d}$,

(ii) $\||u|^{4/d}u - |v|^{4/d}v\|_{H^1(\Omega)} \le C\|u - v\|_{H^2(\Omega)} \left(\|u\|_{H^2(\Omega)} + \|u\|_{H^2(\Omega)}\right)^{4/d}$,

(iii) $\||u|^{4/d}u\|_{H^2(\Omega)} \le C\|u\|_{H^2(\Omega)}^{1+4/d}$.

Proof of Lemma 1. (i) We use the Taylor formula

$$f(u)-f(v)=(u-v)\int_0^1 \partial_z f(tu+(1-t)v)dt+(\overline{u}-\overline{v})\int_0^1 \partial_{\overline{z}} f(tu+(1-t)v)dt \tag{2.2}$$

with the complex function $f(z)=|z|^{4/d}z$. The computation of the derivatives of f shows that

$$|\partial_z f(z)|+|\partial_{\overline{z}} f(z)|\le C|z|^{4/d}.$$

We deduce from (2.2) that for $u,v\in\mathbb{C}$

$$\left||u|^{4/d}u-|v|^{4/d}v\right|\le C|u-v|\left(|u|+|v|\right)^{4/d}.$$

Then, we apply this inequality to the functions u and v, integrate, and the conclusion follows by using Hölder's inequality. This prove the first point.

(ii) First, using (i) and the Sobolev embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ we get

$$\||u|^{4/d}u-|v|^{4/d}v\|_{L^2}\le C\|u-v\|_{H^2}\left(\|u\|_{H^2(\Omega)}+\|v\|_{H^2(\Omega)}\right)^{4/d}. \tag{2.3}$$

Next, a direct computation shows that

$$\partial_i(|u|^{4/d}u)=\Big(\frac{2}{d}+1\Big)(\partial_i u)|u|^{4/d}+\frac{2}{d}(\partial_i\overline{u})|u|^{4/d-2}u^2. \tag{2.4}$$

According to the last identity, we can split $\partial_i(|u|^{4/d}u)-\partial_i(|v|^{4/d}v)$ into two terms. These two terms are treated in the same way. Let us for instance treat the second. We may write

$$(\partial_i\overline{u})|u|^{4/d-2}u^2-(\partial_i\overline{v})|v|^{4/d-2}v^2=A_1+A_2,$$

where

$$A_1=\partial_i(\overline{u}-\overline{v})\left(|u|^{4/d-2}u^2\right),\quad A_2=\partial_i\overline{v}\left(|u|^{4/d-2}u^2-|v|^{4/d-2}v^2\right).$$

But by Hölder's inequality and again the Sobolev embedding of H^2 into L^∞,

$$\|A_1\|_{L^2}\le\|u-v\|_{H^1}\|u\|_{L^\infty}^{4/d}\le\|u-v\|_{H^2}\|u\|_{H^2}^{4/d}.$$

For A_2, we first use (2.2) to get

$$\left||u|^{4/d-2}u^2-|v|^{4/d-2}v^2\right|\le C|u-v|\left(|u|+|v|\right)^{4/d-1}.$$

Hence

$$\begin{aligned}\|A_2\|_{L^2}&\le\|v\|_{H^1}\|u-v\|_{L^\infty}\left(\|u\|_{L^\infty}+\|v\|_{L^\infty}\right)^{4/d-1}\\&\le\|u-v\|_{H^2}\left(\|u\|_{H^2}+\|v\|_{H^2}\right)^{4/d}.\end{aligned}$$

Summing the estimates on A_1, A_2, and (2.3), we obtain the second point.

(iii) The norm $\|\cdot\|_{H^2}$ is equivalent to the norm $\|\cdot\|_{L^2}+\|\nabla^2\cdot\|_{L^2}$. Using again a Sobolev embedding, we deduce that

$$\||u|^{4/d}u\|_{L^2}\leq\|u\|_{L^\infty}^{4/d+1}\leq C\|u\|_{H^2}^{4/d+1}. \tag{2.5}$$

Moreover, deriving the relation (2.4), we get

$$\begin{aligned}\partial_{ij}(|u|^{4/d}u)=\left(\frac{2}{d}+1\right)&\left(\partial_{ij}u|u|^{4/d}+\frac{2}{d}\partial_iu\partial_ju|u|^{4/d-2}\overline{u}+\frac{2}{d}\partial_i\overline{u}\partial_ju|u|^{4/d-2}u\right)\\&+\frac{2}{d}\left(\left(\frac{2}{d}+1\right)\partial_iu\partial_j\overline{u}|u|^{4/d-2}u+\partial_{ij}\overline{u}|u|^{4/d-2}u^2\right.\\&\left.+\left(\frac{2}{d}-1\right)\partial_i\overline{u}\partial_j\overline{u}|u|^{4/d-4}u^3\right).\end{aligned} \tag{2.6}$$

Then using a Hölder's inequality on each term, we get

$$\|\partial_{ij}(|u|^{4/d}u)\|_{L^2}\leq C\left(\|\nabla^2u\|_{L^2}\|u\|_{L^\infty}^{4/d}+\|\nabla u\|_{L^4}^2\|u\|_{L^\infty}^{4/d-1}\right).$$

By the embeddings $H^1\hookrightarrow L^4$ and $H^2\hookrightarrow L^\infty$, we may write

$$\|\partial_{ij}(|u|^{4/d}u)\|_{L^2}\leq C\|u\|_{H^2}^{1+4/d}. \tag{2.7}$$

Gathering (2.5) and (2.7), we obtain the third point. □

Since the supports of φ_k are disjoint, $S_0(t)$ is zero near the points x_k. Therefore, there exists $r>0$ such that

$$\forall t\in[0,T),\ \forall x\in\Omega\setminus\cup_{k=1}^p\overline{B}(x_k,r),\quad S_0(t,x)=0.$$

Using the exponential decay of the ground state Q and its derivatives, we get the existence of C and D such that for every k, t, λ, and α with $|\alpha|\leq 3$,

$$\|\partial^\alpha r_\lambda^k(t)\|_{L^2(\mathbb{R}^2\setminus\overline{B}(x_k,r))}\leq Ce^{-\frac{D}{\lambda(T-t)}}.$$

Therefore, Lemma 1 (iii) and the structure of algebra of $H^2(\Omega\setminus\cup_{k=1}^p\overline{B}(x_k,r))$ allows us to write

$$\|S_0(t)\|_{H^2(\Omega)}=\|S_0(t)\|_{H^2(\Omega\setminus\cup_{k=1}^p\overline{B}(x_k,r))}\leq Ce^{-\frac{\delta}{\lambda(T-t)}}. \tag{2.8}$$

Now, we can introduce the following metric space:

$$\begin{aligned}E_T=\Big\{&u\in L^\infty([0,T),H^2\cap H_0^1),\\&\sup_{0\leq t<T}\left(e^{\frac{\delta}{\lambda(T-t)}}\|u(t)\|_{L^2}\right)+\sup_{0\leq t<T}\left(e^{\frac{\alpha\delta}{\lambda(T-t)}}\|u(t)\|_{H^2}\right)\leq 1\Big\},\end{aligned}$$

equipped with the distance

$$d(u,v)=\sup_{0\leq t<T}\left(e^{\frac{\delta}{\lambda(T-t)}}\|u(t)-v(t)\|_{L^2}\right),$$

where α is a real number such that $0 < \alpha < \min(1, \frac{4}{d} - 1)$. We are going to perform the Banach fixed-point argument in E_T to prove that the map defined by

$$\Phi(u)(t) = i \int_t^T e^{i(t-s)\Delta} \left(S_0(s) + S(u)(s)\right) ds \tag{2.9}$$

has a fixed point. For this, we show that for T small enough and λ big enough, Φ sends E_T into itself and is a contraction.

First, we prove that (E_T, d) is a complete metric space. It suffices to show that E_T is closed in the complete metric space

$$E = \{u \in L^\infty([0,T), L^2) : \exp(\frac{\delta}{\lambda(T-t)})\|u(t)\|_{L^2} \leq 1\}$$

equipped with the distance d. Let (u_n) be a sequence in E_T tending to $u \in E$ for d. Since $v_n(t) := \exp(\frac{\alpha\delta}{\lambda(T-t)})u_n(t)$ is bounded in $L^\infty([0,T), H^2 \cap H_0^1)$, we can extract a subsequence tending to $v \in L^\infty([0,T), H^2 \cap H_0^1)$ for the weak*-topology. Then necessarily, by uniqueness of the limit in $\mathcal{D}'((0,T), H^{-2})$, $v(t) = \exp(\frac{\alpha\delta}{\lambda(T-t)})u(t)$. Using the lower semicontinuity of the norm in $L^\infty([0,T), H^2 \cap H_0^1)$, we have

$$\sup_{0\leq t<T} \left(e^{\frac{\alpha\delta}{\lambda(T-t)}} \|u(t)\|_{H^2}\right) \leq \liminf_{n\to\infty} \sup_{0\leq t<T} \left(e^{\frac{\alpha\delta}{\lambda(T-t)}} \|u_n(t)\|_{H^2}\right);$$

hence, taking the limit inferior in the inequality

$$\sup_{0\leq t<T} \left(e^{\frac{\delta}{\lambda(T-t)}} \|u_n(t)\|_{L^2}\right) + \sup_{0\leq t<T} \left(e^{\frac{\alpha\delta}{\lambda(T-t)}} \|u_n(t)\|_{H^2}\right) \leq 1,$$

we obtain that $u \in E_T$.

Boundedness. Let $T \in (0,1)$, $\lambda \geq 1$ to be chosen later. We prove that Φ sends E_T into itself. We may write

$$S(u) - S(v) = |r_\lambda + v|^{4/d}(r_\lambda + v) - |r_\lambda + u|^{4/d}(r_\lambda + u).$$

Lemma 1 and the Sobolev embedding $H^2 \hookrightarrow L^\infty$ provides

$$\begin{aligned} &\|S(u)(t) - S(v)(t)\|_{L^2} \\ &\leq C\|u(t) - v(t)\|_{L^2} \left(\|u(t)\|_{H^2} + \|v(t)\|_{H^2} + \|r_\lambda(t)\|_{L^\infty}\right)^{4/d}. \end{aligned}$$

But using the explicit formula of r_λ, one can compute its derivatives to get for $k = 0, 1, 2$,

$$\|\nabla^k r_\lambda(t)\|_{L^\infty(\Omega)} \leq \frac{C}{\lambda^{d/2}(T-t)^{d/2+k}}, \tag{2.10}$$

and deduce that if $u, v \in E_T$,

$$\|S(u)(t) - S(v)(t)\|_{L^2} \leq Ce^{-\frac{\delta}{\lambda(T-t)}} d(u,v)\Big(1 + \frac{1}{\lambda^2(T-t)^2}\Big).$$

By integrating this inequality, and using

$$\int_t^T \frac{e^{-\frac{\delta}{\lambda(T-s)}}}{(T-s)^2} ds \leq C\lambda e^{-\frac{\delta}{\lambda(T-t)}},$$

it follows that

$$\|I(u)(t) - I(v)(t)\|_{L^2} \leq Ce^{-\frac{\delta}{\lambda(T-t)}} d(u,v)\Big(T + \frac{1}{\lambda}\Big). \tag{2.11}$$

Now we need to bound $\|I(u)(t)\|_{H^2}$. The norm $\|\cdot\|_{H^2}$ is equivalent to $\|\cdot\|_{L^2} + \|\nabla^2 \cdot\|_{H^2}$. In the formula (2.6), there are two types of terms. The first ones are those containing the product of two first derivatives, namely $\partial_i u \partial_j u |u|^{4/d-2}\overline{u}$, $\partial_i \overline{u} \partial_j u |u|^{4/d-2} u$, $\partial_i u \partial_j \overline{u} |u|^{4/d-2} u$, $\partial_i \overline{u} \partial_j \overline{u} |u|^{4/d-3} u^3$; and the second are those containing a second derivative, namely $\partial_{ij} u |u|^{4/d}$, $\partial_{ij}\overline{u}|u|^{4/d-2}u^2$. Each term of the same type is treated in the same way. Let us begin with the first type, for instance the term $\partial_i \overline{u} \partial_j \overline{u} |u|^{4/d-4} u^3$. We may write

$$\begin{aligned}&\Big\|\partial_i \overline{r_\lambda} \partial_j \overline{r_\lambda} |r_\lambda|^{4/d-4} r_\lambda^3 - \partial_i(\overline{r_\lambda} + \overline{u})\partial_j(\overline{r_\lambda} + \overline{u})|r_\lambda + u|^{4/d-4}(r_\lambda + u)^3\Big\|_{L^2}\\ &\leq B_1 + B_2,\end{aligned}$$

where

$$\begin{aligned}B_1 &= \Big\|\partial_i \overline{r_\lambda} \partial_j \overline{r_\lambda}\left(|r_\lambda|^{4/d-4} r_\lambda^3 - |r_\lambda + u|^{4/d-4}(r_\lambda + u)^3\right)\Big\|_{L^2},\\ B_2 &= \Big\|\left(\partial_i \overline{r_\lambda} \partial_j \overline{u} + \partial_i \overline{u} \partial_j \overline{r_\lambda} + \partial_i \overline{u} \partial_j \overline{u}\right)|r_\lambda + u|^{4/d-4}(r_\lambda + u)^3\Big\|_{L^2}.\end{aligned}$$

Using the estimate on the derivatives of r_λ (2.10) and the inequality

$$\Big||u|^{4/d-4}u^3 - |v|^{4/d-4}v^3\Big| \leq C|u - v|^{4/d-1},$$

we have successively (with always $u \in E_T$).

$$\begin{aligned}B_1 &\leq C\|\partial_i r_\lambda \partial_j r_\lambda |u|^{4/d-1}\|_{L^2} \leq C\|\nabla r_\lambda\|_{L^\infty}^2 \||u|^{4/d-1}\|_{L^2}\\ &\leq \frac{C}{\lambda^d (T-t)^{d+2}}\|u\|_{L^2}^{4/d-1} \leq C\lambda^2 e^{-\frac{\alpha\delta}{\lambda(T-t)}}.\end{aligned}$$

For B_2, Hölder's inequality provides

$$B_2 \leq \||r_\lambda + u|^{4/d-1}\left(\partial_i r_\lambda \partial_j u + \partial_i u \partial_j r_\lambda + \partial_i u \partial_j u\right)\|_{L^2}$$

$$\leq \|r_\lambda + u\|_{L^\infty}^{4/d-1}\left(\|\nabla r_\lambda\|_{L^\infty}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^4}^2\right).$$

But by interpolation between the spaces L^2 and H^2, we remark that there exists $\beta \in (\alpha, 1)$ such that for every $u \in E_T$, and for every $t \in [0, T)$,

$$\|u(t)\|_{H^1} \leq e^{-\frac{\beta\delta}{\lambda(T-t)}}.$$

Hence

$$\begin{aligned}B_2 &\leq C\Big(\frac{1}{\lambda^{2-d/2}(T-t)^{2-d/2}} + 1\Big)\Big(\frac{1}{\lambda^{d/2}(T-t)^{d/2+1}} e^{-\frac{\beta\delta}{\lambda(T-t)}} + e^{-\frac{2\alpha\delta}{\lambda(T-t)}}\Big)\\ &\leq C\lambda e^{-\frac{\alpha\delta}{\lambda(T-t)}}.\end{aligned}$$

Now we treat terms belonging to the second type, for instance the term $\partial_{ij}\overline{u}|u|^{4/d-2}u^2$. We have

$$\|\partial_{ij}\overline{r_\lambda}|r_\lambda|^{4/d-2}r_\lambda^2 - \partial_{ij}(\overline{r_\lambda} + \overline{u})|r_\lambda + u|^{4/d-2}(r_\lambda + u)^2\|_{L^2} \leq C_1 + C_2,$$

where

$$\begin{aligned}C_1 &= \left\|\partial_{ij}\overline{r_\lambda}\left(|r_\lambda|^{4/d-2}r_\lambda^2 - |r_\lambda + u|^{4/d-2}(r_\lambda + u)^2\right)\right\|_{L^2},\\ C_2 &= \left\|\partial_{ij}\overline{u}\left(|r_\lambda + u|^{4/d-2}(r_\lambda + u)^2\right)\right\|_{L^2}.\end{aligned}$$

The inequality

$$\left||z|^{4/d-2}z^2 - |w|^{4/d-2}w^2\right| \leq C|z - w|\left(|z| + |u|\right)^{4/d-1},$$

applied with $z = r_\lambda$ and $w = r_\lambda + u$, allows us to write

$$\begin{aligned}C_1 &\leq \|\nabla^2 r_\lambda\|_{L^\infty}\|u\|_{L^2}\left(\|r_\lambda\|_{L^\infty}^{4/d-1} + \|u\|_{L^\infty}^{4/d-1}\right)\\ &\leq \frac{C}{\lambda^{d/2}(T-t)^{d/2+2}} e^{-\frac{\delta}{\lambda(T-t)}}\Big(\frac{1}{\lambda^{2-d/2}(T-t)^{2-d/2}} + 1\Big) \leq C\lambda^2 e^{-\frac{\alpha\delta}{\lambda(T-t)}}.\end{aligned}$$

For the second bound, we may write

$$C_2 \leq \|\nabla^2 u\|_{L^2}\|r_\lambda + u\|_{L^\infty}^{4/d} \leq Ce^{-\frac{\alpha\delta}{\lambda(T-t)}}\Big(\frac{1}{\lambda^2(T-t)^2} + 1\Big).$$

Summing B_1, B_2, C_1, and C_2, we deduce for $\lambda \geq 1$ and $u \in E_T$,

$$\|S(u)(t)\|_{H^2} \leq Ce^{-\frac{\alpha\delta}{\lambda(T-t)}}\Big(\lambda^2 + \frac{1}{\lambda^2(T-t)^2}\Big), \tag{2.12}$$

and integrating we get

$$\|I(u)(t)\|_{H^2} \leq Ce^{-\frac{\alpha\delta}{\lambda(T-t)}}\Big(T\lambda^2 + \frac{1}{\lambda}\Big). \tag{2.13}$$

Moreover, by (2.8) we obtain the bound

$$\|I_0\|_{H^2} \leq CTe^{-\frac{\delta}{\lambda(T-t)}}.$$

Finally, the latter estimate together with (2.13) and (2.11) applied with $v = 0$ gives for $\lambda \geq 1$ and $u \in E_T$

$$\sup_{0\leq t<T}\Big(e^{\frac{\delta}{\lambda(T-t)}}\|\Phi(u)(t)\|_{L^2}\Big) + \sup_{0\leq t<T}\Big(e^{\frac{\alpha\delta}{\lambda(T-t)}}\|\Phi(u)(t)\|_{H^2}\Big) \leq C\Big(T\lambda^2 + \frac{1}{\lambda}\Big). \tag{2.14}$$

Contraction. Actually, we have already proved the property of contraction of Φ during the proof of the boundedness. Indeed, the estimate (2.11) provides for every $u, v \in E_T$,

$$d\left(\Phi(u), \Phi(v)\right) \leq C\Big(T + \frac{1}{\lambda}\Big)d(u,v). \tag{2.15}$$

Conclusion. First $\Phi(E_T) \subset L^\infty([0,T), H^2 \cap H_0^1)$. Indeed, the bound (2.14) gives $\Phi(E_T) \subset L^\infty([0,T), H^2)$ and it remains to verify that $\Phi(u)(t) \in H_0^1$ almost everywhere. For this, it suffices to prove that for $u \in H^2 \cap H_0^1$, $S(u) \in H_0^1$. But if $u \in H^2 \cap H_0^1$, we can approximate u in the H^2-norm by a sequence $u_n \in C_0^\infty$. By Lemma 1 (ii), we obtain the convergence in H^1-norm of $|u_n|^{4/d}u_n$ to $|u|^{4/d}u$, and this shows that $|u|^{4/d}u \in H_0^1$. Hence $S(u) \in H_0^1$ and $\Phi(E_T) \subset L^\infty([0,T), H^2 \cap H_0^1)$. Moreover, estimates (2.14) and (2.15) prove that we can choose λ big enough and $T > 0$, depending on λ, such that $\Phi(E_T) \subset E_T$ and for every $u, v \in E_T$,

$$d\left(\Phi(u), \Phi(v)\right) \leq \frac{1}{2}d\left(u, v\right). \tag{2.16}$$

Thus, we can apply the Banach fixed-point argument with the function Φ, and this proves the existence of the rest $u \in E_T$ satisfying (2.1). To obtain the continuity in time with values in H^2, we use the integral formulation satisfied by u. Estimates (2.8) and (2.12) show that the map $s \mapsto S_0(s) + (S(u))(s)$ belongs to $L^\infty([0,T), H^2)$, and since u satisfies

$$u(t) = ie^{it\Delta}\int_t^T e^{-is\Delta}\left(S_0(s) + (S(u))(s)\right)ds,$$

we conclude that $u \in C([0,T), H^2)$.

Let us show the second part of the theorem.

(i) For $R > 0$ small enough, we remark that for $x \in \cup_k \overline{B}(x_k, R)$

$$r_\lambda(t,x) = \sum_{k=1}^{p} r_\lambda^k(t,x).$$

Then

$$\begin{aligned}&\Big| \|h_\lambda(t)\|_{L^2(\overline{B}(x_k,R))} - \|r_\lambda^k(t)\|_{L^2(\overline{B}(x_k,R))} \Big| \\ &\leq \|u_\lambda(t)\|_{L^2(\overline{B}(x_k,R))} + \sum_{j\neq k} \|r_\lambda^j(t)\|_{L^2(\overline{B}(x_k,R))}.\end{aligned}$$

Using the fact that the L^2 norm of r_λ^k is only concentrated in x_k with mass $\|Q\|_{L^2(\mathbb{R}^d)}$, and that $\|u_\lambda(t)\|_{L^2}$ converges to 0 when t goes to T_λ, we obtain (i) by a passage to the limit.

(ii) We may write

$$\|h_\lambda(t)\|_{L^2}^2 = \|u_\lambda(t)\|_{L^2}^2 + \sum_{k=1}^p \|\varphi_k r_\lambda^k(t)\|_{L^2(\Omega)}^2 + 2\mathrm{Re}(\langle r_\lambda(t), u_\lambda(t)\rangle_{L^2}).$$

The first and the latter terms go to 0 when t tends to T_λ because $\|u_\lambda(t)\|_{L^2}$ tends to 0, and the second term converges to $p\|Q\|_{L^2}^2$ thanks to the property of concentration of r_λ^i near x_i.

(iii) Let ψ be a continuous function with compact support. Then if we denote

$$I(t) = \int_\Omega |h_\lambda(t,x)|^2\psi(x)dx - \|Q\|_{L^2(\mathbb{R}^d)}^2 \sum_{k=1}^p \psi(x_k)\ ,$$

we have

$$\begin{aligned}|I(t)| &\leq C\int_{\mathbb{R}^d} \Big| |r_\lambda(t,x)|^2\psi(x) - Q^2(x)\sum_{k=1}^p \psi(x_k)\Big| dx + C\int_\Omega |u_\lambda(t,x)|^2 dx \\ &\leq C\sum_{k=1}^p \int_{\mathbb{R}^d} \Big| \left| r_\lambda^k(t,x)\right|^2 \psi(x) - Q^2(x)\psi(x_k)\Big| dx + Ce^{-\frac{2\delta}{\lambda(T_\lambda - t)}} \\ &\quad + C\sum_{k=1}^p \int_{\mathbb{R}^d\setminus\overline{B}(x_k,R)} \Big| (1-\varphi_k^2(x))\psi(x) \left| r_\lambda^k(t,x)\right|^2 \Big| dx \\ &\leq C\sum_{k=1}^p \int_{\mathbb{R}^d} \Big| \left| r_\lambda^k(t,x)\right|^2 \psi(x) - Q^2(x)\psi(x_k)\Big| dx \\ &\quad + C\sum_{k=1}^p \int_{\mathbb{R}^d\setminus\overline{B}(x_k,R)} \left| r_\lambda^k(t,x)\right|^2 dx + Ce^{-\frac{2\delta}{\lambda(T_\lambda - t)}}.\end{aligned}$$

But $|r_\lambda^k(t)|^2$ converges to $\|Q\|_{L^2(\mathbb{R}^d)}^2\delta_{x_k}$ when $t\to T_\lambda$, so the first term goes to 0. The second one does as well by the well-known properties of r_λ^k.

(iv) We have the equality

$$\nabla h_\lambda = \nabla u_\lambda + \sum_{k=1}^{p} r_\lambda^k \nabla \varphi_k + \sum_{k=1}^{p} \varphi_k \nabla r_\lambda^k.$$

Remarking that $\|\nabla u_\lambda(t)\|_{L^2(\Omega)}$ decays to 0 when t goes to T_λ and

$$\| \sum_{k=1}^{p} r_\lambda^k(t) \nabla \varphi_k \|_{L^2(\Omega)}$$

is bounded, we get the equivalence

$$\|\nabla h_\lambda(t)\|_{L^2(\Omega)} \underset{t\to T_\lambda}{\sim} \Big\| \sum_{k=1}^{p} \varphi_k \nabla r_\lambda^k(t) \Big\|_{L^2(\Omega)}.$$

But the φ_k have disjoint supports, so

$$\Big\| \sum_{k=1}^{p} \varphi_k \nabla r_\lambda^k(t) \Big\|_{L^2(\Omega)} = \Big(\sum_{k=1}^{p} \Big\| \varphi_k \nabla r_\lambda^k(t) \Big\|_{L^2(\Omega)}^2 \Big)^{1/2};$$

and for all k,

$$\begin{aligned} \Big\| \varphi_k \nabla r_\lambda^k(t) \Big\|_{L^2(\Omega)} &\underset{t\to T_\lambda}{\sim} \Big\| \nabla r_\lambda^k(t) \Big\|_{L^2(\Omega)} \\ &\underset{t\to T_\lambda}{\sim} \frac{1}{\lambda(T_\lambda - t)} \|\nabla Q\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

We obtain (iv) by summing these equivalences. □

Let us give some remarks about Theorem 2.1.

Remark 1. The existence of blow-up solutions for the L^2-critical focusing nonlinear Schrödinger equation remains true if we replace the bounded domain of $\mathbb{R}^d$ by a Riemannian manifold of dimension $d = 2, 3$ which is locally isometric to an open subset of $\mathbb{R}^d$ near the blow-up points; that is the case of the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Remark 2. For $d = 3, 4$, one can construct solutions of the following L^2-supercritical equation posed on $\mathbb{T}^d$,

$$i\partial_t u + \Delta_{\mathbb{T}^d} u = -|u|^{\frac{4}{d-1}} u, \tag{2.17}$$

blowing up on the union of the p circles. Indeed, for $x_1, \dots, x_p \in \mathbb{T}^{d-1}$, by Theorem 2.1, there exists a solution $u \in C([0, T), H^2(\mathbb{T}^{d-1}))$ blowing up in the p points. Then, we consider the function $v \in C([0, T), H^2(\mathbb{T}^d))$ defined by $v(t, a, b) = u(t, a)$ for $a \in \mathbb{T}^{d-1}$, $b \in \mathbb{T}$. Then v is a solution of (2.17) and blows up in the p circles $\{x_k\} \times \mathbb{T}$. Note that the blow-up on a sphere for the supercritical equation has been studied more precisely in $\mathbb{R}^n$ (see [12]).

Remark 3. One might also find interesting the case where the equation is posed in dimension greater than 3. However, in this case, the nonlinearity is not regular enough to perform the same proof as in the case $d \leq 3$. Indeed, we need to solve the equation in a space included in H^s with $s > d/2 \geq 2$ to get the embedding into L^∞, but if $d \geq 4$ we can not derive the nonlinearity more than two times.

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