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TANGENTIAL FEEDBACK STABILIZATION OF PERIODIC FLOWS IN A 2-D CHANNEL

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Abstract. A 2-D incompressible channel flow with periodic condition along one axis is considered. The stabilization of it is achieved by a boundary feedback controller with tangential velocity observation which acts on the steamwise component of the velocity only. There are no a priori conditions imposed on the viscosity coefficient ν, that is, on the Reynolds number.

1. Introduction

In this paper, we address the problem of boundary control of a viscous incompressible fluid flow in a two-dimensional channel $(x,y)\in(-\infty,\infty)\times(0,1)$, with the walls located at $y=0,1$. The dynamic of the flow is governed by the dimensionless incompressible 2-D Navier-Stokes equations:

$$\begin{cases} u_t-\nu\Delta u+uu_x+vu_y=p_x, \qquad x\in\mathbb{R},\ y\in(0,1),\\ v_t-\nu\Delta v+uv_x+vv_y=p_y, \qquad x\in\mathbb{R},\ y\in(0,1),\\ u_x+v_y=0,\\ u(t,x,0)=0,\ u(t,x,1)=\Psi(t,x),\ v(t,x,0)=v(t,x,1)=0,\\ u(t,x+2\pi,y)=u(t,x,y),\ v(t,x+2\pi,y)\\ \qquad\qquad =v(t,x,y),\ p(t,x+2\pi,y)=p(t,x,y),\\ \forall t\geq 0,\ \forall x\in\mathbb{R},\ \forall y\in(0,1), \end{cases} \tag{1.1}$$

with initial data

$$u(0,x,y)=u_0(x,y),\ v(0,x,y)=v_0(x,y),\ x\in\mathbb{R},\ y\in(0,1).$$

Here $(u(t,x,y),v(t,x,y))$ is the velocity field, $p(t,x,y)$ is the unknown pressure, and ν is the viscosity coefficient. To avoid dealing with an infinitely long

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channel, we assume that the velocity field and the pressure are 2π-periodic in the first spatial coordinate x.

In order to compute the equilibrium state, we put the boundary control to be zero, i.e., $u = 0$, at $y = 1$, $\forall x \in \mathbb{R}$, and consider a steady-state flow with zero wall-normal velocity, i.e., $(U(x, y), 0)$. From the divergence free we first get that $U_x \equiv 0$; this means that $U = U(y)$. Substituting U into equation (1.1) we obtain

$$-\nu U''(y) = p^e_x(x, y), \quad p^e_y(x, y) \equiv 0, \tag{1.2}$$

where p^e is the equilibrium pressure corresponding to the steady-state solution $(U, 0)$ (by $'$ we mean the derivative with respect to the second spatial coordinate y). One can easily deduce then

$$U(y) = C(y^2 - y), \quad y \in (0, 1), \tag{1.3}$$

where $C \in \mathbb{R}_-$. In the following, we take $C = -\frac{a}{2\nu}$, where $a \in \mathbb{R}_+$.

This is a parabolic laminar flow profile. It is well known that the stability property of the stationary flow $(U, 0)$ varies with the Reynolds number $\frac{1}{\nu}$ (for ν small the flow is unstable, for ν big enough the flow is stable).

Our aim here is the stabilization of this flow profile by a boundary controller on the wall $y = 1$, that is,

$$u(t, x, 1) = \Psi(t, x), \quad t \geq 0, \ x \in \mathbb{R}.$$

There is no action, however, in $y = 0$, for the normal component v or inside the channel. Therefore, only the streamwise component u is controlled on the wall $y = 1$.

By the substitutions $(u, v) \to (u, v) - (U, 0)$ and $p \to p - p^e$, we are readily led via (1.1) and (1.2) to the study of null stabilization of the system

$$\begin{cases} u_t - \nu\Delta u + u_x U + vU' + uu_x + vu_y = p_x, \ x \in \mathbb{R}, \ y \in (0, 1), \\ v_t - \nu\Delta v + v_x U + uv_x + vv_y = p_y, \ x \in \mathbb{R}, \ y \in (0, 1), \\ u_x + v_y = 0, \\ u(t, x, 0) = 0, u(t, x, 1) = \Psi(t, x), \ v(t, x, 0) = v(t, x, 1) = 0, \\ u(t, x + 2\pi, y) = u(t, x, y), \ v(t, x + 2\pi, y) = v(t, x, y), \\ p(t, x + 2\pi, y) = p(t, x, y), \ \forall t \geq 0, \ \forall x \in \mathbb{R}, \ \forall y \in (0, 1), \end{cases} \tag{1.4}$$

with initial data

$$u(0, x, y) = u^0(x, y) = u_0(x, y) - U(y), \ v(0, x, y) = v^0(x, y) = v_0(x, y),$$

$x \in \mathbb{R}$, $y \in (0, 1)$. The linearization of (1.4) is the following system:

$$\begin{cases} u_t - \nu\Delta u + u_x U + vU' = p_x, \\ v_t - \nu\Delta v + v_x U = p_y, \\ u_x + v_y = 0, \\ u(t,x,0)=0, u(t,x,1)=\Psi(t,x),\ v(t,x,0) = v(t,x,1) = 0,\ \forall x \in \mathbb{R}, \\ u(t,x+2\pi,y) = u(t,x,y),\ v(t,x+2\pi,y) = v(t,x,y), \\ p(t,x+2\pi,y) = p(t,x,y),\ \forall t > 0,\ \forall x \in \mathbb{R},\ \forall y \in (0,1), \end{cases} \tag{1.5}$$

with initial data u^0, v^0.

This work is a continuation of [12], where the case of normal boundary controllability is considered. The main difference between that paper and the present work is that in [12] the normal velocity is controlled on the wall $y = 1$, while here we control the streamwise velocity on the wall $y = 1$. By controlling only the streamwise component on the wall $y = 1$ we have tangential boundary conditions for the velocity field; thus, if we prove the exponential asymptotical feedback stability of the linearized system (1.5), one can apply then the fixed-point method presented in (Theorem 5.1, [6]) in order to show the feedback stability of the nonlinear system (1.4), also. On the contrary, in [12], nothing can be said about the stability of the nonlinear system because of the nontangential boundary conditions.

The main result here (see Theorem 4.1 below) amounts to saying that the feedback asymptotical exponential stability of (1.4) can be achieved by a tangential boundary finite-dimensional feedback controller of the form

$$\Psi(t,x) = -\nu \sum_{|k|\le S} \frac{1}{ik}(L_k^{-2} R_k L_k v_k(t))''(1) e^{ikx},$$

where

$$v_k(t,y) = \int_0^{2\pi} v(t,x,y) e^{-ikx} dx,\ |k| \le S;$$

more precisely, for every initial data $(u^0, v^0) \in U_\rho$, the corresponding solution of the closed-loop system (1.4) satisfies

$$\|(u(t), v(t))\|_W \le M e^{-\omega t} \|(u^0, v^0)\|_W,\ \forall t > 0,$$

for some $M, \omega > 0$. Here S is given by relation (2.10) below, and the neighbourhood of the origin, U_ρ, and the space W are given in Theorem 4.1 below.

The expression of the stabilizable feedback controller involves linear operators $R_k : X \to X$, which are self-adjoint and satisfy Riccati algebraic equations in X:

$$\langle L_k^{-1} R_k z_0, L_k^{-1} \mathbf{A}_k z_0 \rangle + \frac{1}{2}\nu^2 |(L_k^{-2} R_k z_0)''(1)|^2 = \frac{1}{2}|L_k^{-1} z_0|^2, \ \forall z_0 \in X,$$

for all $|k| \leq S$, S given by (2.10). $\mathbf{A}_k = F_k L_k^{-1}$, where F_k is given by (2.5) and L_k is given by (2.4). $X = (H^2(0,1) \cap H_0^1(0,1))'$.

In a few words, our approach is the following one: we reduce first, as in [7] via the Fourier functional setting, the linear system (1.5) to an infinite parabolic system and prove the boundary stabilization of it (see Theorem 3.2 and notation (3.2) below). Next, using a standard technique—minimization of a cost functional—we prove the feedback exponential stability of the infinite parabolic system (see Theorem 3.3 below). Then, we plug that feedback controller, determined in the previous step, into the linear system (1.5), and prove the feedback asymptotical stability of it (see Theorem 3.1 below). Finally, using the fixed-point method presented in (Theorem 5.1, [6]), we show the feedback stability of the nonlinear system (1.4) also (see Theorem 4.1 below).

2. The Fourier functional setting

Let $L^2(Q)$, $Q = (0, 2\pi) \times (0,1)$, be the space of all functions $u \in L^2_{loc}(\mathbb{R} \times (0,1))$ which are 2π-periodic in x. These functions are characterized by their Fourier series:

$$u(x,y) = \sum_{k \in \mathbb{Z}} u_k(y) e^{ikx}, \ u_k = \bar{u}_{-k}, \ u_0 = 0,$$

$$\sum_{k \in \mathbb{Z}} \int_0^1 |u_k(y)|^2 dy < \infty.$$

The norm in $L^2(Q)$ is defined as

$$|u|_{L^2(Q)} = \Big(\sum_{k \in \mathbb{Z}} |u_k|^2_{L^2(0,1)} \Big)^{\frac{1}{2}}.$$

Since there is no danger of confusion we shall denote by $\|\cdot\|$ the norm in $L^2(Q)$ and in $L^2(0,1)$ also.

We set $(L^2(Q))^2 = \left\{(u,v) : u, v \in L^2(Q)\right\}$. We denote also by $\|\cdot\|$ the norm in $(L^2(Q))^2$, defined as $\|(u,v)\| = (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$. Finally, we set

$$H_\pi = \left\{(u,v) \in (L^2(Q))^2 : v_k(0) = v_k(1) = 0,\ iku_k + v_k' = 0,\ k \in \mathbb{Z},\ k \neq 0\right\}. \tag{2.1}$$

We define the norm in H_π to be the norm in $(L^2(Q))^2$, defined above.

Since the modes $u_0 = 0$ for all $u \in L^2(Q)$, from now on we only consider $k \in \mathbb{Z}$, with $k \neq 0$.

We now return to system (1.5) and rewrite it in terms of Fourier coefficients $(u_k)_{k\in\mathbb{Z}}$, $(v_k)_{k\in\mathbb{Z}}$, $(p_k)_{k\in\mathbb{Z}}$, and $(\psi_k)_{k\in\mathbb{Z}}$. We have

$$\left\{\begin{array}{c} (u_k)_t - \nu u_k'' + (\nu k^2 + ikU)u_k + U'v_k = ikp_k,\ \text{a.e. in } (0,1), \\ (v_k)_t - \nu v_k'' + (\nu k^2 + ikU)v_k = p_k',\ \text{a.e. in } (0,1), \\ iku_k + v_k' = 0,\ \text{a.e. in } (0,1), \\ u_k(0) = 0,\ u_k(1) = -\frac{1}{ik}\psi_k,\ v_k(0) = v_k(1) = 0. \end{array}\right. \tag{2.2}$$

with initial data u_k^0, v_k^0. Here $u = \sum_{k\in\mathbb{Z}} u_k(t,y)e^{ikx}$, $v = \sum_{k\in\mathbb{Z}} v_k(t,y)e^{ikx}$, and $p = \sum_{k\in\mathbb{Z}} p_k(t,y)e^{ikx}$, $\Psi = \sum_{k\in\mathbb{Z}} -\frac{1}{ik}\psi_k(t)e^{ikx}$ (we consider the Fourier modes of Ψ in the form $-\frac{1}{ik}\psi_k$ because of the facilitation of the computations).

Reducing p_k from the first two equations from (2.2), we obtain

$$\begin{gathered} ik(v_k)_t - ik\nu v_k'' + ik^2(\nu k + iU)v_k - (u_k')_t \\ +\nu u_k''' - k(\nu k + iU)u_k' - ikU'u_k - U'v_k' - U''v_k = 0. \end{gathered}$$

Taking $u_k = -\frac{1}{ik}v_k'$, we have

$$\begin{gathered} ik(v_k)_t - ik\nu v_k'' + ik^2(\nu k + iU)v_k + \frac{1}{ik}(v_k'')_t - \frac{\nu}{ik}v_k^{iv} \\ +\frac{1}{i}(\nu k + iU)v_k'' - U''v_k = 0, t \geq 0, y \in (0,1). \end{gathered}$$

Finally,

$$\left\{\begin{array}{c} (v_k'' - k^2v_k)_t - \nu v_k^{iv} + (2\nu k^2 + ikU)v_k'' - k(\nu k^3 + ik^2U + iU'')v_k = 0, \\ t \geq 0,\ y \in (0,1), \\ v_k(0) = v_k(1) = 0,\ v_k'(0) = 0,\ v_k'(1) = \psi_k(t). \end{array}\right. \tag{2.3}$$

In what follows we shall consider H to be the complexified space of $L^2(0,1)$. We denote also by $\|\cdot\|$, the norm in H and by $\langle\cdot,\cdot\rangle$, the scalar product. We shall denote by $H^m(0,1)$, $m = 1, 2, \dots$ the standard Sobolev spaces on $(0,1)$ and

$$H_0^1 = \left\{v \in H^1(0,1) : v(0) = v(1) = 0\right\},$$

$$H_0^2(0,1)=\left\{v\in H^2(0,1)\cap H_0^1(0,1): v'(0)=v'(1)=0\right\},$$
$$H_0^3(0,1)=\left\{v\in H^3(0,1)\cap H_0^2: v''(0)=v''(1)=0\right\}.$$

For each $0\neq k\in\mathbb{Z}$, we denote by $L_k:\mathcal{D}(L_k)\subset H\to H$ and $F_k:\mathcal{D}(F_k)\subset H\to H$ the operators

$$L_kv=-v''+k^2v,\ \mathcal{D}(L_k)=H_0^1\cap H^2(0,1), \tag{2.4}$$

$$F_kv=\nu v^{iv}-(2\nu k^2+ikU)v''+k(\nu k^3+ik^2U+iU'')v, \tag{2.5}$$
$$\mathcal{D}(F_k)=H^4(0,1)\cap H_0^2(0,1).$$

With this notation, equation (2.3) becomes

$$\begin{cases} (L_kv_k)_t+F_kv_k=0,\ t\geq 0,\ y\in(0,1),\\ v_k(0)=v_k(1)=0,\ v_k'(0)=0,\ v_k'(1)=\psi_k(t).\end{cases} \tag{2.6}$$

Consider the solution $w_k=w_k(t,y)$ of the equation

$$\begin{cases} \theta_kw_k+F_kw_k=0,\ y\in(0,1),\ t\geq 0,\\ w_k(0)=w_k(1)=0,\ w_k'=0,\ w_k'(1)=\psi_k.\end{cases} \tag{2.7}$$

(For θ_k positive and sufficiently large, there exists a unique solution w_k for equation (2.7)). Then, subtracting (2.6) and (2.7), we obtain

$$(L_kv_k)_t+F_k(v_k-w_k)-\theta_kw_k=0,\ t\geq 0.$$

Equivalently,

$$(L_k(v_k-w_k))_t+F_k(v_k-w_k)=\theta_kw_k-(L_k(w_k))_t, \tag{2.8}$$
$$v_k-w_k\in\mathcal{D}(F_k).$$

In order to represent equation (2.8) as an abstract boundary control system, we consider the operators $\mathbf{A}_k:\mathcal{D}(\mathbf{A}_k)\subset H\to H$ defined by

$$\mathbf{A}_k=F_kL_k^{-1}, \tag{2.9}$$
$$\mathcal{D}(\mathbf{A}_k)=\left\{v\in H: L_k^{-1}v\in\mathcal{D}(F_k)\right\}.$$

We have

Lemma 2.1. *The operator $-\boldsymbol{A}_k$ generates a C_0-analytic semigroup on H and for each $\lambda\in\rho(-\boldsymbol{A}_k)$ (the resolvent set of $-\boldsymbol{A}_k$), $(\lambda I+\boldsymbol{A}_k)^{-1}$ is compact. Moreover, one has $\sigma(-\boldsymbol{A}_k)\subset\{\lambda\in\mathbb{C}:\Re\lambda\leq 0\}$, $\forall|k|>S$, where*

$$S=\frac{1}{\sqrt{\nu}}\left(1+\frac{\sqrt{a}}{\sqrt{2\nu}}\right)^{\frac{1}{2}}. \tag{2.10}$$

Here $\sigma(-\boldsymbol{A}_k)$ is the spectrum of $-\boldsymbol{A}_k$.

Proof. The proof can be found in (Lemma 1, [7]). □

Remark 2.1. Let us observe that, if we define the operators $\mathbf{O}_k : \mathcal{D}(\mathbf{O}_k) \subset H \to H$ as $\mathbf{O}_k v = L_k^{-1} F_k v$, $\forall v \in \mathcal{D}(\mathbf{O}_k) = \left\{ v \in H : F_k v \in \mathcal{D}(L_k^{-1}) \right\}$ one can easily obtain, as in Lemma 2.1, that $-\mathbf{O}_k$ generates a C_0-analytic semigroup on H, $\forall k \in \mathbb{Z}$ and

$$\sigma(-\mathbf{O}_k) \subset \{\lambda \in \mathbb{C} : \Re\lambda \leq 0\}, \forall k \text{ such that } |k| > S, \tag{2.11}$$

where S is given by $S = \frac{1}{\sqrt{\nu}}(1 + \frac{\sqrt{a}}{\sqrt{2\nu}})^{\frac{1}{2}}$.

Hence, for $|k| > S$, $-\mathbf{O}_k$ satisfies the spectrum-determined growth condition

$$\|e^{-\mathbf{O}_k t}\|_{L(H,H)} \leq C_\delta e^{-\delta t}, \ \forall t \geq 0, \tag{2.12}$$

for some $C_\delta, \delta > 0$.

For $|k| > S$, we write equation (2.6) with boundary controller $\psi_k \equiv 0$ as

$$(L_k v_k)_t + F_k v_k = 0, \ t \geq 0, \ y \in (0,1), \tag{2.13}$$
$$v_k(0) = 0, \ v_k(1) = 0, \ v_k'(0) = v_k'(1) = 0.$$

Since the operators L_k^{-1} and $\frac{d}{dt}$ commute, if we apply L_k^{-1} to the equation (2.13), we get that

$$(v_k)_t + \mathbf{O}_k v_k = 0, \ t \geq 0, \ y \in (0,1), \tag{2.14}$$
$$v_k(0) = 0, \ v_k(1) = 0, \ v_k'(0) = v_k'(1) = 0.$$

It is easy to see that, via (2.12), we have

$$\|v_k\| \leq C_\delta e^{-\delta t} \|v_k^0\|, \ \forall t \geq 0, \tag{2.15}$$

which implies that the exponential asymptotical stability of (2.6) (equivalently (2.3)) holds true for $|k| > S$, without any boundary controller. This means that we have to control the system (2.6) (equivalently (2.3)) for $|k| \leq S$ only.

Now, coming back to system (2.8), we set

$$y_k(t) = L_k(v_k(t) - w_k(t))$$

and write it as

$$\begin{aligned} y_k(t) &= e^{-\mathbf{A}_k t} y_k(0) + \int_0^t e^{-\mathbf{A}_k (t-s)} (\theta_k w_k(s) - (L_k(w_k(s))_s) ds \\ &= e^{-\mathbf{A}_k t} y_k(0) - L_k(w_k(t)) + e^{-\mathbf{A}_k t} L_k(w_k(0)) + \\ &\quad + \int_0^t e^{-\mathbf{A}_k (t-s)} (\theta_k(w_k(s)) + (\tilde{F}_k(w_k(s))) ds, \end{aligned} \tag{2.16}$$

where $\tilde{F}_k : H \to (\mathcal{D}(F_k^*))'$ is the extension of F_k to H defined by

$$(\tilde{F}_k v, \psi) = \int_0^1 v(y) F_k^* \psi(y) dy. \tag{2.17}$$

$\forall \psi \in \mathcal{D}(F_k) = \mathcal{D}(F_k^*)$.

The space $(\mathcal{D}(F_k^*))'$ is the completion of the space H in the norm $|||f||| = ||(\lambda I + F_k)^{-1} f||$, for some $\lambda \in \mathbb{R}$ sufficiently large. Where F_k^* is the dual operator of F_k.

Define similarly $\tilde{\mathbf{A}}_k$, the extension of $\mathbf{A}_k$ to H. Likewise $-\mathbf{A}_k$, the operator $-\tilde{\mathbf{A}}_k$ generates e C_0-analytic semigroup on H. In the same way is defined the extension of L_k to an operator from H to $(H_0^1(0,1) \cap H^2(0,1))'$, again denoted L_k.

Then, the above equation (2.16) can be rewritten as

$$\frac{d}{dt}(L_k v_k(t)) + \tilde{\mathbf{A}}_k(L_k(v_k)(t)) = (\theta_k + \tilde{F}_k)(w_k(t)), t \geq 0. \tag{2.18}$$

For each $\psi \in \mathbb{C}$, we denote by $D_k \psi = w \in H^4(0,1)$ the solution to the equation

$$\begin{cases} \theta_k w + F_k w = 0, \forall y \in (0,1), \\ w(0) = w(1) = 0,\, w'(0) = 0,\, w'(1) = \psi. \end{cases} \tag{2.19}$$

(The operator D_k is called the Dirichlet map associated with $\theta_k + F_k$.) It is easy to see that the dual $((\theta_k + \tilde{F}_k) D_k)^*$ is given by

$$((\theta_k + \tilde{F}_k) D_k)^* \phi = -\nu \phi''(1), \tag{2.20}$$

for all $\phi \in H^4(0,1)$, $\phi(0) = \phi(1) = 0$, $\phi'(0) = 0$.

With this notation, equation (2.18) can be rewritten as

$$\frac{d}{dt}(L_k v_k(t)) + \tilde{\mathbf{A}}_k(L_k(v_k)(t)) = (\theta_k + \tilde{F}_k)(D_k \psi_k(t)),\ t > 0, \tag{2.21}$$

and initial data v_k^0.

Remark 2.2. Note that the systems (2.21), (2.6), and (2.3) are all equivalent.

Remark 2.3. Equation (2.21) is understood in the following weak sense:

$$\langle \frac{d}{dt} L_k v_k(t), \phi \rangle + \langle L_k v_k, \mathbf{A}_k^* \phi \rangle = \langle D_k \psi_k(t), (\theta_k + F_k^*) \phi \rangle,\ \forall \phi \in \mathcal{D}(\mathbf{A}_k^*).$$

2.1. **Further results concerning the linear operators $\mathbf{A}_k$.** For every $0 \neq k \in \mathbb{Z}$, denote by $(\lambda_j^k)_{j=1}^{\infty}$ the eigenvalues of $-\mathbf{A}_k$ (λ_j^k is repeated according to its multiplicity m_j^k). Hence, $(\bar{\lambda}_j^k)_{j=1}^{\infty}$ are the eigenvalues of the dual operator $-\mathbf{A}_k^*$ of $-\mathbf{A}_k$. Denote by $\{\phi_j^k : j = 1, 2, \dots\}$ and $\{\phi_j^{*k} : j = 1, 2, \dots\}$ the corresponding eigenfunctions for $-\mathbf{A}_k$ and $-\mathbf{A}_k^*$, respectively.

Remark 2.4. Concerning the eigenvalues and eigenfunctions of the extended operator $-\tilde{\mathbf{A}}_k$, it is known that they coincide with the eigenvalues and eigenfunctions of $-\mathbf{A}_k$. The same holds true for the dual operators $-\tilde{\mathbf{A}}_k^*$.

As mentioned before (see Remark 2.1), we have to stabilize system (2.21) (equivalently (2.6)) for $|k| \leq S$, only. So, let us consider $k \in \mathbb{Z}$ such that $|k| \leq S$. From Lemma 2.1 we know that $-\mathbf{A}_k$ generates a C_0-analytic semigroup and has a compact resolvent in H, so it has only a finite number M_k of eigenvalues λ_j^k with $\Re\lambda_j^k \geq 0$, the unstable eigenvalues. Denote by $N_k = m_1^k + m_2^k + \cdots + m_{M_k}^k$ (m_j^k is the multiplicity of $\lambda_j^k, j = 1, 2, \dots, M_k$).

The algebraic multiplicity of λ_j^k is the dimension of the range of the projection operator

$$P_j = -\frac{1}{2\pi i}\int_{\Gamma_j} (\lambda I + \mathbf{A}_k)^{-1} d\lambda,$$

where Γ_j is a smooth, closed curve encircling λ_j^k. The geometric multiplicity of λ_j^k is the dimension of the eigenfunction space corresponding to λ_j^k. In general, the algebraic multiplicity is greater than the geometric multiplicity, but when they are equal, the eigenvalue is called semi-simple. Here we shall assume that the following assumption holds.

(A_1) *All the eigenvalues λ_j^k, with $j = 1, \dots, M_k$, are semi-simple.*

We shall see that, at the end of the day, this assumption is not absolutely necessary for the construction of the stabilizing controller, but it simplifies however the argument.

Next, we will see that the geometric multiplicity of each eigenvalue of the operators $-\mathbf{A}_k$ is 1. Thus, if an eigenvalue is semi-simple then it is simple. This is important when we will compute the Kalman matrix of controllability, and this is the reason why we need only one controller on the wall $y = 1$, and not on both the walls (see Remark 3.2 below).

Proposition 2.1. *Let λ be an eigenvalue of the linear operator $-\boldsymbol{A}_k$. If λ is semi-simple, then it is simple.*

Proof. First we show the next lemmas.

Lemma 2.2. *Let $\lambda \in \mathbb{C}$ and ϕ^* be a solution to the equation*

$$\bar{\lambda}\phi^* + \boldsymbol{A}_k^*\phi^* = 0 \text{ in } (0,1), \tag{2.22}$$
$$\phi^*(0) = \phi^*(1) = 0,\ (\phi^*)'(0) = (\phi^*)'(1) = 0.$$

If $(\phi^*)''(1) = 0$, then $\phi^* \equiv 0$.

Proof. The proof follows the same ideas as in (Lemma 2.2, [12]), so, it will be omitted. □

Lemma 2.3. *The geometric multiplicity of each eigenvalue λ of $-\boldsymbol{A}_k$ is 1.*

Proof. The proof follows the same ideas as in (Lemma 2.3, [12]), so, it will be omitted. □

The proof of Proposition 2.1 follows immediately from the definition of a semi-simple eigenvalue and Lemma 2.3. □

Remark 2.5. By Proposition 2.1 and assumption (A_1), we have that the multiplicities m_j^k, $j = 1, 2, \ldots, M_k$ are all equal to 1 and $M_k = N_k$. Moreover, it follows by the assumption (A_1) that the system $\{\phi_j^{*k}\}_{j=1}^{N_k}$ can be chosen in such a way that

$$\langle \phi_l^k, \phi_j^{*k} \rangle = \delta_{lj},\ l, j = 1, \ldots, N_k.$$

We can conclude now with the next remark.

Remark 2.6. Remark 2.4 and Remark 2.5 imply that for all $|k| \leq S$, operator $-\tilde{\mathbf{A}}_k$ has a finite number of unstable eigenvalues $(\lambda_j^k)_{j=1}^{N_k}$, which are simple. Moreover, the corresponding eigenfunctions $(\phi_j^k)_{j=1}^{N_k}$ and $(\phi_j^{*k})_{j=1}^{N_k}$ of $-\tilde{\mathbf{A}}_k$ and $-\tilde{\mathbf{A}}_k^*$, respectively, can be chosen in such a way that

$$\langle \phi_l^k, \phi_j^{*k} \rangle = \delta_{lj},\ l, j = 1, \ldots, N_k.$$

Lemma 2.4. *The algebraic multiplicity of each unstable eigenvalue λ_j^k, $j = 1, \ldots, N_k$ of $-\boldsymbol{A}_k$ is ≤ 2.*

Proof. The proof can be found in (Lemma 2.4, [12]). □

And, finally, we have the following:

Proposition 2.2. *For all $|k| \leq S$, $D_k : \mathbb{C} \to H$ is a linear continuous operator.*

Proof. One can easily adapt for this case the proof of (Proposition 2.2, [12]). □

3. FEEDBACK STABILIZATION OF THE LINEARIZED SYSTEM (1.5)

The aim of this section is to give a proof of the next theorem, which proves the feedback controllability of the linear system (1.5).

Theorem 3.1. *There exists a finite-dimensional feedback controller* ψ *of the form*

$$\psi(t,x) = -\nu \sum_{|k|\leq S} \frac{1}{ik}(L_k^{-2}R_kL_kv_k(t))''(1)e^{ikx}, \tag{3.1}$$

where

$$v_k(t,y) = \int_0^{2\pi} v(t,x,y)e^{-ikx}dx,\ |k| \leq S,$$

such that once inserted into equation (1.5), *the corresponding solution of the closed-loop system* (1.5) *satisfies*

$$\|(u(t),v(t))\|^2 \leq Ce^{-\alpha t}\|(u^0,v^0)\|^2,\ t \geq 0,$$

for some $C,\alpha > 0$*. Here* $R_k : X \to X$ *are linear self-adjoint operators which satisfy Riccati algebraic equations in* X*,*

$$\langle L_k^{-1}R_kz_0, L_k^{-1}\boldsymbol{A}_kz_0\rangle + \frac{1}{2}\nu^2|(L_k^{-2}R_kz_0)''(1)|^2 = \frac{1}{2}|L_k^{-1}z_0|^2,\ \forall z_0 \in X,$$

for all $|k| \leq S$*,* S *given by* (2.10). $\boldsymbol{A}_k = F_kL_k^{-1}$*, where* F_k *is given by* (2.5) *and* L_k *is given by* (2.4). $X = (H^2(0,1) \cap H_0^1(0,1))'$.

In order to prove Theorem 3.1, we first control the system (2.21) (equivalently (2.6)), for every $|k| \leq S$, such that we have exponential asymptotical stability of the system, that is, Theorem 3.2 below (see notation (3.2)).

Let us fix $k \in \mathbb{Z}$ such that $0 < |k| \leq S$, S given by relation (2.10).

For simplicity we are going to omit the index k and the symbol $\tilde{}$ (since there is no danger of confusion); also, we are going to denote some symbols, i.e.,

$$-\tilde{\mathbf{A}}_k = \mathbf{A},\quad L_kv_k = z,\quad (\theta_k + \tilde{F}_k)D_k = \mathbf{B},\quad \psi_k = \psi. \tag{3.2}$$

With this notation, equation (2.21) becomes

$$\frac{d}{dt}z(t,y) - \mathbf{A}z(t,y) = \mathbf{B}\psi(t),\quad t > 0,\quad z(0,y) = z_0(y), \tag{3.3}$$

where $z_0(y) = L_kv_k^0(y)$.

By virtue of Remark 2.4 and Remark 2.6 we have that the eigenvalues of $\mathbf{A}$ are $\lambda_j = \lambda_j^k$, $j = 1,2,\ldots$, and the eigenfunctions $\phi_j = \phi_j^k$, $j = 1,2,\ldots$. The eigenfunctions of the dual operator $\mathbf{A}^*$, of $\mathbf{A}$, are $\phi_j^* = \phi_j^{*k}$, $j = 1,2,\ldots$.

Moreover, the operator $\mathbf{A}$ has a finite number of unstable eigenvalues $(\lambda_j)_{j=1}^N$, which are simple, and the corresponding eigenfunctions $(\phi_j)_{j=1}^N$ and $(\phi_j^*)_{j=1}^N$ can be chosen in such a way that $\langle \phi_l, \phi_j^* \rangle = \delta_{lj}$, $l, j = 1, \dots, N$. We denote by $Z_N^u = \text{span}\,\{\phi_j\}_{j=1}^N$, and by $Z_N^s = \text{span}\,\{\phi_j\}_{j=N+1}^\infty$.

Denote by P_N the projection and its adjoint P_N^*, defined by

$$P_N = -\frac{1}{2\pi i}\int_\Gamma (\lambda I - \mathbf{A})^{-1} d\lambda;\ P_N^* = -\frac{1}{2\pi i}\int_{\tilde{\Gamma}} (\lambda I - \mathbf{A}^*)^{-1} d\lambda,$$

where Γ (respectively, its conjugate $\tilde{\Gamma}$) separates the unstable spectrum from the stable one of $\mathbf{A}$ (respectively, $\mathbf{A}^*$). We set

$$\mathbf{A}_N^u = P_N \mathbf{A} \text{ and } \mathbf{A}_N^s = (I - P_N)\mathbf{A} \tag{3.4}$$

for the restrictions of $\mathbf{A}$ to Z_N^u and Z_N^s, respectively. These projections commute with $\mathbf{A}$. We then have that the spectra of $\mathbf{A}$ on Z_N^u and Z_N^s coincide with $\{\lambda_j\}_{j=1}^N$ and $\{\lambda_j\}_{j=N+1}^\infty$, respectively. (For more details about the projection P_N see [11].)

Moreover, since $\mathbf{A}$ generates a C_0-analytic semigroup on H, then its restriction $\mathbf{A}_N^s$ to Z_N^s generates likewise a C_0-analytic semigroup on Z_N^s. This implies that $\mathbf{A}_N^s$ satisfies the spectrum-determined growth condition on Z_N^s, and so we have

$$\|e^{\mathbf{A}_N^s t}\|_{L(H,H)} \leq C_{\alpha_0} e^{-\alpha_0 t},\ \forall t \geq 0, \tag{3.5}$$

for some $\alpha_0 < |\Re \lambda_{N+1}|$. The system (3.3) can accordingly be decomposed as $z = z_N + \zeta_N$, $z_N = P_N z$, $\zeta_N = (I - P_N)z$, where applying P_N and $(I - P_N)$ on (3.3), we obtain

$$\text{on } Z_N^u\text{: } \frac{d}{dt} z_N - \mathbf{A}_N^u z_N = P_N(\mathbf{B}\psi),\ z_N(0) = P_N z_0, \tag{3.6}$$

$$\text{on } Z_N^s\text{: } \frac{d}{dt} \zeta_N - \mathbf{A}_N^s \zeta_N = (I - P_N)(\mathbf{B}\psi),\ \zeta_N(0) = (I - P_N) z_0, \tag{3.7}$$

respectively.

3.1. Stabilization of the unstable z_N-system (3.6) on Z_N^u.

Lemma 3.1. *There exist $\alpha_1 > 0$ and a controller ψ such that once inserted in* (3.6), *it yields the estimate*

$$\|z_N(t)\| \leq C_{\alpha_1} e^{-\alpha_1 t} \|z_0\|,\ t \geq 0;$$

here z_N is the solution to (3.6), *for the corresponding controller ψ. Moreover, the controller ψ can be chosen of class C^1 and such that*

$$|\frac{d}{dt}\psi(t)| + |\psi(t)| \leq C_{\alpha_1} e^{-\alpha_1 t}\|z_0\|,\ t \geq 0.$$

Proof. Let us decompose z_N as

$$z_N(t,y) = \sum_{j=1}^{N} z^j(t)\phi_j(y).$$

We introduce that z_N in equation (3.6), and obtain

$$\sum_{j=1}^{N} \Big\{ (\frac{d}{dt} z^j(t))\phi_j - z^j \mathbf{A}\phi_j \Big\} = P_N(\mathbf{B}\psi).$$

We multiply (scalarly) the above equation with ϕ_l^*, and obtain

$$\frac{d}{dt}(z^l) - \lambda_l z^l = (P_N \mathbf{B}\psi, \phi_l^*),\ l = 1, \ldots, N. \tag{3.8}$$

Let us observe that we can assume $P_N^*(\phi_l^*) = \phi_l^*$, $l = 1, \ldots, N$. We have

$$(P_N \mathbf{B}\psi, \phi_l^*) = (\mathbf{B}\psi, P_N^* \phi_l^*) = \psi \mathbf{B}^* \phi_l^* =$$
$$= \text{ by (2.20) } = -\psi\nu(\phi_l^*)''(1),\ l = 1, \ldots, N.$$

Denote by

$$b_l = -\nu(\phi_l^*)''(1),\ l = 1, \ldots, N. \tag{3.9}$$

We set

$$\mathbf{z} = \operatorname{col} \|z^l\|_{l=1}^N, \tag{3.10}$$
$$B = \operatorname{col} \|b_l\|_{l=1}^N, \tag{3.11}$$

and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix}. \tag{3.12}$$

With this notation, equation (3.8) becomes

$$\frac{d}{dt}\mathbf{z} - \Lambda \mathbf{z} = B\psi. \tag{3.13}$$

Remark 3.1. Note that B is a column matrix, and by $B\psi$ we mean the product between a scalar and a column matrix. This is because, in our case $\psi = \psi(t)$, only.

So, the question is whether the pair $\{\Lambda, B\}$ is controllable. To see this we shall test the Kalman's criterion of controllability. The Kalman controllability matrix is $[B, \Lambda B, \Lambda^2 B, \dots, \Lambda^{N-1}B] = \begin{pmatrix} b_1 & \lambda_1 b_1 & \cdots & \lambda_1^{N-1} b_1 \\ b_2 & \lambda_2 b_2 & \cdots & \lambda_2^{N-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_N & \lambda_N b_N & \cdots & \lambda_N^{N-1} b_N \end{pmatrix}$. The Kalman matrix is of size $N \times N$. We shall prove that it is of full rank.

It is easy to see that the rank of the Kalman matrix $[\Lambda, B]$ is full if and only if $b_l \neq 0$, $l = 1 \dots, N$, since the Kalman matrix is of Vandermonde type. By Lemma 2.2, we have $(\phi_l^*)''(1) \neq 0$, for all $l = 1, \dots, N$, since $\phi_l^* \neq 0$, $l = 1, \dots, N$. Hence, $b_l \neq 0$, $l = 1, \dots, N$ (see (3.9)), so the Kalman's controllability condition is satisfied.

Remark 3.2. ***The general case of eigenvalues.*** The general case of eigenvalues (i.e., the case when the eigenvalues are not all semi-simple) can be treated in the same manner as in (Remark 3.2, [12]), obtaining that the Kalman's controllability condition is satisfied in this case also.

Since the Kalman's criterion of controllability is satisfied, we know from finite-dimensional theory that there exists a vector q in $\mathbb{C}^N$ such that the spectrum of the matrix $[\Lambda + Bq]$ may be arbitrarily preassigned, in particular to lie on the left half-plane $\{\lambda : \Re\lambda < -\Re\lambda_{N+1}\}$, as desired. Thus, returning form $\mathbb{C}^N$ to Z_N^u, there exists a vector Ξ in Z_N^u such that $\psi = \langle z_N, \Xi \rangle$, whereby the closed-loop system corresponding to (3.6) is given by

$$\frac{d}{dt} z_N - \mathbf{A}_N^u z_N = \langle z_N, \Xi \rangle, \tag{3.14}$$

rewritten as

$$\frac{d}{dt} z_N = \hat{\mathbf{A}}^u z_N,\ z_N = e^{\hat{\mathbf{A}}^u t} z_N(0), \tag{3.15}$$

where $\hat{\mathbf{A}}^u = \mathbf{A}_N^u + \Xi$.

In conclusion, there exist $\alpha_1 > 0$ and $\Xi \in Z_N^u$ such that the solution to (3.6) corresponding to the controller $\psi = \langle z_N, \Xi \rangle$ satisfies the estimate

$$|\psi(t)| + \|z_N(t)\| = \|q z_N(t)\| + \|z_N(t)\| \leq$$

$$\leq (|q| + 1)\|e^{\hat{\mathbf{A}}^u t} z_N(0)\| \leq C_{\alpha_1} e^{-\alpha_1 t} \|z_N(0)\|,\ t \geq 0.$$

Moreover, since the solution z_N is of class C^1, the controller is of class C^1 and satisfies the exponential decay claimed. □

3.2. **Stabilization of system (3.3).**

Theorem 3.2. *There exists a controller ψ such that the corresponding solution z to the system* (3.3) *and the controller have the exponential decay*

$$|\frac{d}{dt}\psi(t)| + |\psi(t)| \leq C_{\alpha_1} e^{-\alpha_1 t}\|z_0\|, \ \|z(t)\| \leq C_{\alpha_0} e^{-\alpha_0 t}\|z_0\|, \ \forall t \geq 0,$$

where α_0 is given by (3.5), $0 < \alpha_0 < \Re\lambda_{N+1}$, *and $\alpha_1 > 0$ is given by Lemma* 3.1.

Proof. The proof can be found in (Theorem 3.2, [12]) □

Remark 3.3. By Theorem 3.2 and the notation (3.2), we have that there exists a controller ψ such that once inserted into equation (3.3), the corresponding solution of (3.3) satisfies the estimate

$$\|L_k v_k(t)\| \leq C_{\alpha_0} e^{-\alpha_0 t}\|L_k v_k^0\|, \ \forall t \geq 0. \tag{3.16}$$

Taking the scalar product between $L_k v$ and v, for some $v \in H$, we get that

$$\|v'\|^2 + k^2\|v\|^2 = \langle L_k v, v\rangle.$$

Hence, by the Schwarz inequality, we have

$$\|v\|^2 \leq |(L_k v, v)| \leq \|L_k v\|\|v\|.$$

This implies that

$$\|v\| \leq \|L_k v\|. \tag{3.17}$$

We replace C_{α_0} by $C_{\alpha_0}\|L_k\|_{L(H,H)}$. Then, relations (3.16) and (3.17) imply that

$$\|v_k(t)\| \leq C_{\alpha_0} e^{-\alpha_0 t}\|v_k^0\|, \ t \geq 0. \tag{3.18}$$

Returning to the notation (3.2), relation (3.18) can be rewritten as

$$\|L_k^{-1} z(t)\| \leq C_{\alpha_0} e^{-\alpha_0 t}\|L_k^{-1} z_0\|, \ \forall t \geq 0. \tag{3.19}$$

3.3. **Feedback stabilization of the system (3.3).** By Remark 3.3, relation (3.19), we see that the controller ψ stabilizes system (3.3) in the $X = (H^2(0,1) \cap H_0^1(0,1))'$ topology. This suggests considering system (3.3) in the space X and looking for a feedback representation of the controller ψ by the standard technique—minimization of a cost functional. More precisely, we have the next result.

Theorem 3.3. *There exists a feedback controller of the form*

$$\psi = \nu(L_k^{-2} R_k z)''(1),$$

such that, once inserted into equation (3.3), *the corresponding solution of the closed-loop system* (3.3) *satisfies*

$$\|L_k^{-1}z(t)\| \leq C_\gamma e^{-\gamma t}\|L_k^{-1}z_0\|,\ \forall t \geq 0,$$

for some $C_\gamma, \gamma > 0$. *Here* $R_k : X \to X$ *is a linear self-adjoint operator which satisfies the next Riccati algebraic equation in* X,

$$-\langle L_k^{-1}R_k z_0, L_k^{-1}\boldsymbol{A}z_0\rangle + \frac{1}{2}\nu^2|(L_k^{-2}R_k z_0)''(1)|^2 = \frac{1}{2}|L_k^{-1}z_0|^2,\ \forall z_0 \in X.$$

Proof. Consider the optimization problem

$$\phi(z_0) = \min \frac{1}{2}\int_0^\infty (\|L_k^{-1}z(t)\|^2 + |\psi(t)|^2)dt, \tag{3.20}$$

subject to $\psi \in L^2(0,\infty;X)$ and

$$\frac{d}{dt}z(t) - \mathbf{A}z(t) = \mathbf{B}\psi,\ z(0) = z_0. \tag{3.21}$$

Let us first show that the optimization problem is well-posed on the state space X; i.e., $\phi(z_0) < \infty$, $\forall z_0 \in X$.

We must show that with $z_0 \in X$ arbitrary, there exists some control $\psi \in L^2(0,\infty;X)$ such that the corresponding solution z of (3.21) satisfies $z \in L^2(0,\infty;X)$. Indeed, Remark 3.3 provides the wanted result (see relation (3.19)). From the exponential stability we deduce also that there exists some constant a_2 such that

$$\phi(z_0) \leq a_2\|L_k^{-1}z_0\|^2,\ \forall z_0 \in X. \tag{3.22}$$

It is easy to see that the map $\phi(z) \to z \in X$ is continuous; thus, $\|z\| \leq c\phi(z)$. This, with relation (3.22), shows that there exist constants a_1 and a_2 such that

$$a_1\|z_0\|^2 \leq \phi(z_0) \leq a_2\|z_0\|^2,\ \forall z_0 \in X. \tag{3.23}$$

Thus, by (3.23), there is a linear nonnegative self-adjoint operator $R_k : X \to X$ associated with the linear symmetric form $\phi(\cdot)$, such that

$$\phi(z_0) = \tfrac{1}{2}\langle R_k z_0, z_0\rangle_X,\ \forall z_0 \in X. \tag{3.24}$$

$R_k \in L(X,X)$. By the dynamic programming principle, for each $0 < t < T$, the optimal solution (ψ^*, z^*) to (3.20), (3.21) is also the solution to the optimization problem

$$\min\Big\{\frac{1}{2}\int_t^T (\|L_k^{-1}z(s)\|^2 + |\psi(s)|^2)ds + \phi(z(T)), \tag{3.25}$$
$$\text{subject to (3.20)},\ z(t) = z^*(t)\Big\},$$

$z^*(t) \in X$ as initial condition, where $z^*(T) \in X$ as well.

By the maximum principle, we obtain that

$$\psi^*(t) = \mathbf{B}^* q_T = -\nu q_T''(1), \text{ a.e. } t \in (0,T),$$
$$L_k^{-2} R_k z^*(t) = -q_T(t), \ \forall t \in [0,T], \tag{3.26}$$
$$\psi^*(t) = \mathbf{B}^*(L_k^{-2} R_k z^*(t)), \ \forall t \geq 0,$$

where q_T is the solution to the dual equation

$$\frac{d}{dt} q_T + \mathbf{A}^* q_T = L_k^{-2} z, \ \forall t \in (0,T),$$
$$q_T(T) = -L_k^{-2} R_k z^*(T). \tag{3.27}$$

Finally, we show that R_k is a solution to a Riccati-type equation. To this end, we first notice that, again by the dynamic programming principle and (3.24), we have

$$\frac{1}{2}\langle R z^*(t), z^*(t)\rangle = \phi(z^*(t)) = \frac{1}{2}\int_t^\infty (\|L_k^{-1} z^*(s)\|^2 + |\psi^*(s)|^2) ds, \ \forall t \geq 0. \tag{3.28}$$

Differentiating (3.28) in t and using the self-adjointness of R_k on X and equation (3.21) yields

$$-(L_k^{-1}\mathbf{A} z^*(t), L_k^{-1} R_k z^*(t)) + \frac{1}{2}\nu^2 |(L_k^{-2} R_k z^*(t))''(1)|^2 = \frac{1}{2}\|L_k^{-1} z^*(t)\|^2, \ t \geq 0, \tag{3.29}$$

which implies, by setting $t = 0$, that R_k satisfies the Riccati equation

$$(L_k^{-1}\mathbf{A} z_0, L_k^{-1} R_k z_0) + \frac{1}{2}\nu^2 |(L_k^{-2} R_k z_0)''(1)|^2 = \frac{1}{2}||L_k^{-1} z_0||^2, \ \forall z_0 \in X. \tag{3.30}$$

In the end, using the classical Datko's Theorem, we finally obtain the exponential decay claimed. □

Remark 3.4. Applying Theorem 3.3 for all $|k| \leq S$ (see notation (3.2)), we obtain that for all $|k| \leq S$, there exist constants $C_{\gamma_k}, \gamma_k > 0$ and a feedback controller ψ_k of the form $\psi_k(t) = \nu(L_k^{-2} R_k L_k^{-1} v_k(t))''(1)$, such that, once inserted into the equation (2.21), the corresponding solution of the closed-loop system (2.21) satisfies the exponential decay

$$\|v_k(t)\| \leq C_{\gamma_k} e^{-\gamma_k t} \|v_k^0\|, \ \forall t \geq 0. \tag{3.31}$$

3.4. **Feedback stabilization of the linear system (1.5).** Next, we will prove Theorem 3.1.

Proof of Theorem 3.1. For convenience we will again write down Theorem 3.1.

Theorem 3.4. *There exists a finite-dimensional feedback controller* ψ *of the form*

$$\psi(t,x) = -\nu \sum_{|k|\leq S} \frac{1}{ik}(L_k^{-2}R_kL_kv_k(t))''(1)e^{ikx}, \tag{3.32}$$

where

$$v_k(t,y) = \int_0^{2\pi} v(t,x,y)e^{-ikx}dx,\ |k| \leq S,$$

such that once inserted into equation (1.5), *the corresponding solution of the closed-loop system* (1.5) *satisfies*

$$\|(u(t),v(t))\|^2 \leq Ce^{-\alpha t}\|(u^0,v^0)\|^2,\ t\geq 0,$$

for some $C,\alpha>0$. *Here* $R_k : X \to X$ *are linear self-adjoint operators which satisfy Riccati algebraic equations in* X,

$$\langle L_k^{-1}R_kz_0, L_k^{-1}\boldsymbol{A}_kz_0\rangle + \frac{1}{2}\nu^2|(L_k^{-2}R_kz_0)''(1)|^2 = \frac{1}{2}|L_k^{-1}z_0|^2,\ \forall z_0\in X,$$

for all $|k|\leq S$, S *given by* (2.10). $\boldsymbol{A}_k = F_kL_k^{-1}$, *where* F_k *is given by* (2.5) *and* L_k *is given by* (2.4). $X = (H^2(0,1)\cap H_0^1(0,1))'$.

Proof. Let us define the feedback controller (see Remark 3.4)

$$\Psi(t,x) = -\nu \sum_{|k|\leq S} \frac{1}{ik}(L_k^{-2}R_kL_kv_k(t))''(1)e^{ikx}, \tag{3.33}$$

where

$$v_k(t,y) = \int_0^{2\pi} v(t,x,y)e^{-ikx}dx,\ |k| \leq S.$$

Here R_k is given by Theorem 3.3.

If we plug that controller into system (1.5), we have by Remark 3.4, relation (3.31), that

$$\|v_k(t)\|^2 \leq C_{\gamma_k}^2 e^{-2\gamma_k t}\|v_k^0\|^2,\ \forall t\geq 0,\ \forall |k|\leq S. \tag{3.34}$$

By Remark 2.1, relation (2.15), we have that

$$\|v_k(t)\|^2 \leq C_\delta^2 e^{-2\delta t}\|v_k^0\|^2,\ \forall t\geq 0,\ \forall |k|> S. \tag{3.35}$$

Thus, relations (3.34) and (3.35) imply that

$$\sum_k \|v_k(t)\|^2 \leq Ce^{-\beta_1 t} \sum_k \|v_k^0\|^2, \ \forall t \geq 0, \tag{3.36}$$

where $C = \max\left\{C_{\gamma_k}^2 : |k| \leq S, C_\delta^2\right\}$ and $\beta_1 = 2\min\{\gamma_k : |k| \leq S, \delta\}$.

Now, taking into account the definition of the norm in $L^2(0, 2\pi) \times (0, 1))$, relation (3.36) implies that

$$\|v(t)\|^2 \leq Ce^{-\beta_1 t}\|v^0\|^2, \ t \geq 0. \tag{3.37}$$

In order to estimate u, we will use the linearized system (1.5), which we will write down again for convenience:

$$\begin{cases} u_t - \nu\Delta u + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} = \frac{\partial p}{\partial x}, \\ v_t - \nu\Delta v + U\frac{\partial v}{\partial x} = \frac{\partial p}{\partial y}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \ x \in \mathbb{R}, \ y \in (0,1), \ t \geq 0, \end{cases} \tag{3.38}$$

with boundary-value conditions

$$u(t, x + 2\pi, y) = u(t, x, y), \ v(t, x + 2\pi, y) = v(t, x, y),$$

$$p(t, x + 2\pi, y) = p(t, x, y), \ u(t, x, 0) = 0, \ u(t, x, 1) = \Psi(t, x),$$

$$v(t, x, 0) = 0, \ v(t, x, 1) = 0, \ \forall x \in \mathbb{R}, \ y \in (0, 1), \ t \geq 0.$$

Multiplying (scalarly) the first equation of (3.38) by u and the second by v and taking into account the 2π-periodicity of u and v and the fact of being divergence free, we get

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \|v(t)\|^2) + \nu(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2) + \int_Q u(t)v(t)U'(y)dx\,dy = 0, \tag{3.39}$$

$t \geq 0$. By Poincaré's inequality, since the trace of u is zero, we have

$$\|u(t)\|^2 \leq \mathcal{C}\|\nabla u(t)\|^2, \ t \geq 0. \tag{3.40}$$

Hence, (3.39) together with (3.40) yields

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \|v(t)\|^2) + \frac{1}{\mathcal{C}}\nu(\|u(t)\|^2) + \int_Q u(t)v(t)U'(y)dx\,dy \leq 0. \tag{3.41}$$

We estimate $\int_Q u(t)v(t)U'(y)dx\,dy$ using the next inequality:

$$|qw| \leq \frac{1}{2}\Big(K|q|^2 + \frac{1}{K}|w|^2\Big),$$

for $0 < K < \frac{2\nu}{\mathcal{C}}$, by taking $q = u(t)$ and $w = v(t)U'$. This yields

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \|v(t)\|^2) + \frac{\nu}{\mathcal{C}}\|u(t)\|^2 \leq \frac{K}{2}\|u(t)\|^2 + \frac{a^2}{K8\nu^2}\|v(t)\|^2. \quad (3.42)$$

Hence

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \|v(t)\|^2) + \beta_2\|u(t)\|^2 \leq \frac{a^2}{K8\nu^2}\|v(t)\|^2, \quad (3.43)$$

where $0 < \beta_2 = \frac{\nu}{\mathcal{C}} - \frac{K}{2}$.

Then, using estimate (3.37), we finally obtain from (3.43) that

$$\|u(t)\|^2 + \|v(t)\|^2 \leq Ce^{-\alpha t}(\|u^0\|^2 + \|v^0\|^2),\ t \geq 0, \quad (3.44)$$

for some $C, \alpha > 0$. □

4. Feedback stabilization of the nonlinear system (1.4)

In order to prove that we can obtain the exponential stability of the nonlinear system (1.4) by the feedback controller Ψ defined in Theorem 3.1, we will write the system (1.4) more compactly. For this we provide the following notation, which we shall use for the rest of this paper.

Remember the nonlinear problem (1.1), with boundary controller:

$$\begin{gathered}
u_t - \nu\Delta u + uu_x + vu_y = p_x,\ x \in \mathbb{R},\ y \in (0,1),\\
v_t - \nu\Delta v + uv_x + vv_y = p_y,\ x \in \mathbb{R},\ y \in (0,1),\\
u_x + v_y = 0, \qquad (4.1)\\
u(t,x,0) = 0,\ u(t,x,1) = \Psi(t,x),\ v(t,x,0) = v(t,x,1) = 0,\\
u(t,x+2\pi,y) = u(t,x,y),\ v(t,x+2\pi,y) = v(t,x,y),\ p(t,x+2\pi,y) = p(t,x,y),\\
\forall t \geq 0,\ \forall x \in \mathbb{R},\ \forall y \in (0,1).
\end{gathered}$$

Denote by $Y = (u,v) \in (L^2(Q))^2$, $\mathcal{X} = (x,y) \in \mathbb{R}^2$, and $\Omega = Q \times (0,\infty)$, where $Q = (0,2\pi) \times (0,1)$. $\partial Q = \Sigma_1 \cup \Sigma_2$, where Σ_1 is the wall $y = 1$ and $\Sigma_2 = \partial Q \setminus \Sigma_1$. Introduce the operator $\mathcal{G} : L^2(0,2\pi) \to \left(L^2(\partial Q)\right)^2$ as

$$\mathcal{G}(\Psi) = \begin{cases} (\Psi(x),0), & \text{if } (x,y) \in \Sigma_1,\\ (0,0), & \text{if } (x,y) \in \Sigma_2, \end{cases} \quad (4.2)$$

for all $\Psi \in L^2(0,2\pi)$.

With this notation, equations (4.1) can be written as

$$\begin{gathered}
Y_t(t,\mathcal{X}) - \nu\Delta Y(t,\mathcal{X}) + (Y\cdot\nabla)Y(t,\mathcal{X}) = \nabla p(t,\mathcal{X}) \text{ in } \Omega;\\
\nabla \cdot Y = 0 \text{ in } \Omega;\\
Y = \mathcal{G}(\Psi(t)),\ \text{on } \partial Q, t \geq 0; \qquad (4.3)\\
Y(0,\mathcal{X}) = Y_0(\mathcal{X}) = (u_0(\mathcal{X}), v_0(\mathcal{X})) \text{ in } Q.
\end{gathered}$$

Remember the steady-state solution $(U, 0)$, denoted here by Y_e, which satisfies the next equation:

$$\begin{aligned} -\nu\Delta Y_e + (Y_e \cdot \nabla)Y_e &= \nabla p_e \text{ in } Q; \\ \nabla \cdot Y_e &= 0 \text{ in } Q; \\ Y_e &= 0 \text{ on } \partial Q. \end{aligned} \tag{4.4}$$

It is shown in (Section 3.1, [5]) that the translated equation of (4.3)(i.e., $Y \Rightarrow Y - Y_e$, $p \Rightarrow p - p_e$) projected on the space H_π defined by (2.1) is given by

$$Y_t - \mathcal{A}Y + \mathcal{B}Y = -\mathcal{A}D\mathcal{G}\Psi \in [\mathcal{D}(\mathcal{A}^*)]',\ Y_0 \in H_\pi, \mathcal{G}(\Psi) \cdot n \equiv 0 \text{ on } \partial Q,\ t \geq 0, \tag{4.5}$$

where the operator $\mathcal{A}$ is the extension by transposition $\mathcal{A} : H_\pi \to [\mathcal{D}(\mathcal{A}^*)]'$, duality with respect to H_π given by (2.1) as a pivot space, of the original (differential) Oseen operator

$$\mathcal{A} = -(\nu A + A_0),\ \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) = (H^2(Q))^2 \cap H_\pi \to H_\pi; \tag{4.6}$$

$$AY = P(\Delta Y), \forall Y \in \mathcal{D}(A);\ A_0 Y = P((Y_e \cdot \nabla)Y - (Y \cdot \nabla)Y_e),\ \mathcal{D}(A_0) = \mathcal{D}(A^{\frac{1}{2}}); \tag{4.7}$$

and

$$\mathcal{B}Y = P((Y \cdot \nabla)Y),\ \forall Y \in \mathcal{D}(A^{\frac{1}{2}}). \tag{4.8}$$

Here $P : (L^2(Q))^2 \to H_\pi$ is the Leray projector ([9], p. 9).

The definition of the Dirichlet map D in (4.5) is given as follows (Section 3.1 complemented by Appendix A.2, [5]). To begin with, we introduce for convenience the differential expression

$$\mathbb{A}Y = -\nu\Delta Y + (Y_e \cdot \nabla)Y + (Y \cdot \nabla)Y_e. \tag{4.9}$$

Next, we consider the stationary Stokes/Oseen problem

$$\left\{ \begin{array}{c} \mathcal{Z} = D_k g \Leftrightarrow (k + \mathbb{A})\mathcal{Z} = \nabla p^* \text{ in } Q, \\ \nabla \cdot \mathcal{Z} = 0, \text{ in } Q, \\ \mathcal{Z} = \left\{ \begin{array}{c} (g(x), 0), \text{ if } (x, y) \in \Sigma_1, \\ (0, 0), \text{ if } (x, y) \in \Sigma_2. \end{array} \right. \end{array} \right. \tag{4.10}$$

The corresponding linear system is

$$Y_t - \mathcal{A}Y = -\mathcal{A}D\mathcal{G}\Psi \in [\mathcal{D}(\mathcal{A}^*)]',\ Y_0 \in H_\pi, \mathcal{G}(\Psi) \cdot n \equiv 0 \text{ on } \partial Q,\ t \geq 0. \tag{4.11}$$

Remark 4.1. Observe that system (4.5) is equivalent to system (1.4) and system (4.11) is equivalent to system (1.5).

Finally, the main result of this paper is the next theorem.

Theorem 4.1. *Let* $W = \mathcal{D}(A^{\frac{1}{4}})$. *Let* $\mathcal{U}_\rho = \{(u^0, v^0) \in W : \|(u^0, v^0)\|_W \leq \rho\}$. *With reference to Theorem* 3.1, *the feedback controller*

$$\psi(t,x) = -\nu \sum_{|k|\leq S} \frac{1}{ik}(L_k^{-2}R_kL_kv_k(t))''(1)e^{ikx}$$

once inserted into system (1.4), *there exists* ρ *sufficiently small such that for all initial data* $(u^0, v^0) \in U_\rho$ *there is a unique solution* $(u, v) \in C([0,\infty); W) \cap L^2(0,\infty; Z)$ *to the closed-loop system* (1.4) *which satisfies*

$$\|(u(t), v(t))\|_W \leq Me^{-\omega t}\|(u^0, v^0)\|_W,\ \forall t > 0,$$

for some $M, \omega > 0$. *Here* $Z = \mathcal{D}(A^{\frac{3}{4}})$. A *is defined by* (4.7).

Remark 4.2. If system (4.5) (equivalently (4.3)) is feedback stabilizable, then it follows that system (1.4) is feedback stabilizable. Hence it is enough to prove the feedback stability of (4.5).

By Theorem 3.1 we have that the feedback controller

$$\Psi = FY,\ FY = -\nu \sum_{|k|\leq S} \frac{1}{ik}(L_k^{-2}R_kL_kv_k(t))''(1)e^{ikx},$$

exponentially stabilizes the linear system (4.11); more precisely, we have

$$\|e^{-\mathcal{A}_Ft}Y_0\| \leq Ce^{-\alpha t}\|Y_0\|,\ t \geq 0,\ Y_0 \in H_\pi, \tag{4.12}$$

where $\mathcal{A}_FY = \mathcal{A}Y - \mathcal{A}D\mathcal{G}FY$.

We introduce that controller into equation (4.5), and obtain

$$Y_t + \mathcal{B}Y = \mathcal{A}_FY. \tag{4.13}$$

The corresponding variation-of-parameters formula of (4.13) is

$$Y(t) = e^{\mathcal{A}_Ft}Y_0 - (NY)(t);\ (NY)(t) = \int_0^t e^{\mathcal{A}_F(t-\tau)}(\mathcal{B}Y)(\tau)d\tau. \tag{4.14}$$

Theorem 4.2. *If* $\rho > 0$ *is sufficiently small, for each* $Y_0 \in W$ *with* $\|Y_0\|_W \leq \rho$, *the problem* (4.14) *is well posed on* W *with unique solution* $Y \in C([0,\infty]; W) \cap L^2(0,\infty; Z)$. *Moreover, these local solutions satisfy the following local exponential decay: there exist constants* $M \geq 1$ *and* $\omega > 0$, *independent of* $\rho > 0$, *such that*

$$\|Y(t)\| \leq Me^{-\omega t}\|Y_0\|,\ t \geq 0.$$

Here W *and* Z *are the spaces introduced in Theorem* 4.1.

Proof. We shall apply Theorem 5.1 from [6]. In order to do that we have to show that the hypotheses $(H.1)$, $(H.1i)$, $(H.1ii)$, and $(H.1iii)$ from [6] hold true in our case (see [6], p. 2712); more precisely, we must show that with the notation above we have the following:

- (i) $\mathcal{A}_F$ generates an analytic semigroup $e^{\mathcal{A}_F t}$ on W, which is uniformly exponentially stable on W; i.e.,
$$\|e^{\mathcal{A}_F t}\|_{L(W,W)} \leq Ce^{-\alpha t},\ t \geq 0,$$
for some $C, \alpha > 0$.
- (ii) For each $Y_0 \in W$, we have $e^{\mathcal{A}_F t} Y_0 \in L^2(0, \infty; Z)$; thus, for some positive constant c,
$$\int_0^\infty \|e^{\mathcal{A}_F t} Y_0\|_Z^2\, dt \leq c\|Y_0\|_W^2,\ \forall Y_0 \in W.$$

To this end, we have the next lemma.

Lemma 4.1. *We have*

$$\int_0^\infty \|e^{-(\mathcal{A}_F - \alpha)t} Y_0\|_Z^2\, dt \leq c\|Y_0\|_W,\ \forall Y_0 \in W, \tag{4.15}$$

$$\|e^{-\mathcal{A}_F t} Y_0\|_W \leq Ce^{-\alpha t}\|Y_0\|_W,\ \forall t \geq 0. \tag{4.16}$$

Proof. The function $V(t) = e^{-(\mathcal{A}_F - \alpha)t} Y_0$ is the solution to the equation

$$\frac{dV}{dt} + \nu AV + A_0 V + \mathcal{A}DFV = \alpha V,\ V(0) = Y_0.$$

If we multiply the latter by $A^{\frac{1}{2}} V$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|V(t)\|_W^2 + \nu\|A^{\frac{3}{4}} V(t)\|^2 \leq C\left(\|A^{\frac{1}{2}} V(t)\|^2 + \|A^{\frac{1}{2}} V(t)\|\|V(t)\|\right),\ t > 0.$$

By an interpolation inequality, we have

$$\|A^{\frac{1}{2}} V\|^2 \leq \|A^{\frac{3}{4}} V\|^{\frac{2}{3}} \|V\|^{\frac{4}{3}} \leq \frac{\nu}{2}\|A^{\frac{3}{4}} V\|^2 + C\|V\|^2,$$

and this yields

$$\frac{d}{dt}\|V(t)\|_W^2 + \frac{\nu}{2}\|A^{\frac{3}{4}} V(t)\|^2 \leq C_1\|V(t)\|^2,\ \forall t > 0.$$

Taking into account that, by relation (4.12),

$$\|V(t)\| \leq Ce^{-\alpha t}\|Y_0\|,\ \forall t \geq 0,$$

the conclusion follows immediately. ☐

As mentioned before, in order to prove Theorem 4.2 we apply Theorem 5.1 from [6]. Since the proof is exactly the same, we only sketch the idea.

First, for any $r > 0$, introduce the ball of radius r, centered at the origin, of the space $L^2(0,\infty;Z)$:

$$S(0,r) = \Big\{ f \in L^2(0,\infty;Z) : |f|_{L^2(0,\infty;Z)} = \Big\{ \int_0^\infty \|f(t)\|_Z \, dt \Big\}^{\frac{1}{2}} \le r \Big\}.$$

Next, for any $Y_0 \in W$ and $z \in L^2(0,\infty;Z)$, introduce the map

$$(\Lambda z)(t) = e^{\mathcal{A}_F t} Y_0 - (Nz)(t);\ (Nz)(t) = \int_0^t e^{\mathcal{A}_F (t-\tau)} (\mathcal{B} z)(\tau) d\tau.$$

Remark 4.3. Concerning the nonlinear term $\mathcal{B}$, we have by ([6], Lemma 5.4), the next key estimate:

$$\|\mathcal{B} z\|_W \le h \|z\|_Z^2,\ \forall z \in Z. \tag{4.17}$$

Using (4.17) we get the next estimate for Λ.

Lemma 4.2. *For any $Y_0 \in W$ and $z \in L^2(0,\infty;Z)$, the map Λ satisfies the following estimate:*

$$|\Lambda z|^2_{L^2(0,\infty;Z)} \le 2c\|Y_0\|^2_W + 2ch^2 \left[\int_0^\infty \|z(t)\|_Z^2 dt \right]^2, \tag{4.18}$$

where $c > 0$ is the constant in (4.15), *and $h > 0$ is the constant identified in* (4.17).

Further, imposing some constraints on c, h, and r, and using Lemma 4.2, we obtain the next lemma.

Lemma 4.3. *Let $z \in S(0,r)$, where $r > 0$ is chosen to satisfy the following constraints:*

$$2c\|Y_0\|^2_W \le \frac{1}{2} r^2;\ 2ch^2 \le \frac{1}{2} r^2 \quad \text{or} \quad r \le \frac{1}{2\sqrt{ch}}. \tag{4.19}$$

Then

$$|\Lambda z|^2_{L^2(0,\infty;Z)} \le r^2.$$

Lemma 4.3 says that Λ maps $S(0,r)$ into $S(0,r)$. In order to prove the existence of the solution, we shall apply the contraction-mapping principle. For this we have the next proposition.

Proposition 4.1. *Let $Y_0 \in W$ and $z_1, z_2 \in S(0,r)$. Then the operators Λ and N satisfy*

$$|\Lambda z_1 - \Lambda z_2|_{L^2(0,\infty;Z)} = |Nz_1 - Nz_2|_{L^2(0,\infty;Z)} \le 2\sqrt{c}hr|z_1 - z_2|_{L^1(0,\infty;Z)}, \tag{4.20}$$

where $c > 0$ is the constant in (4.15) *and $h > 0$ is the constant in* (4.17).

The main results, via Proposition 4.1, are the next two propositions.

Proposition 4.2. *Let $r < \frac{1}{2\sqrt{c}h}$, c be as in* (4.15), *and h be as in* (4.17). *We have*

(i) *the operator Λ is a contraction on $S(0,r)$;*

(ii) *for any $Y_0 \in W$, $\|y_0\|_W \le r/2\sqrt{c}$, there exists a unique solution $Y \in S(0,r) \subset L^2(0,\infty;Z)$ of*

$$\Lambda Y = Y; \text{ that is, } Y(t) = e^{\mathcal{A}_F t}Y_0 - \int_0^t e^{\mathcal{A}_F(t-\tau)}(\mathcal{B}Y)(\tau)d\tau; \tag{4.21}$$

(iii) *more precisely, problem* (4.14) (*equivalently* (4.13)) *has a unique solution Y for any Y_0 as in (ii), and r specified there, satisfying*

$$Y \in C([0,\infty); B(0,b)) \cap L^2(0,\infty;Z),$$

where $B(0,b)$ is the ball in W defined by

$$B(0,b) = \{f \in W : \|f\|_W \le b\}, \ b = \frac{C}{2ch},$$

C in (4.16), *c in* (4.15), *h in* (4.17), *continuously in $Y_0 \in W$:*

$$\int_0^\infty \|Y(t)\|_Z^2 \, dt \le c_{1r}\|Y_0\|_W^2; \ c_{1r} = \frac{2c}{1-2ch^2r^2} > 0. \tag{4.22}$$

Proposition 4.3. *Let $Y_0 \in W$, $\|Y_0\| < r/2\sqrt{c}$ for $r < 1/(2\sqrt{c}h)$, c be as in* (4.15), *and h be as in* (4.17). *Then, with reference to the unique solution $Y(t)$ of* (4.14) *asserted in Proposition* 4.2, *we have that there exist constants $M \ge 1$ and $\omega > 0$ such that*

$$\|Y(t)\|_W \le Me^{-\omega t}\|Y_0\|_W, \ t \ge 0. \tag{4.23}$$

The proof of Theorem 4.1 follows immediately from Remark 4.2 and Theorem 4.2. □

Remark 4.4. It should be emphasized that the Riccati equation (3.30), which provides the feedback controller (3.33), is easily manageable from a computational point of view since it is associated with a parabolic boundary control system on $(0,1)$, whose structure is identical for different wave numbers k.

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