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GENERALIZED STRICHARTZ ESTIMATES ON PERTURBED WAVE EQUATION AND APPLICATIONS ON STRAUSS CONJECTURE

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Abstract. In this paper we prove a general Strichartz estimate for certain perturbed wave equations, and here we can drop the nontrapping hypothesis and handle trapping obstacles with some loss of derivatives for data in the local energy decay estimates. We then give the obstacle version of the sharp life span for semilinear wave equations when $n = 3, p < p_c$, by using the real interpolation method, and by getting corresponding finite time Strichartz estimates (see Section 3). Finally, as another application, we get the Strauss conjecture for semilinear wave equations with several convex obstacles when $n = 3, 4$ (see Section 4).

1. Introduction and main result

The purpose of this paper is to prove a general Strichartz estimate for certain perturbed wave equations under known local energy decay estimates, and, as applications, we get the Strauss conjecture for several convex obstacles in $n = 3, 4$. Our results improve on earlier work in Hidano, Metcalfe, Smith, Sogge and Zhou [15]. Firstly, we can drop the nontrapping hypothesis and handle trapping obstacles with some loss of derivatives for data in the local energy decay estimates (see (1.2) below). The hypothesis (1.2) is fulfilled in many cases in the nontrapping case where there is local decay of energy with no loss of derivatives (see [34], [24], [32], [3], [25]). (1.2) is also known to hold in several examples involving hyperbolic trapped rays (see [16], [17], [9]). Secondly, we give the obstacle version of the sharp life span for semilinear wave equations when $n = 3$ and $p < p_c$, by using the real interpolation between the KSS estimate and the endpoint trace lemma, and by getting a corresponding finite time Strichartz estimate. This blowup result complements the Strauss conjecture stated in [15], which only dealt with the global existence part ($p > p_c$) (see Section 3). Lastly, we are able

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to use the general Strichartz estimates we have gained to get the Strauss conjecture for some perturbed semilinear wave equations with trapped rays when $n = 3, 4$ (see Section 4).

We consider wave equations on an exterior domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} (\partial_t^2 - \Delta_g)u = F(t,x) \text{ on } \mathbb{R}_+ \times \Omega, \\ u|_{t=0} = f,\ \partial_t u|_{t=0} = g, \\ (Bu)(t,x) = 0, \text{ on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.1}$$

where for simplicity we take B to be either the identity operator or the inward pointing normal derivative ∂_v. The operator Δ_g is the Laplace-Beltrami operator associated with a smooth, time independent Riemannian metric $g_{jk}(x)$ which we assume equals the Euclidean metric δ_{jk} for $|x| \geq R$, for some R. The set Ω is assumed to be either all of $\mathbb{R}^n$, or the complement of a subset of $|x| < R$ with smooth boundary. Note that here we do not require that $\mathbb{R}^n \backslash \Omega$ be nontrapping.

We will make the following local decay assumption.

Hypothesis B. Fix the boundary operator B and the exterior domain $\Omega \subset \mathbb{R}^n$ as above. We then assume that given $R_0 > 0$

$$\int_0^S (\|u(t,\cdot)\|^2_{H^1(|x|<R_0)} + \|\partial_t u(t,\cdot)\|^2_{L^2(|x|<R_0)})\, dt \\ \lesssim \|f\|^2_{\dot H^{1+\varepsilon}} + \|g\|^2_{\dot H^{\varepsilon}} + \int_0^S \|F(s,\cdot)\|^2_{\dot H^{\varepsilon}}\, ds, \tag{1.2}$$

where u is a solution of (1.1) with data (f, g) and forcing term F that both vanish for $|x| > R_0$.

Remark 0.1. We assume S to be a finite time T or ∞. Moreover, although ε could be any real number without affecting our techniques much, here we assume $\varepsilon \geq 0$ is an arbitrarily small number (which is all we need for now) throughout the paper for clarity of explanation. Note that, when $\varepsilon = 0$ and $T = \infty$, it is just the case in [15]. More specifically, when the obstacle is nontrapping, and Δ is the standard Euclidean Laplacian, we will have that local energy decays exponentially in odd dimensions $n \geq 3$ and polynomially in even dimensions except $n = 2$ ([23]); for $n = 2$, local energy decays like $O((\log(2+t))^{-2}(1+t)^{-1})$ ([34]). These results imply (1.2). When Δ_g is a time-independent variable coefficient compact perturbation of Δ, one also has that (1.2) is valid for the Dirichlet-wave equation for $n \geq 3$ as well as for $n = 2$ if $\partial\Omega \neq \emptyset$ ([32],[3]). On the other hand, when there are trapped rays, it is known that a uniform decay rate is generally not possible ([26]), but we can get some local energy decay by trading some derivatives in the initial data. Ikawa, for example, got the following exponential decay when $n = 3$ and there are several convex obstacles which are far apart (see [17]):

$$\left\|u'(t,x)\right\|_{L^2_x(|x|<1)} \lesssim e^{-at}\left\|u'(0,x)\right\|_{\dot{H}^2(|x|<1)},$$

where a is a constant By interpolation between this estimate and the standard energy estimates, it is easy to get

$$\left\|u'(t,x)\right\|_{L^2_x(|x|<1)} \lesssim e^{-ct}\left(\|f\|_{\dot{H}^{1+\varepsilon}(|x|<1)} + \|g\|_{\dot{H}^{\varepsilon}(|x|<1)}\right), \tag{1.3}$$

where c is a constant, which implies our Hypothesis B. When there is only one hyperbolic trapped ray, Christianson ([9]) also showed that for all odd dimensions $n \geq 3$ we have the local energy decay

$$\left\|u'(t,x)\right\|_{L^2_x(|x|<1)} \lesssim e^{-t^{1/2}/C}\left\|u'(0,x)\right\|_{\dot{H}^{1+\varepsilon}(|x|<1)}, \tag{1.4}$$

which gives Hypothesis B as well. Further work in this direction can be seen in [4], [5], [8], [11].

In order to deal with the extra derivatives in (1.2), we introduce a Sobolev-type norm as follows.

Definition 0.2. *Define $\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)$ (and $\tilde{H}^\gamma_\varepsilon(\Omega)$) to be the space with norm*

$$\|h\|_{\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)} = \left\||D|^\gamma(1-\Delta)^{\frac{\varepsilon}{2}}h\right\|_{L^2_x(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n}\left||\xi|^\gamma(1+|\xi^2|)^{\frac{\varepsilon}{2}}\hat{h}(\xi)\right|^2 d\xi\Big)^{1/2}. \tag{1.5}$$

Notice that, if $0 \leq \varepsilon_1 < \varepsilon_2$, then $\tilde{H}^\gamma_{\varepsilon_2} \subset \tilde{H}^\gamma_{\varepsilon_1}$. We also notice that, when $\varepsilon \geq 0$, the above norm is equivalent to the following useful form:

$$\|h\|_{\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)} \approx \|h\|_{\dot{H}^\gamma(|\xi|<1)} + \|h\|_{\dot{H}^{\gamma+\varepsilon}(|\xi|>1)} \approx \|h\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|h\|_{\dot{H}^{\gamma+\varepsilon}(\mathbb{R}^n)}. \tag{1.6}$$

Remark 0.3. Similarly we can define the norm on a manifold Ω as in [15] and [28]. Roughly speaking,

$$\|f\|_{\tilde{H}^\gamma_\varepsilon(\Omega)} = \|\beta f\|_{\tilde{H}^\gamma_\varepsilon(\Omega')} + \|(1-\beta)f\|_{\tilde{H}^\gamma_\varepsilon(\mathbb{R}^n)},$$

where $\beta \in \mathbb{C}^\infty_0$ is supported in $|x| < 2R$ and equals 1 in $|x| < R$, and Ω' is the embedding of $\Omega \cap \{|x| < 2R\}$ into the torus obtained by periodic extension of $\Omega \cap [-2R, 2R]^n$, so that $\partial\Omega' = \partial\Omega$. Here the spaces $\tilde{H}^\gamma_\varepsilon(\Omega')$ are defined by a spectral decomposition of $\Delta_{\mathrm{g}}|_{\Omega'}$ subject to the boundary condition B.

When $\varepsilon = 0$, it is easy to see our definitions coincide with the homogenous Sobolev spaces and we have $\tilde{H}^\gamma_0(\mathbb{R}^n) = \dot{H}^\gamma(\mathbb{R}^n)$, $\tilde{H}^\gamma_0(\Omega) = \dot{H}^\gamma(\Omega)$.

Now we can redefine "admissible" using the above Sobolev-type norm.

Definition 0.4. *We say that (X,γ,η,p) is almost admissible if it satisfies*
i) Minkowski almost Strichartz estimates

$$\|u\|_{L_t^p X([0,S]\times\mathbb{R}^n)} \lesssim A(S)\left(\|u(0,\cdot)\|_{\dot H^{\gamma}(\mathbb{R}^n)} + \|\partial_t u(0,\cdot)\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}\right), \tag{1.7}$$

where $A(S)$ is a function of S and equals a constant when $S=\infty$;
ii) Local almost Strichartz estimates for Ω

$$\|u\|_{L_t^p X([0,1]\times\Omega)} \lesssim \|u(0,\cdot)\|_{\tilde H^{\gamma}_{\eta}(\Omega)} + \|\partial_t u(0,\cdot)\|_{\tilde H^{\gamma-1}_{\eta}(\Omega)}. \tag{1.8}$$

Notice that the factor $A(S)$ only depends on the time S, and we only consider when the time S is large and $A(S)\gtrsim 1$. Also notice that here we assume a weaker local Strichartz estimates by losing some derivatives in the regularity of initial data, which probably will happen when there are broken rays in the manifold. We also assume $\eta\ge 0$ is an arbitrarily small number in our theorems, and actually in our applications we only need the case where $\eta=0$.

We will assume $1-n/2<\gamma<n/2$ throughout, so that $(H_\gamma,\dot H_\gamma)$ and $(H_{1-\gamma},\dot H_{1-\gamma})$ are comparable pairs for functions supported in a ball. Besides, for $\beta\in C_0^\infty(\mathbb{R}^n)$, with $\beta=1$ on a neighborhood of $\mathbb{R}^n\backslash\Omega$, we assume that

$$\|(1-\beta)f\|_{X(\Omega)} \approx \|(1-\beta)f\|_{X(\mathbb{R}^n)}.$$

Now we will state our main Strichartz estimates.

Theorem 0.5. *Let $n>2$ and assume that (X,γ,η,p) is almost admissible with*

$$p>2 \text{ and } \gamma\in[-\tfrac{n-3}{2},\tfrac{n-1}{2}). \tag{1.9}$$

Then if Hypothesis B is valid and if u solves (1.1) *with forcing term $F=0$, we have the abstract Strichartz estimates*

$$\|u\|_{L_t^p X([0,S]\times\Omega)} \lesssim A(S)(\|f\|_{\tilde H^{\gamma}_{\varepsilon+\eta}(\Omega)} + \|g\|_{\tilde H^{\gamma-1}_{\varepsilon+\eta}(\Omega)}). \tag{1.10}$$

Remark 0.6. We need $-(n-3)/2\le\gamma<(n-1)/2$ since we will use Lemma 0.9, which requires $\gamma+\varepsilon+\eta\le\frac{n-1}{2}$ for a sufficiently small number ε, thus precisely $\gamma\in[-(n-3)/2,(n-1)/2-\varepsilon-\eta]$. On the other hand, when ε is allowed to take large values, which depends on our local energy estimates, we can easily adapt our arguments to show

$$\|u\|_{L_t^p X([0,S]\times\Omega)} \lesssim A(S)(\|f\|_{\tilde H^{\gamma}_{2\varepsilon+\eta}(\Omega)} + \|g\|_{\tilde H^{\gamma-1}_{2\varepsilon+\eta}(\Omega)})$$

under the assumption $\gamma\in[-(n-3)/2,(n-1)/2]$.

Next we will see two corollaries that involve adding forcing terms to the equation.

Corollary 0.7. *Assume that* (X,γ,η,p) *and* $(Y,1-\gamma,\eta,r)$ *are almost admissible and that Hypothesis B is valid. Also assume that* (1.9) *holds for* (X,γ,η,p) *and* $(Y,1-\gamma,\eta,r)$. *Then we have the following global abstract Strichartz estimates for the solution of* (1.1)*:*

$$\begin{aligned}&\|u\|_{L^p_t X([0,S]\times\Omega)} \\ &\lesssim A(S)(\|f\|_{\tilde H^\gamma_{\varepsilon+\eta}(\Omega)}+\|g\|_{\tilde H^{\gamma-1}_{\varepsilon+\eta}(\Omega)})+A^2(S)\|\Lambda^{2(\varepsilon+\eta)}F\|_{L^{r'}_t Y'([0,S]\times\Omega)},\end{aligned} \tag{1.11}$$

where $\Lambda=(1-\Delta)^{1/2}$, r' *denotes the conjugate exponent to* r *and* $\|\cdot\|_{Y'}$ *is the dual norm to* $\|\cdot\|_Y$.

Proof. Since

$$\|h\|_{\tilde H^\gamma_\varepsilon}=\||D|^\gamma\Lambda^\varepsilon h\|_{L^2}\,,\quad \varepsilon\ge 0,$$

it is easy to see that the dual norm is

$$\|h\|_{(\tilde H^\gamma_\varepsilon)'}=\|h\|_{\tilde H^{-\gamma}_{-\varepsilon}}=\big\||D|^{-\gamma}\Lambda^{-\varepsilon}h\big\|_{L^2}\,.$$

To prove (1.11), we may assume by (1.10) that the initial data vanishes. If $|D|=\sqrt{-\Delta_{\mathrm g}}$ is the square root of minus the Laplacian (with the boundary conditions B), then we need to show

$$\Big\|\int_0^t e^{-i(t-s)|D|}|D|^{-1}F(s,\,\cdot\,)\,ds\,\Big\|_{L^p_tX([0,S]\times\Omega)} \lesssim A^2(S)\|\Lambda^{2(\varepsilon+\eta)}F\|_{L^{r'}_tY'([0,S]\times\Omega)}. \tag{1.12}$$

We have $p>2>r'$, so (1.10) and an application of the Christ-Kiselev lemma (cf. [10]) implies that it suffices to prove the estimate

$$\begin{aligned}&\Big\|\int_0^S e^{-is|D|}|D|^{-1}F(s,\,\cdot\,)\,ds\Big\|_{\tilde H^\gamma_{\varepsilon+\eta}(\Omega)} \\ &=\Big\|\int_0^S e^{-is|D|}|D|^{-1+\gamma}\Lambda^{\varepsilon+\eta}F(s,\,\cdot\,)\,ds\Big\|_{L^2(\Omega)}\lesssim A(S)\|\Lambda^{2(\varepsilon+\eta)}F\|_{L^{r'}_tY'([0,S]\times\Omega)}.\end{aligned} \tag{1.13}$$

Note that duality of (1.10) for $(Y,1-\gamma,\eta,r)$ gives

$$\Big\|\int_0^S e^{-is|D|}F(s,\,\cdot\,)\,ds\,\Big\|_{\tilde H^{\gamma-1}_{-\varepsilon-\eta}(\Omega)}\lesssim A(S)\|F\|_{L^{r'}_tY'([0,S]\times\Omega)};$$

i.e.,

$$\Big\|\int_0^S e^{-is|D|}|D|^{\gamma-1}\Lambda^{-\varepsilon-\eta}F(s,\,\cdot\,)\,ds\,\Big\|_{L^2(\Omega)}\lesssim A(S)\|F\|_{L^{r'}_tY'(\mathbb R_+\times\Omega)}. \tag{1.14}$$

Now (1.13) follows from (1.14). ☐

As a special case of (1.11) when the spaces X and Y are the standard Lebesgue spaces, we have the following obstacle version of Strichartz estimates in Minkowski space obtained by Keel-Tao [19].

Corollary 0.8. *Suppose that $n \geq 3$ and that Hypothesis B is valid. Suppose that $p, r > 2$, $q, s \geq 2$ and that*

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{r'} + \frac{n}{s'} - 2$$

and

$$\frac{2}{p} + \frac{n-1}{q}, \frac{2}{r} + \frac{n-1}{s} \leq \frac{n-1}{2}.$$

Then if the local Strichartz estimate (1.8) *holds respectively for* $(L^q(\Omega),\gamma,\eta,p)$ *and* $\big(L^s(\Omega), 1-\gamma, \eta, r\big)$, *it follows that, when u solves* (1.1),

$$\|u\|_{L^p_t L^q_x(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\tilde{H}^{\gamma}_{\varepsilon+\eta}(\Omega)} + \|g\|_{\tilde{H}^{\gamma-1}_{\varepsilon+\eta}(\Omega)} + \|(1-\Delta)^{\varepsilon+\eta}F\|_{L^{r'}_t L^{s'}_x(\mathbb{R}_+\times\Omega)}.$$

Note that the estimates we obtain here do not cover the endpoint case $q = 2$ or $\tilde{q} = 2$, or the case $n = 2$, which are proved to be true for $\mathbb{R}^n$ and $\Delta_g = \Delta$ by Keel-Tao [19]. As for the assumption (1.8), Smith and Sogge [27] showed that it holds when Ω is the exterior of a geodesically convex obstacle, and their results apply to the case where there are finitely many convex obstacles by finite propagation of speed. More work in this direction can be found in [6], [7], [2], [29], but only partial results with smaller range of q, r, γ have been gained.

2. Proof of Theorem 0.5

In this section we will see how local Strichartz estimates and global Strichartz estimates in Minkowski space imply the (almost) global Strichartz estimates in a general domain Ω. The first lemma is key to achieving Proposition 0.10 and will be used in Theorem 0.5 as well.

Lemma 0.9. *Fix $\beta \in C_0^\infty(\mathbb{R}^n)$ and assume that $\gamma \leq \frac{n-1}{2}$. Then*

$$\int_{-\infty}^{\infty} \left\|\beta(\cdot)(e^{it|D|}f)(t,\cdot)\right\|^2_{H^\gamma(\mathbb{R}^n)} \, dt \lesssim \|f\|^2_{\dot{H}^\gamma(\mathbb{R}^n)}, \tag{2.1}$$

if $|D| = \sqrt{-\Delta}$.

Proof. Refer to Lemma 2.2 in [28]. □

Now we introduce a result which will be used to control the solution of (1.1) away from the obstacle.

Proposition 0.10. *Consider the wave equation*

$$\begin{cases} (\partial_t^2 - \Delta)u &= F(t,x) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ u|_{t=0} &= f, \\ \partial_t u|_{t=0} &= g. \end{cases} \tag{2.2}$$

Let w be a solution with $f = g = 0$, and assume that (1.7) *is valid whenever v is a solution of the homogeneous wave equation. Assume further that $p > 2, \gamma \geq (n-3)/2$. Then, if*

$$F(t,x) = 0 \quad \text{if } |x| > 2R,$$

we have

$$\|w\|_{L_t^p X([0,S]\times\mathbb{R}^n)} \lesssim A(S)\, \|F\|_{L_t^2 \dot{H}^{\gamma-1}([0,S]\times\mathbb{R}^n)}\,. \tag{2.3}$$

Proof. When $S = \infty$, this is just Proposition 2.1 in [15]. Moreover, their argument is easily modified to give the proof when $S = T$ is finite. □

The next lemma gives two useful local decay estimates. The first one is necessary to prove Theorem 0.5, and the second one will be applied in the next two sections.

Lemma 0.11. *Let u solve* (1.1) *and assume that Hypothesis B holds. Let $\beta \in C_0^\infty(\mathbb{R}^n)$ equal 1 on a neighborhood of $\mathbb{R}^n \backslash \Omega$. Then we have the following estimates:*

i) if f, g, and F are supported in $|x| < 2R$,

$$\begin{aligned} &\|\beta u\|_{L_t^\infty H_B^\gamma([0,S]\times\Omega)} + \|\beta\partial_t u\|_{L_t^\infty H_B^{\gamma-1}([0,S]\times\Omega)} + \|\beta u\|_{L_t^2 H_B^\gamma([0,S]\times\Omega)} \\ &\qquad + \|\beta\partial_t u\|_{L_t^2 H_B^{\gamma-1}([0,S]\times\Omega)} \\ &\lesssim \|f\|_{\dot{H}_B^{\gamma+\varepsilon}(\Omega)} + \|g\|_{\dot{H}_B^{\gamma+\varepsilon-1}(\Omega)} + \|F\|_{L_t^2 \dot{H}_B^{\gamma+\varepsilon-1}([0,S]\times\Omega)}\,; \end{aligned} \tag{2.4}$$

ii) if F is supported in $|x| < R$, $\gamma < \frac{n-1}{2}$,

$$\begin{aligned} &\|u\|_{L_t^\infty \dot{H}_B^\gamma([0,S]\times\Omega)} + \|\partial_t u\|_{L_t^\infty \dot{H}_B^{\gamma-1}([0,S]\times\Omega)} + \|\beta u\|_{L_t^2 H_B^\gamma([0,S]\times\Omega)} \\ &\qquad + \|\beta\partial_t u\|_{L_t^2 H_B^{\gamma-1}([0,S]\times\Omega)} \\ &\lesssim \|f\|_{\tilde{H}_\varepsilon^\gamma(\Omega)} + \|g\|_{\tilde{H}_\varepsilon^{\gamma-1}(\Omega)} + \|F\|_{L_t^2 \dot{H}_B^{\gamma+\varepsilon-1}([0,S]\times\Omega)}\,. \end{aligned} \tag{2.5}$$

Proof. First note that the space $H_B^\gamma(\Omega)$ is the usual Dirichlet space with compatibility conditions on the boundary of Ω satisfied, and we are assuming that the required compatibility conditions on data are met throughout the paper (see for example in [28]), therefore write $H^\gamma(\Omega)$ for short elsewhere.

i) Note that f, g, and F are supported in a ball, the L_t^2 estimate in the case $\gamma = 1$ is just (1.2), and then by elliptic regularity of the operator Δ_g,

$$\begin{aligned}
&\|\beta u\|_{L_t^2 H_x^3} + \|\beta \partial_t u\|_{L_t^2 H_x^2} \\
&\lesssim \|\Delta_g(\beta u)\|_{L_t^2 H_x^1} + \|\beta u\|_{L_t^2 H_x^1} + \|\Delta_g(\beta \partial_t u)\|_{L_t^2 L_x^2} + \|\beta \partial_t u\|_{L_t^2 L_x^2} \\
&\lesssim \|\beta \Delta_g u\|_{L_t^2 H_x^1} + \|[\Delta_g, \beta] u\|_{L_t^2 H_x^1} + \|\beta u\|_{L_t^2 H_x^1} \\
&\qquad + \|\beta \partial_t \Delta_g u\|_{L_t^2 L_x^2} + \|[\Delta_g, \beta] \partial_t u\|_{L_t^2 L_x^2} + \|\beta \partial_t u\|_{L_t^2 L_x^2} .
\end{aligned} \tag{2.6}$$

Since $\Delta_g u$ solves the equation with data $(\Delta_g f, \Delta_g g)$ and forcing term $\Delta_g F$, we get

$$\begin{aligned}
&\|\beta \Delta_g u\|_{L_t^2 H_x^1} + \|\beta \partial_t \Delta_g u\|_{L_t^2 L_x^2} \lesssim \|\Delta_g f\|_{\dot H_x^{1+\varepsilon}} + \|\Delta_g g\|_{\dot H_x^{\varepsilon}} + \|\Delta_g F\|_{L_t^2 \dot H_x^{\varepsilon}} \\
&\lesssim \|f\|_{\dot H_x^{3+\varepsilon}} + \|g\|_{\dot H_x^{2+\varepsilon}} + \|F\|_{L_t^2 \dot H_x^{2+\varepsilon}} .
\end{aligned} \tag{2.7}$$

Also, notice that $[\Delta_g, \beta] u = \beta_1 \partial_x u + \beta_2 u$, where $\beta_i \in C_0^\infty,\ i = 1, 2$ have support belonging to $\mathrm{supp}(\beta)$. Thus,

$$\begin{aligned}
&\|[\Delta_g, \beta] u\|_{L_t^2 H_x^1} + \|[\Delta_g, \beta] \partial_t u\|_{L_t^2 L_x^2} \lesssim \|\beta_3 u\|_{L_t^2 H_x^2} + \|\beta_3 \partial_t u\|_{L_t^2 H_x^1} \\
&\lesssim \|\beta_3 u\|_{L_t^2 H^1}^{\theta} \|\beta_3 u\|_{L_t^2 H^3}^{1-\theta} + \|\beta_3 \partial_t u\|_{L_t^2 L_x^2}^{\theta} \|\beta_3 \partial_t u\|_{L_t^2 H_x^2}^{1-\theta} ,
\end{aligned} \tag{2.8}$$

where $\beta_3 \in C_0^\infty$ has support in $\mathrm{supp}(\beta_1) \cap \mathrm{supp}(\beta_2)$, and θ is any real number in $(0, 1)$.

Based on (2.6), (2.7) and (2.8), we get that the L_t^2 estimate is true for $\gamma = 3$, and similarly holds for $\gamma = 5, 7, 9...$ and moreover for $\gamma \in \mathbb{R}$ by duality and interpolation; i.e.,

$$\begin{aligned}
&\|\beta u\|_{L_t^2 H_B^{\gamma}([0,S]\times\Omega)} + \|\beta \partial_t u\|_{L_t^2 H_B^{\gamma-1}([0,S]\times\Omega)} \\
&\qquad \lesssim \|f\|_{\dot H^{\gamma+\varepsilon}(\Omega)} + \|g\|_{\dot H^{\gamma+\varepsilon-1}(\Omega)} + \|F\|_{L_t^2 \dot H_B^{\gamma+\varepsilon-1}([0,S]\times\Omega)} .
\end{aligned} \tag{2.9}$$

By Duhamel's principle, the inhomogeneous solution v satisfies

$$\|\beta v\|_{L_t^2 H_B^{\gamma}([0,S]\times\Omega)} + \|\beta \partial_t v\|_{L_t^2 H_B^{\gamma-1}([0,S]\times\Omega)} \lesssim \|F\|_{L_t^1 \dot H_B^{\gamma+\varepsilon-1}([0,S]\times\Omega)} .$$

By duality of the above estimate, energy estimates and elliptic regularity, we get

$$\begin{aligned}
&\|u\|_{L_t^\infty \dot H_B^{\gamma}([0,S]\times\Omega)} + \|\partial_t u\|_{L_t^\infty \dot H_B^{\gamma-1}([0,S]\times\Omega)} \\
&\qquad \lesssim \|f\|_{\dot H^{\gamma}(\Omega)} + \|g\|_{\dot H^{\gamma-1}(\Omega)} + \|F\|_{L_t^2 \dot H_B^{\gamma+\varepsilon-1}([0,S]\times\Omega)} .
\end{aligned} \tag{2.10}$$

Now (2.4) is a result of (2.9) and (2.10).

ii) We first handle the L_t^2 bounds. For the homogeneous solution v, we can assume $f = g = 0$ for $|x| \le 3R/2$ by i). Decompose $v = (1 - \eta) v_0 + \tilde v$, where

$\eta \in C_0^\infty(\mathbb{R}^n)$ equals 1 for $|x| < R$ and vanishes for $|x| > 3R/2$, and v_0 solves the homogeneous wave equation in $\mathbb{R}^n \times \mathbb{R}$. It is easy to see $(1-\eta)v_0$ solves the Cauchy problem for the Minkowski space wave equation with initial data $((1-\eta)f, (1-\eta)g)$ and forcing term $G = \Delta\eta v_0 + 2\nabla\eta \cdot \nabla v_0$; $\tilde{v}$ solves the wave equation with initial data $(0,0)$ and forcing term $-G$. Since G is supported in $R < |x| < 2R$, we get the L_t^2 bounds for $(1-\eta)v_0$ and $\tilde{v}$ by Lemma 0.9 and i).

We get the L_t^2 bounds for the inhomogeneous solution w from i) since F is still compactly supported.

Similarly to i), the L_t^∞ bounds for $u = v + w$ follow also from energy estimates, elliptic regularity and duality. □

The last proposition is a result of Proposition 0.10 and (2.4).

Proposition 0.12. *Let u solve* (1.1) *and assume that*

$$f(x) = g(x) = F(t,x) = 0, \quad \text{when } |x| > 2R. \tag{2.11}$$

If (X, γ, η, p) is almost admissible with $p > 2, \gamma \geq -\frac{n-3}{2}$, and Hypothesis B holds, then we have

$$\|u\|_{L_t^p X([0,S]\times\Omega)} \lesssim A(S)(\|f\|_{\dot{H}^{\gamma+\varepsilon+\eta}} + \|g\|_{\dot{H}^{\gamma+\varepsilon+\eta-1}} + \|F\|_{L_t^2 \dot{H}^{\gamma+\varepsilon+\eta-1}}). \tag{2.12}$$

Proof. Fix $\beta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\beta(x) = 1$, $|x| \leq 3R$ and write

$$u = v + w, \quad \text{where } v = \beta u, \ w = (1-\beta)u.$$

Then w solves the free wave equation

$$(\partial_t^2 - \Delta)w = [\beta, \Delta]u, \qquad w|_{t=0} = \partial_t w|_{t=0} = 0.$$

Notice that $[\beta, \Delta]u$ is compactly supported, so an application of Proposition 0.10 shows that $\|w\|_{L_t^p X}$ is dominated by $A(S)\|\rho u\|_{L_t^2 \dot{H}_B^\gamma}$ if $\rho \in C_0^\infty$ equals one on the support of β. Therefore, by (2.4), $\|w\|_{L_t^p X}$ is dominated by the right-hand side of (2.12) with $\eta = 0$.

For $v = \beta u$, we decompose it in time t and write $v = \sum_{j=-\infty}^\infty \varphi(t-j)v$, where $\varphi \in C_0^\infty((-1,1))$. Let $v_j = \varphi(t-j)v$ for $j \geq 1$ and $v_0 = v - \sum_{j=1}^\infty v_j$. Then v_j solves

$$\begin{cases} (\partial_t^2 - \Delta_{\mathrm{g}})v_j = G_j \\ Bv_j(t,x) = 0, \quad x \in \partial\Omega \\ v_j(0,\cdot) = \partial_t v_j(0,\cdot) = 0, \end{cases}$$

where $G_j = -\varphi(t-j)[\Delta_g, \beta]u + [\partial_t^2, \varphi(t-j)]\beta u + \varphi(t-j)F$. Also v_0 solves the equation with $G_0 = -\tilde{\varphi}[\Delta_g, \beta]u + [\partial_t^2, \tilde{\varphi}]\beta u + \tilde{\varphi}F$ and initial date (f, g).

Since G_j with $j \geq 0$ vanishes if t is not in $[j-1, j+1]$ or if $|x| > 3R$, by the local Strichartz estimates (1.8) and Duhamel, we get, for $j = 1, 2, \dots,$

$$\|v_j\|_{L_t^p X([0,S]\times\Omega)} \lesssim \int_0^S \|G_j(s,\,\cdot\,)\|_{\tilde{H}_\eta^{\gamma-1}}\, ds \lesssim \|G_j\|_{L_t^2 H_B^{\gamma+\eta-1}}.$$

Similarly,

$$\|v_0\|_{L_t^p X(\mathbb{R}_+\times\Omega)} \lesssim \|f\|_{\tilde{H}_\eta^{\gamma}} + \|g\|_{\tilde{H}_\eta^{\gamma-1}} + \|G_0\|_{L_t^2 H_B^{\gamma+\eta-1}}.$$

Since $p > 2$, by (2.4) and the disjoint support of G_j, we have

$$\begin{aligned}
\|v\|^2_{L_t^p X([0,S]\times\Omega)} &\lesssim \sum_{j=0}^{\infty} \|v_j\|^2_{L_t^p X([0,S]\times\Omega)} \\
&\lesssim \sum_{j=1}^{\infty} \|G_j\|^2_{L_t^2 H_B^{\gamma+\eta-1}([0,S]\times\Omega)} + \|v_0\|_{L_t^p X(\mathbb{R}_+\times\Omega)} \\
&\lesssim \|f\|^2_{\dot{H}^{\gamma+\varepsilon+\eta}} + \|g\|^2_{\dot{H}^{\gamma+\varepsilon+\eta-1}} + \|F\|^2_{L_t^2 \dot{H}^{\gamma+\varepsilon+\eta-1}}, \\
&\lesssim A^2(S)(\|f\|^2_{\dot{H}^{\gamma+\varepsilon+\eta}} + \|g\|^2_{\dot{H}^{\gamma+\varepsilon+\eta-1}} + \|F\|^2_{L_t^2 \dot{H}^{\gamma+\varepsilon+\eta-1}}),
\end{aligned}$$

which finishes the proof of Proposition 0.12. □

Proof of Theorem 0.5: By Proposition 0.12 we can assume that the initial data for u vanishes when $|x| < 3R/2$. Then we use a cutoff function $\beta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\beta(x) = 1$, $|x| \leq R$, and $\beta(x) = 0$, $|x| > 3R/2$, and write

$$u = u_0 - v = (1-\beta)u_0 + (\beta u_0 - v),$$

where u_0 solves the Cauchy problem for the Minkowski space wave equation. By the free estimate (1.7), Proposition 0.10, Lemma 0.9 and energy estimates,

$$\begin{aligned}
\|(1-\beta)u_0\|_{L_t^p X([0,S]\times\mathbb{R}^n)} &\lesssim A(S)(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}} + \|G\|_{L_t^2 \dot{H}_B^{\gamma-1}}) \\
&\lesssim A(S)(\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}),
\end{aligned} \tag{2.13}$$

where $G = \Box_g((1-\beta)u_0) = \Box((1-\beta)u_0) = \Delta\beta \cdot u_0 + 2\nabla\beta \cdot \nabla u_0$ is supported in $R < |x| < 3R/2$.

Now consider $\tilde{u} = \beta u_0 - v$, which has forcing term $-G$ and zero initial data. Again by Proposition 0.12 and Lemma 0.9,

$$\begin{aligned}
\|\beta u_0 - v\|_{L_t^p X([0,S]\times\Omega)} &\lesssim A(S)\|G\|_{L_t^2 \dot{H}_B^{\gamma+\varepsilon+\eta-1}} \\
&\lesssim A(S)\,\|\rho u_0\|_{L_t^2 H^{\gamma+\varepsilon+\eta}} \quad \text{(here } \rho \text{ is a } C_0^\infty \text{ function.)} \\
&\lesssim A(S)(\|f\|_{\dot{H}^{\gamma+\varepsilon+\eta}} + \|g\|_{\dot{H}^{\gamma+\varepsilon+\eta-1}}).
\end{aligned} \tag{2.14}$$

Based on (2.13) and (2.14), we have the theorem proved. □

3. Application 1: Sharp life span bounds for $p < p_c$ when $n = 3$

First let us describe the wave equation we will consider:

$$\begin{cases} (\partial_t^2 - \Delta_g)u = F_p(u(t,x)) \text{ on } \mathbb{R}_+ \times \Omega \\ u|_{t=0} = f, \partial_t u|_{t=0} = g, \\ (Bu)(t,x) = 0, \text{ on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{3.1}$$

with B as above; the set Ω is assumed to be either all of $\mathbb{R}^3$, or else $\Omega = \mathbb{R}^3 \backslash \kappa$ where κ is a compact subset of $|x| < R$ with smooth boundary. Also we assume κ is nontrapping in the sense that any geodesic restricted to $|x| < R$ has bounded length.

We will assume that the nonlinear term behaves like $|u|^p$ when u is small, and so we assume that

$$\sum_{0 \le j \le 2} |u|^j |\partial_u^j F_p(u)| \lesssim |u|^p, \tag{3.2}$$

when u is small.

On the basis of the discussion in the first section (Remark 0.1), we will assume Hypothesis B holds with $\varepsilon = 0, S = T$. Now if we set

$$\{Z\} = \{\partial_l, x_j\partial_k - x_k\partial_j : 1 \le l \le n, 1 \le j < k \le n\},$$

then we have the following existence theorem for (3.1).

Theorem 0.13. *Let $n = 3$, and fix $\Omega \subset \mathbb{R}^n$ and the boundary operator B as above. Assume further that Hypothesis B is valid with $\varepsilon = 0$. Then, if $2 < p < p_c = 1 + \sqrt{2}$, $\gamma = \frac{1}{2} - \frac{1}{p}$, and if*

$$T_{\varepsilon',p} = c\varepsilon'^{\frac{p(p-1)}{p^2-2p-1}},$$

then there exists an $\varepsilon_0 > 0$ depending on Ω, B and p so that (3.1) has an almost global solution in $[0, T_{\varepsilon'}] \times \Omega$, satisfying $(Z^\alpha u(t,\cdot), \partial_t Z^\alpha u(t,\cdot)) \in \dot{H}_B^\gamma \times \dot{H}_B^{\gamma-1}$, $|\alpha| \le 2$, $t \in [0, T_{\varepsilon'}]$, whenever the initial data satisfies the boundary conditions of order 2, and

$$\sum_{|\alpha| \le 2} (\|Z^\alpha f\|_{\dot{H}_B^\gamma(\Omega)} + \|Z^\alpha g\|_{\dot{H}_B^{\gamma-1}(\Omega)}) < \varepsilon' \tag{3.3}$$

with $0 < \varepsilon' < \varepsilon_0$.

In the case where $\Omega = \mathbb{R}^3$ and $\Delta_{\rm g} = \Delta$ it is known that $p > p_c$ is necessary for global existence (see John [18]). John [17] also established the global existence theorem for $p > p_c = 1+\sqrt{2}$. For the local existence result, Lindblad [22] handled the case $1 < p < 1+\sqrt{2}$ in $\mathbb{R}^3$, then Zhou [35] obtained the case $p = 1+\sqrt{2}$. In their works it was also shown that the lifespan estimates given are sharp.

On the other hand, when the data is spherically symmetrical and $n = 3$, Sogge [30] and Hidano [14] obtained the sharp local well-posedness theorem for the Minkowski wave equation respectively by using some radial estimates. It is also shown in [30] that the regularity $\gamma = 1/2 - 1/p$ is sharp for radial data.

For nontrapping obstacles, Hidano, Metcalfe, Smith, Sogge and Zhou [15] dealt with the global existence part (i.e., $p > p_c$) for (3.1) with $n = 3, 4$.

Here we will use the real interpolation method to get the local existence theorem for (3.1) when the perturbation is nontrapping. Before handling the obstacle problem we will first give an alternative proof for the Minkowski space results, which involves an interpolation between the following two estimates.

Lemma 0.14. *(A variant of the KSS estimate) For $n \geq 3$, let u solve the homogeneous wave equation* (2.2) *($F = 0$) in the Minkowski space. Then we have*

$$\left\| \langle x\rangle^a e^{it|D|} f \right\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^n)} \lesssim B(T)\|f\|_{\dot{H}^0}, \tag{3.4}$$

where

$$B(T) = \begin{cases} T^{(1/2+a)}, & \text{if } \; -1/2 < a \leq 0 \\ (\log((2+T)))^{1/2}, & \text{if } \; a = -1/2 \\ Constant, & \text{if } \; a < -1/2. \end{cases} \tag{3.5}$$

In particular, for $-1/2 < a \leq 0$, we have

$$\left\| |x|^a e^{it|D|} f \right\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^n)} \lesssim T^{(1/2+a)}\|f\|_{\dot{H}^0}. \tag{3.6}$$

Proof. Actually the cases where $a \leq -1/2$ have been well set up in Du, Metcalfe, Sogge, Zhou [12], and can be adapted to handle the case $-1/2 < a \leq 0$. Specifically, considering $u = -i\sum_{j=1}^n \partial_j v_j, v_j = \check{F}[\hat{u}(t,\xi)\frac{\xi_j}{|\xi|^2}]$, the lemma follows from

$$\left\| \langle x\rangle^a v' \right\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^n)} \lesssim B(T)\|f\|_{\dot{H}^1}, \quad a \leq 0.$$

But this is from the estimate

$$\|v'(t,x)\|_{L_t^2 L_x^2([0,\infty]\times|x|<1)} \lesssim \|v'(0,x)\|_{L_x^2}, \tag{3.7}$$

and a scaling argument for $|x| < T$, the energy inequality for $|x| > T$. For the proof of (3.7), refer to Keel, Smith and Sogge [20].

As for (3.6), we just need to take care of the case where $|x| < 1$, but that is just a direct result of Lemma 0.9 and a scaling argument for a partition of $\{x : 0 < |x| < 1\}$. See [14] for details. □

In what follows, we will employ (3.6) to do the interpolations for simplicity, while we remark that the weaker estimate (3.4) can actually lead us to the same existence theorem as well by the same argument.

The next estimate is a result of the complex interpolation between (3.6) and the endpoint trace lemma.

Proposition 0.15. *For $n = 3$, let u solve the homogeneous wave equation* (2.2) *in Minkowski space. Then we have*

$$\left\| |x|^{(1+2a)/3} e^{it|D|} f \right\|_{L_t^3 L_r^3 L_\omega^2([0,T]\times\mathbb{R}^3)} \lesssim T^{(1+2a)/3} \|f\|_{\dot{B}^{1/6}_{2,3/2}}, \tag{3.8}$$

for $-1/2 < a \le 0$.

Here, and in what follows, we are using the mixed-norm notation with respect to the volume element

$$\|h\|_{L_r^q L_\omega^p} = \left(\int_0^\infty \left(\int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \right)^{q/p} r^{n-1} dr \right)^{1/q}$$

for finite exponents and

$$\|h\|_{L_r^\infty L_\omega^p} = \sup_{r>0} \left(\int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \right)^{1/p}.$$

Also, the homogeneous Besove space $\dot{B}^s_{p,q}$ is defined as

$$\|f\|_{\dot{B}^s_{p,q}} = \|2^{js} P_j f\|_{l_j^q(j\in\mathbb{Z}) L_x^p},$$

where $f = \sum_j P_j f$ is the Littlewood-Paley decomposition.

Proof. Recall that we have the endpoint trace lemma (see [13]):

$$\left\| |x|^{\frac{n-1}{2}} e^{it|D|} f \right\|_{L_t^\infty L_r^\infty L_\omega^2([0,T]\times\mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}}.$$

Now we use the complex interpolation between this estimate and (3.6) for $n = 3$, and set $\theta = 1/3$. Noting $\dot{B}^0_{2,2} = \dot{H}^0$ and using the fact that $(\dot{B}^0_{2,2}, \dot{B}^{1/2}_{2,1})_{[\theta]} = \dot{B}^{1/6}_{2,3/2}$ for $\theta = 1/3$ (see Section 6.4 in [1]), we get the desired estimate (3.8) for $-1/2 < a \le 0$. □

Next we will cite some notation and results in [1] and [33]. Let A_0, A_1 be Banach spaces, define the real interpolation space $(A_0, A_1)_{\theta,q}$ for $0 < \theta < 1$ and $1 \le q \le \infty$ via the norm:

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \|a\|_{(A_0,A_1)_{\theta,q;K}} = \Big(\int_0^\infty (t^{-\theta} K(t,a))^q dt/t \Big)^{1/q},$$

where $K(t,a) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + \|a_1\|_{A_1})$. Now if we let

$$\begin{aligned} &A_0 = \dot{B}^0_{2,2}, \quad A_1 = \dot{B}^{1/6}_{2,3/2}, \\ &B_0 = L^2_{t,r} L^2_\omega([0,T] \times [0,\infty) \times S^2, r^{2+2a} dtdrdw), \\ &B_1 = L^3_{t,r} L^2_\omega([0,T] \times [0,\infty) \times S^2, r^{3+2a} dtdrdw), \end{aligned}$$

then, by (3.6) and (3.8), we have

$$Tf = e^{it|D|} f : \bar{A} \to \bar{B}, \quad \text{where } \bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1), \tag{3.9}$$

and

$$M_0 \lesssim T^{1/2+a}, \; M_1 \lesssim T^{2/3(1/2+a)}, \quad \text{where } M_i = \|T\|_{A_j,B_j}, \; j = 0,1. \tag{3.10}$$

Now we can state the main weighted Strichartz estimates as follows.

Proposition 0.16. *For $n = 3$, let u solve the homogeneous wave equation* (2.2) *in Minkowski space* ($F = 0$). *Then we have*

$$\big\| |x|^{(-1/2-\gamma)/p} u \big\|_{L^p_t L^p_r L^2_\omega([0,T]\times\mathbb{R}^3)} \lesssim T^{(-p+1/p+2)/p} (\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}), \tag{3.11}$$

where $\gamma = 1/2 - 1/p$ and $2 < p < 1 + \sqrt{2}$.

Proof. The result when the data is radial was shown in [14]. Here we will use a different method to handle the nonradial case.

Since $K_{\theta,q}$ is an exact interpolation functor of exponent θ (Theorem 3.1.2 in [1]), from (3.9) and (3.10) we get

$$\|Tf\|_{\bar{B}_{\theta,2}} \le M_0^{1-\theta} M_1^\theta \|f\|_{\bar{A}_{\theta,2}} \lesssim T^{(1-\frac{1}{3}\theta)(\frac{1}{2}+a)} \|f\|_{\bar{A}_{\theta,2}}, \tag{3.12}$$

if $-1/2 < a \le 0$. To proceed, we note that from Theorem 6.4.5 in [1] we have $(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,r} = B^{s^*}_{pr}$, if $s_0 \ne s_1$, $0 < \theta < 1$, $r, q_0, q_1 \ge 1$ and $s^* = (1-\theta)s_0 + \theta s_1$. Set $a = (-p + 1/p + 1)/2$ and $\theta = 3 - 6/p$, then we have $0 < \theta < 1$ and $-1/2 < a \le 0$ since $2 < p < 1 + \sqrt{2}$. Thus, we see

$$RHS \ of \ (3.12) \lesssim T^{(-p+1/p+2)/p} \|f\|_{\dot{B}^{1/2-1/p}_{2,2}} = T^{(-p+1/p+2)/p} \|f\|_{\dot{H}^{1/2-1/p}}. \tag{3.13}$$

On the other hand, we can use the fact (Theorem 3.4.1(b) in [1]) that $\bar{A}_{\theta,q} \subset \bar{A}_{\theta,r}$, if $q \le r$, and bilinear weighted interpolation (Section 1.18.5 in [33])

$$\Big(\big(L^{p_0}_{t,r} L^2_\omega, w_0(r) dtdrdw\big), \big(L^{p_1}_{t,r} L^2_\omega, w_1(r) dtdrdw\big) \Big)_{\theta,p} = \big(L^{p}_{t,r} L^2_\omega, w(r) dtdrdw\big),$$

if $1/p = (1-\theta)/p_0 + \theta/p_1,\ w(r) = w_0^{p(1-\theta)/p_0} w_1^{p\theta/p_1}$.

Since $p > 2$, we also have

$$\begin{aligned} LHS\ of\ (3.12) \gtrsim \|Tf\|_{\bar{B}_{\theta,p}} &= \|Tf\|_{(L^p_{t,r} L^2_\omega, r^{1+1/p} dtdrdw)} \\ &= \big\| |x|^{\frac{-1+1/p}{p}} Tf \big\|_{L^p_t L^p_r L^2_w([0,T]\times\mathbb{R}^n)}. \end{aligned} \tag{3.14}$$

Now (3.11) is just a result of (3.13) and (3.14). □

As a result, by the arguments to follow, (3.11) is strong enough to show the local existence of solutions as described in Theorem 0.13 in the Minkowski space case.

To prove the obstacle version of this result, we define $X = X_{\gamma,p}(\mathbb{R}^n)$ to be the space with norm defined by

$$\|h\|_{X_{\gamma,p}} = \|h\|_{L^{s_\gamma}(|x|<2R)} + (A(T))^{-1} \Big\| |x|^{-1/2-\gamma/p} h \Big\|_{L^p_r L^2_\omega(|x|>2R)}, \tag{3.15}$$

with $A(T) = T^{\frac{-p+\frac{1}{p}+2}{p}}$ and $s_\gamma = \frac{2n}{n-2\gamma}$.

Using the space X defined just now, we can prove the following estimate provided $\gamma = 1/2 - 1/p$ and $p \ge 2$:

$$\|u\|_{L^p_t X([0,T]\times\mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}, \tag{3.16}$$

when u solves $\Box u = 0$ with initial data (f, g).

Indeed, the contribution of the second part of the norm in (3.15) is controlled by (3.11), and the contribution of the first term is due to Sobolev estimates and an interpolation between L^2_t and L^∞_t in (2.5) (note that $\varepsilon = 0$ in our case).

Furthermore, by finite propagation speed of the wave equation, Sobolev estimates and interpolation between (2.5), we have the local estimate for solutions of (1.1) with $F = 0$:

$$\|u\|_{L^p_t X([0,1]\times\Omega)} \lesssim (\|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}}), \tag{3.17}$$

where $p \ge 2$.

From (3.16) and (3.17), we see that $(X, \gamma, 0, p)$ is admissible. By Theorem 0.5, we therefore obtain the following proposition.

Proposition 0.17. *For $n = 3$, let u be a solution of* (1.1) *with $F = 0$, and let Ω be such a domain as described in the beginning of this section. Moreover, assume that*

$$\gamma = \tfrac{1}{2} - \tfrac{1}{p}, \quad p \in (2, 1 + \sqrt{2}). \tag{3.18}$$

Then

$$\|u\|_{L_t^p X([0,T]\times\Omega)} \lesssim \|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}}. \tag{3.19}$$

From the above proposition we get the following useful corollary.

Corollary 0.18. *For $n = 3$, let u be a solution of* (1.1)*, and let Ω be such a domain as described in the beginning of this section. Moreover, assume the condition* (3.18)*. Then*

$$\begin{aligned}
&\|u\|_{L_t^p L_x^{s_\gamma}([0,T]\times|x|<2R)} + (A(T))^{-1} \left\| |x|^{(-1/2-\gamma)/p} u \right\|_{L_t^p L_r^p L_\omega^2(|x|>2R)} \\
&\lesssim \|f\|_{\dot H^\gamma} + \|g\|_{\dot H^{\gamma-1}} + \|F\|_{L_t^1 L_x^{s'_{1-\gamma}}([0,T]\times\{|x|<2R\})} \\
&+ \left\| |x|^{-1/2-\gamma} F \right\|_{L_t^1 L_r^1 L_\omega^2([0,T]\times\{|x>2R|\})}.
\end{aligned} \tag{3.20}$$

Proof. This is an immediate consequence of Duhamel's principle, Sobolev estimates and the following estimate (originated in [21], see also (3.7)in [15]):

$$\|\varphi\|_{\dot H^{\gamma-1}} \lesssim \left\| |x|^{-n/2+1-\gamma} \varphi \right\|_{L_r^1 L_\omega^2}, \quad \text{if } \tfrac{1}{2} < 1 - \gamma < \tfrac{n}{2}.$$

Here the condition $1/2 < 1 - \gamma < n/2$ is satisfied owing to (3.18). $\square$

If we set $\Gamma = \{\partial_t, Z\}$, then we can easily adapt such an argument as in [15] (see page 15-17) to get the following higher order estimates of (3.20) and (2.5):

$$\begin{aligned}
&\sum_{|\alpha|\le 2} \Big(\|\Gamma^\alpha u\|_{L_t^p L_x^{s_\gamma}([0,T]\times\{|x|<2R\})} \\
&+ A(T)^{-1} \left\| |x|^{-1/2-\gamma/p} \Gamma^\alpha u \right\|_{L_t^p L_r^p L_\omega^2([0,T]\times\{|x|>2R\})} \Big) \\
&\lesssim \sum_{|\alpha|\le 2} (\|Z^\alpha f\|_{\dot H^\gamma} + \|Z^\alpha g\|_{\dot H^{\gamma-1}}) + \sum_{|\alpha|\le 2} (\|\Gamma^\alpha F\|_{L_t^1 L_x^{s'_{1-\gamma}}([0,T]\times\{|x|<2R\})} \\
&+ \left\| |x|^{-1/2-\gamma} \Gamma^\alpha F \right\|_{L_t^1 L_r^1 L_\omega^2([0,T]\times\{|x>2R|\})}).
\end{aligned} \tag{3.21}$$

$$\sum_{|\alpha|\le 2} (\|\Gamma^\alpha u\|_{L_t^\infty \dot H_B^\gamma([0,T]\times\Omega)} + \|\partial_t \Gamma^\alpha u\|_{L_t^\infty \dot H_B^{\gamma-1}([0,T]\times\Omega)}) \tag{3.22}$$

$$\lesssim \sum_{|\alpha|\le 2}(\|Z^\alpha f\|_{\dot H^\gamma}+\|Z^\alpha g\|_{\dot H^{\gamma-1}})+\sum_{|\alpha|\le 2}(\|\Gamma^\alpha F\|_{L^1_tL_x^{s'_{1-\gamma}}([0,T]\times\{|x|<2R\})}$$
$$+\left\||x|^{-1/2-\gamma}\Gamma^\alpha F\right\|_{L^1_tL^1_rL^2_\omega([0,T]\times\{|x>2R|\})}).$$

Now we set

$$M_k(T)=\sum_{|\alpha|\le 2}\Big(\|\Gamma^\alpha u_k\|_{L^p_tL_x^{s_\gamma}([0,T]\times\{|x|<2R\})}$$
$$+A(T)^{-1}\left\||x|^{-1/2-\gamma/p}\Gamma^\alpha u_k\right\|_{L^p_tL^p_rL^2_\omega([0,T]\times\{|x|>2R\})}\Big),$$

where $u_k, k\ge 0$ is the solution of

$$\left\{\begin{array}{l}(\partial_t^2-\Delta_g)u_k=F_p(u_{k-1}(t,x))\text{ on }\mathbb{R}_+\times\Omega\\ u_k|_{t=0}=f,\\ \partial_t u_k|_{t=0}=g,\\ (Bu_k)(t,x)=0,\text{ on }\mathbb{R}_+\times\partial\Omega.\end{array}\right.\tag{3.23}$$

By the same iteration argument as followed in Section 4, we obtain Theorem 0.13.

Note. If we use the KSS estimate for $a=-1/2$ instead of $-1/2<a\le 0$, and the same complex interpolation method and the same real interpolation method as above, we will get Proposition 0.16 with $p=1+\sqrt{2}, a=-1/2$ and $A(T)=(log(2+T))^{1/p}$. Furthermore we get the local well posedness for the critical power $p=1+\sqrt{2}$ with $T_\varepsilon=exp(C\varepsilon^{-(p-1)})$, but this life span is not optimal (the optimal one should be $T_\varepsilon=\exp(C\varepsilon^{-p(p-1)})$).

4. Application 2: Strauss conjecture on semilinear wave equations with finitely many obstacles

We will consider wave equations of the form

$$\begin{cases}(\partial_t^2-\Delta_{\rm g})u(t,x)=F_p\big(u(t,x)\big), & (t,x)\in\mathbb{R}_+\times\Omega\\ Bu=0, \quad \text{on }\mathbb{R}_+\times\partial\Omega\\ u(0,x)=f(x),\quad \partial_t u(0,x)=g(x), & x\in\Omega,\end{cases}\tag{4.1}$$

with B described as in the first section. $\Omega=\mathbb{R}^n\backslash\bigcup_{i=1}^m\kappa_i$ where $\kappa_i(i=1,2,\cdot,m)$ are disjoint compact convex subsets of $|x|<R$ with smooth boundary. We will assume that the nonlinear term behaves like $|u|^p$ when u is small, and so we assume that

$$\sum_{0\le j\le 2}|u|^j\left|\partial_u^jF_p(u)\right|\lesssim|u|^p,\tag{4.2}$$

when u is small.

Ikawa [17] managed to show that solutions of (4.1) with $n = 3$, $\Delta_g = \Delta$, $B = I$, and $F_p(u) = 0$ have exponential decay estimates with a loss of 2 derivatives of data. To assure this, we need some technical assumptions on the obstacles (see page 3-4 in [17]), which we will assume are satisfied here. Now, interpolating between this estimate and the energy estimate we get an estimate of the form

$$\left\| u'(t,x) \right\|_{L^2_x(|x|<1)} \lesssim e^{-ct} \left\| u'(0,x) \right\|_{\dot{H}^{\varepsilon}(|x|<1)},$$

for any positive number ε.

This motivates us to study (4.1) under Hypothesis B.

In the next theorem we are abusing Hypothesis B a little by assuming it is true for $n = 4$. Actually there has been no polynomially local energy decay set up for even dimensions when there are trapped rays, which could be expected though. And Burq did show that local energy decays at least logarithmically with some loss in derivatives([4]).

Theorem 0.19. *Let $n = 3$ or 4, and fix $\Omega \subset \mathbb{R}^n$ and boundary operator B as above. Assume further that Hypothesis B is valid for an arbitrarily small $\varepsilon > 0$. Let $p = p_c$ be the positive root of*

$$(n-1)p^2 - (n+1)p - 2 = 0. \tag{4.3}$$

If

$$p_c < p < (n+3)/(n-1), \ \gamma = \tfrac{n}{2} - \tfrac{2}{p-1}, \tag{4.4}$$

then there exists an $\varepsilon_0 > 0$ depending on Ω, B and p and ε so that (4.1) has a global solution satisfying $(Z^\alpha u(t,\cdot), \partial_t Z^\alpha u(t,\cdot)) \in \dot{H}^\gamma_B \times \dot{H}^{\gamma-1}_B$, $|\alpha| \le 2$, $t \in \mathbb{R}_+$, whenever the initial data satisfies the boundary conditions of order 2, and

$$\sum_{|\alpha|\le 2} \left(\|Z^\alpha f\|_{\tilde{H}^\gamma_{2\varepsilon}(\Omega)} + \|Z^\alpha g\|_{\tilde{H}^{\gamma-1}_{2\varepsilon}(\Omega)} \right) < \varepsilon' \tag{4.5}$$

with $0 < \varepsilon' < \varepsilon_0$. On the other hand, if

$$n = 3, \ \gamma = \tfrac{1}{2} - \tfrac{1}{p}, \ p \in (2, 1+\sqrt{2}) \tag{4.6}$$

and

$$T_{\varepsilon'} = c\varepsilon'^{\frac{p(p-1)}{p^2-2p-1}}, \tag{4.7}$$

then there exists a unique solution in $[0, T_{\varepsilon'}) \times \Omega$ such that

$$(Z^\alpha u(t,\cdot), \partial_t Z^\alpha u(t,\cdot)) \in \dot{H}^\gamma_B \times \dot{H}^{\gamma-1}_B$$

under the condition (4.5).

Before we turn to the proof of this existence theorem, we will first employ Theorem 0.5 to get important estimates that will be used.

Define $X = X_{\gamma,p}(\mathbb{R}^n)$ to be the space with the norm defined by

$$\|h\|_{X_{\gamma,p}} = \|h\|_{L^{s_\gamma}(|x|<2R)} + (A(S))^{-1} \left\| |x|^{-n/2+1-\gamma/p} h \right\|_{L^p_r L^2_\omega(\{|x|>2R\})}, \quad (4.8)$$

where $s_\gamma = 2n/n-2\gamma$. When $n=3, p<p_c, \gamma = \frac{1}{2}-\frac{1}{p}$, we have $S=T$ and $A(T)$ is as defined in the last section; when $n=3,4,\ p>p_c$ and $\gamma = n/2-2/p-1$ we have $S=\infty$ and $A(S)$ is a constant.

Now, by using (3.11), a known result (3.6) in [15] and energy estimates, we can adapt the argument in Section 3 to get the following proposition.

Proposition 0.20. *For $n=3$ or 4, let u be a solution of* (1.1) *with $F=0$, and assume condition* (4.4) *or* (4.6) *is satisfied. Then*

$$\|u\|_{L^p_t X([0,S]\times\Omega)} \lesssim \|f\|_{\tilde{H}^\gamma_\varepsilon} + \|g\|_{\tilde{H}^{\gamma-1}_\varepsilon}. \quad (4.9)$$

Based on the above proposition, it is easy to get the following corollary with forcing term added.

Corollary 0.21. *For $n=3,4$, let u be a solution of* (1.1)*, and assume condition* (4.4) *or* (4.6) *is satisfied. Then*

$$\begin{aligned}
&\|u\|_{L^p_t L^{s_\gamma}_x([0,S]\times\{|x|<2R\})} + (A(S))^{-1} \left\| |x|^{-n/2+1-\gamma/p} u \right\|_{L^p_t L^p_r L^2_\omega([0,S]\times\{|x|>2R\})} \\
&\lesssim \|f\|_{\tilde{H}^\gamma_\varepsilon} + \|g\|_{\tilde{H}^{\gamma-1}_\varepsilon} + \|F\|_{L^1_t L_x^{s'_{1-\gamma-\varepsilon}}(\mathbb{R}_+\times\{|x|<2R\})} \\
&\quad + \left\| |x|^{-n/2+1-\gamma} F \right\|_{L^1_t L^1_r L^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}. \qquad (4.10)
\end{aligned}$$

Proof. By (4.9), we can assume $f=g=0$. By Duhamel's principle, we have

$$LHS \lesssim \|F\|_{L^1_t \tilde{H}^{\gamma-1}_\varepsilon(\mathbb{R}_+\times\Omega)} \lesssim \|F\|_{L^1_t \dot{H}^{\gamma-1}(\mathbb{R}_+\times\Omega)} + \|F\|_{L^1_t \dot{H}^{\gamma+\varepsilon-1}(\mathbb{R}_+\times\Omega)}.$$

Recall that the dual version of the trace lemma and Sobolev embedding gives (see (3.16) of [15]):

$$\|g\|_{\dot{H}^{\gamma-1}} \lesssim \left\| |x|^{-n/2+1-\gamma} g \right\|_{L^1_r L^2_\omega(\{|x|>2R\})} + \|g\|_{L^{s'_{1-\gamma}}(\{|x|<2R\})}, \quad (4.11)$$

if $1/2 < 1-\gamma < n/2$. Here the condition $1/2<1-\gamma<n/2$ is satisfied owing to (4.4) or (4.6). If we use (4.11), then we get

$$\begin{aligned}
&\|F\|_{L^1_t \dot{H}^{\gamma-1}([0,S]\times\Omega)} + \|F\|_{L^1_t \dot{H}^{\gamma+\varepsilon-1}([0,S]\times\Omega)} \\
&\lesssim \left\| |x|^{-n/2+1-\gamma} F \right\|_{L^1_t L^1_r L^2_\omega([0,S]\times\{|x|>2R\})} + \|F\|_{L^1_t L_x^{s'_{1-\gamma}}([0,S]\times\{|x|<2R\})}
\end{aligned}$$

$$+\left\||x|^{-n/2+1-\gamma-\varepsilon}F\right\|_{L^1_tL^1_rL^2_\omega([0,S]\times\{|x|>2R\})}+\|F\|_{L^1_tL_x^{s'_{1-\gamma-\varepsilon}}([0,S]\times\{|x|<2R\})}$$
$$\lesssim\left\||x|^{-n/2+1-\gamma}F\right\|_{L^1_tL^1_rL^2_\omega([0,S]\times\{|x|>2R\})}+\|F\|_{L^1_tL_x^{s'_{1-\gamma-\varepsilon}}([0,S]\times\{|x|<2R\})},$$

when $\varepsilon>0$ is small enough, which completes the proof. □

By modifying the proof of corresponding estimates in [15], we get the following higher-order estimates of (2.5) and (4.10), which are key to proving the existence theorem.

Proposition 0.22. (*Higher order Energy and Strichartz Estimates*). *Suppose that data* (f,g,F) *satisfies the* $H^2_B\times H^1_B\times H^1_B$ *boundary conditions. Under the conditions in Corollary* 0.21, *the following estimates hold:*

$$\sum_{|\alpha|\le 2}\left(\|\Gamma^\alpha u\|_{L^\infty_t\dot H^\gamma_B}+\|\partial_t\Gamma^\alpha u\|_{L^\infty_t\dot H^{\gamma-1}_B}\right)\lesssim\sum_{|\alpha|\le 2}\left(\|Z^\alpha f\|_{\tilde H^\gamma_{2\varepsilon}}+\|Z^\alpha g\|_{\tilde H^{\gamma-1}_{2\varepsilon}}\right)$$
$$+\sum_{|\alpha|\le 2}\Big(\||x|^{-\frac{n}{2}+1-\gamma}\Gamma^\alpha F\|_{L^1_tL^1_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}\tag{4.12}$$
$$+\|\Gamma^\alpha F\|_{L^1_tL_x^{s'_{1-\gamma-2\varepsilon}}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Big),$$

and

$$\sum_{|\alpha|\le 2}\left(\||x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma}\Gamma^\alpha u\|_{L^p_tL^p_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}+\|\Gamma^\alpha u\|_{L^p_tL^{s_\gamma}_x(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\right)$$
$$\lesssim\sum_{|\alpha|\le 2}\left(\|Z^\alpha f\|_{\tilde H^\gamma_{2\varepsilon}}+\|Z^\alpha g\|_{\tilde H^{\gamma-1}_{2\varepsilon}}\right)\tag{4.13}$$
$$+\sum_{|\alpha|\le 2}\Big(\||x|^{-\frac{n}{2}+1-\gamma}\Gamma^\alpha F\|_{L^1_tL^1_rL^2_\omega(\mathbb{R}_+\times\{|x|>2R\})}$$
$$+\|\Gamma^\alpha F\|_{L^1_tL_x^{s'_{1-\gamma-2\varepsilon}}(\mathbb{R}_+\times\{x\in\Omega:|x|<2R\})}\Big).$$

Proof. We will first deal with the Cauchy data for $\Gamma^\alpha u$. This is clear if Γ^α is replaced by Z^α. On the other hand, the Cauchy data is $(g,\Delta_g f+F(0,\cdot))$ for $\partial_t u$ and $(\Delta_g f+F(0,\cdot),\Delta_g g+\partial_t F(0,\cdot))$ for $\partial_t^2 u$, so we have

$$\|g\|_{\tilde H^\gamma_\varepsilon}+\|\Delta_{\mathrm g} f\|_{\tilde H^{\gamma-1}_\varepsilon\cap\tilde H^\gamma_\varepsilon}+\|F\|_{L^\infty_t\tilde H^{\gamma-1}_\varepsilon\cap L^\infty_t\tilde H^\gamma_\varepsilon}+\|\partial_t F\|_{L^\infty_t\tilde H^{\gamma-1}_\varepsilon}+\|\Delta_g g\|_{\tilde H^{\gamma-1}_\varepsilon}$$
$$\lesssim\sum_{|\alpha|\le 2}\left(\|Z^\alpha f\|_{\tilde H^\gamma_\varepsilon}+\|Z^\alpha g\|_{\tilde H^{\gamma-1}_\varepsilon}\right)+\sum_{|\alpha|\le 2}\|\Gamma^\alpha F\|_{L^1_t\tilde H^{\gamma-1}_\varepsilon},$$

where we use Sobolev embedding in the time variable t for $(F, \partial_t F)$. If we use (4.11) to control the last term $\sum_{|\alpha|\le 2} \|\Gamma^\alpha F\|_{L^1_t \tilde{H}^{\gamma-1}_\varepsilon}$, then we get (4.12) and (4.13) for the Cauchy data part of Γu.

Let us now give the argument for (4.13). Fix $\beta_0 \in C_0^\infty$ satisfying $\beta_0 = 1$ for $|x| \le R$ and vanishing for $|x| > 2R$. Let

$$\Gamma^\alpha u = (1-\beta_0)\Gamma^\alpha u + \beta_0 \Gamma^\alpha u = v + w.$$

Since Γ commutes with $\Box_g$ when $|x| \ge R$, we have

$$\begin{cases} \Box_g v = (1-\beta_0)\Gamma^\alpha F - [\beta_0, \Delta_{\mathrm{g}}]\Gamma^\alpha u, \\ v(0,\,\cdot\,) = ((1-\beta_0)\Gamma^\alpha u(0,\,\cdot\,),\ \partial_t v(0,\,\cdot\,) = \partial_t (1-\beta_0)\Gamma^\alpha u(0,\,\cdot\,). \end{cases}$$

The initial data has been taken care of from the discussion above, and the first nonlinear term is dominated by the right-hand side of (4.13) by (4.10). For the second nonlinear term, we use Proposition 0.10 and control it by

$$\sum_{|\alpha|\le 2} \|\,[\beta_0, \Delta_{\mathrm{g}}]\Gamma^\alpha u\|_{L^2_t H^{\gamma-1}_B} \lesssim \sum_{j\le 2} \|\beta_1 \partial_t^j u\|_{L^2_t H^{\gamma+2-j}_B}, \tag{4.14}$$

assuming that β_1 equals one on the support of β_0 and is supported in $R < |x| < 2R$. Noting that $[\Box_g, \partial_t^2] = 0$, if we use (2.5) for $\partial_t^2 u$ and Duhamel's principle for the forcing term $\partial_t^2 F$, we can control $\|\beta_1 \partial_t^2 u\|_{L^2_t H^\gamma_B}$ by the right-hand side of (4.13). On the other hand, by Cauchy-Schwarz and Parseval's formula,

$$\|\beta_1 \partial_t u\|^2_{L^2_t H^{\gamma+1}_B} \lesssim \|\beta_1 \partial_t^2 u\|_{L^2_t H^\gamma_B}\, \|\beta_1 u\|_{L^2_t H^{\gamma+2}_B}.$$

Thus it suffices to dominate $\|\beta_1 u\|_{L^2_t H^{\gamma+2}_B}$. By elliptic regularity of the operator Δ_g, we have

$$\begin{aligned} \|\beta_1 u\|_{L^2_t H^{\gamma+2}_B} &\lesssim \|\beta_2 \Delta_{\mathrm{g}} u\|_{L^2_t H^\gamma_B} + \|\beta_2 u\|_{L^2_t H^\gamma_B} \\ &\lesssim \|\beta_2 \partial_t^2 u\|_{L^2_t H^\gamma_B} + \|\beta_2 u\|_{L^2_t H^\gamma_B} + \|\beta_2 F\|_{L^2_t H^\gamma_B}, \end{aligned}$$

where $\beta_2 \in C_0^\infty$ equals one on the support of β_1 and is supported in the set where $|x| < 2R$. The first two terms are dominated as above using (2.5) and Duhamel's principle. For the last term, Sobolev embedding and duality yields

$$\begin{aligned} \|\beta_2 F\|_{L^2_t H^\gamma_B} &\lesssim \sum_{|\alpha|\le 1} \|\partial_x^\alpha F\|_{L^2_t L^{s'_{1-\gamma}}(\mathbb{R}_+ \times \{x\in\Omega : |x|\le 2R\})} \\ &\lesssim \sum_{|\alpha|\le 2} \|\partial_{t,x}^\alpha F\|_{L^1_t L^{s'_{1-\gamma-\varepsilon}}(\mathbb{R}_+ \times \{x\in\Omega : |x|\le 2R\})}. \end{aligned} \tag{4.15}$$

Thus, we are done with the proof of (4.13) when $\Gamma^\alpha u$ is replaced by v.

For $w=\beta_0\Gamma^\alpha u$, the coefficients of Γ are bounded on the support of β_0, so by Sobolev embedding

$$\begin{aligned}\sum_{|\alpha|\le 2}\|\beta_0\Gamma^\alpha u\|_{L^p_tL^{s_\gamma}_x(\mathbb{R}_+\times\Omega)} &\lesssim \sum_{|\alpha|\le 2}\|\beta_1\Gamma^\alpha u\|_{L^p_t\dot H^\gamma_B}\\ &\lesssim \sum_{|j|\le 2}\Big(\|\beta_1\partial_t^j u\|_{L^2_tH_B^{\gamma+2-j}}+\|\beta_1\Gamma^j u\|_{L^\infty_t\dot H^\gamma_B}\Big).\end{aligned}$$

The first term is dominated as above, and the bound for the second term comes from (4.12), so we are done with the proof of (4.13).

Now we turn to the proof of (4.12).

As before we first consider the inequality where $\Gamma^\alpha u$ is replaced by $v=(1-\beta_0)\Gamma^\alpha u$ in (4.12). The inequality involving initial data has been taken care of in the first paragraph of the proof, and the first nonlinear term is from energy estimates in $\mathbb{R}^n$, Duhamel's principle and (4.11). For the remaining term by (2.5) we see that it is controlled by

$$\sum_{|\alpha|\le 2}\|\,[\beta_0,\Delta_{\mathrm g}]\Gamma^\alpha u\|_{L^2_tH_B^{\gamma+\varepsilon-1}}\lesssim\sum_{j\le 2}\|\beta_1\partial_t^j u\|_{L^2_tH_B^{\gamma+\varepsilon+2-j}}. \tag{4.16}$$

By almost the same argument as above we get the desired bound in (4.12).

Now we are only left with $w=\beta_0\Gamma^\alpha u$. First notice that the left-hand side of (4.12) with w is dominated by $\sum_{j\le 3}\|\beta_1\partial_t^j u\|_{L^\infty_tH_B^{2+\gamma-j}}$. For the case $j=0,1$, since

$$\begin{cases}\Box_{\mathrm g}(\beta_1 u)=\beta_1F+[\Delta_{\mathrm g},\beta_2]u\\ (\beta_1u,\partial_t\beta_1u)|_{t=0}=(\beta_1f,\beta_1g),\end{cases}$$

we use (2.4) with the Duhamel formula to bound

$$\begin{aligned}&\|\beta_1u\|_{L^\infty_tH_B^{\gamma+2}}+\|\beta_1\partial_tu\|_{L^\infty_tH_B^{\gamma+1}}\\ &\qquad\lesssim\|\beta_1f\|_{H_B^{\gamma+2}}+\|\beta_1g\|_{H_B^{\gamma+1}}+\|\beta_2u\|_{L^2_tH_B^{\gamma+\varepsilon+2}}+\|\beta_1F\|_{L^1_tH_B^{\gamma+\varepsilon+1}}.\end{aligned}$$

The term on the right involving u was controlled previously; on the other hand, by Sobolev embedding,

$$\|\beta_1F\|_{L^1_tH_B^{\gamma+\varepsilon+1}}\lesssim\sum_{|\alpha|\le 2}\|\partial_x^\alpha F\|_{L^1_tL_x^{s'_{1-\gamma-\varepsilon}}}.$$

To handle the terms for $j=2,3$ we use the equation to bound

$$\sum_{j=2,3}\|\beta_1\partial_t^j u\|_{L^\infty_tH_B^{2+\gamma-j}}\le\sum_{j=0,1}\Big(\|\beta_1\partial_t^j\Delta_{\mathrm g}u\|_{L^\infty_tH_B^{\gamma-j}}+\|\beta_1\partial_t^jF\|_{L^\infty_tH_B^{\gamma-j}}\Big).$$

The terms involving $\Delta_{\mathrm{g}} u$ are dominated by $\|\beta_2 \partial_t^j u\|_{L_t^\infty H_B^{\gamma+2-j}}$ with $j = 0, 1$. The terms involving F are controlled for $j = 1$ by the Sobolev embedding theorem, and for $j = 0$ by observing that (4.15) holds with L_t^2 replaced by L_t^∞. This completes the proof of (4.12). $\square$

Proof of Theorem 0.19:
We will adapt the argument from [15]. First, let u_0 solve the Cauchy problem (1.1) with $F = 0$. We iteratively define u_k, for $k \geq 1$, by solving

$$\begin{cases} (\partial_t^2 - \Delta_{\mathrm{g}}) u_k(t,x) = F_p(u_{k-1}(t,x)), \quad (t,x) \in \mathbb{R}_+ \times \Omega \\ u_k(0,\,\cdot\,) = f,\ \partial_t u_k(0,\,\cdot\,) = g \\ (Bu_k)(t,x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

Our aim is to show that, if the constant $\varepsilon' > 0$ in (4.5) is small enough, then so is

$$\begin{aligned} M_k = \sum_{|\alpha| \leq 2} \Big(&\|\Gamma^\alpha u_k\|_{L_t^\infty \dot H_B^\gamma([0,S]\times\Omega)} + \|\partial_t \Gamma^\alpha u_k\|_{L_t^\infty \dot H_B^{\gamma-1}([0,S]\times\Omega)} \\ &+ (A(S))^{-1} \big\| |x|^{\frac{-\frac{n}{2}+1-\gamma}{p}} \Gamma^\alpha u_k \big\|_{L_t^p L_r^p L_\omega^2([0,S]\times\{|x|>2R\})} \\ &+ \|\Gamma^\alpha u_k\|_{L_t^p L_x^{s_\gamma}([0,S]\times\{x\in\Omega:\,|x|<2R\})} \Big) \end{aligned}$$

for every $k = 0, 1, 2, \ldots$.

For $k = 0$, it follows by (4.12) and (4.13) that $M_0 \leq C_0 \varepsilon'$, with C_0 a fixed constant. More generally, (4.12) and (4.13) yield that

$$\begin{aligned} M_k \leq C_0\varepsilon' + C_0 \sum_{|\alpha|\leq 2} \Big(&\big\| |x|^{-\frac{n}{2}+1-\gamma} \Gamma^\alpha F_p(u_{k-1}) \big\|_{L_t^1 L_r^1 L_\omega^2(\mathbb{R}_+\times\{|x|>2R\})} \\ &+ \|\Gamma^\alpha F_p(u_{k-1})\|_{L_t^1 L_x^{s'_{1-\gamma-2\varepsilon}}(\mathbb{R}_+\times\{x\in\Omega:\,|x|<2R\})} \Big). \end{aligned} \tag{4.17}$$

Note that our assumption (4.2) on the nonlinear term F_p implies that for small v

$$\sum_{|\alpha|\leq 2} |\Gamma^\alpha F_p(v)| \lesssim |v|^{p-1} \sum_{|\alpha|\leq 2} |\Gamma^\alpha v| + |v|^{p-2} \sum_{|\alpha|\leq 1} |\Gamma^\alpha v|^2.$$

Furthermore, since u_k will be locally of regularity $H_B^{\gamma+2} \subset L^\infty$ and F_p vanishes at 0, it follows that $F_p(u_k)$ satisfies the B boundary conditions if u_k does.

Since the collection Γ contains vectors spanning the tangent space to S^{n-1}, by Sobolev embedding for $n = 3, 4$ we have

$$\|v(r\,\cdot)\|_{L^\infty_\omega} + \sum_{|\alpha|\le 1} \|\Gamma^\alpha v(r\,\cdot)\|_{L^4_\omega} \lesssim \sum_{|\alpha|\le 2} \|\Gamma^\alpha v(r\,\cdot)\|_{L^2_\omega}.$$

Consequently, for fixed $t, r > 0$

$$\sum_{|\alpha|\le 2} \|\Gamma^\alpha F_p(u_{k-1}(t, r\,\cdot))\|_{L^2_\omega} \lesssim \sum_{|\alpha|\le 2} \|\Gamma^\alpha u_{k-1}(t, r\,\cdot)\|^p_{L^2_\omega}.$$

Thus the first summand in the right-hand side of (4.17) is dominated by $C_1\big(A(S)M_{k-1}\big)^p$.

We next observe that, since $s_\gamma > 2$ and $n \le 4$, it follows by Sobolev embedding on $\{\Omega \cap |x| < 2R\}$ that

$$\|v\|_{L^\infty(x\in\Omega:|x|<2R)} + \sum_{|\alpha|\le 1} \|\Gamma^\alpha v\|_{L^4(x\in\Omega:|x|<2R)} \lesssim \sum_{|\alpha|\le 2} \|\Gamma^\alpha v\|_{L^{s_\gamma}(x\in\Omega:|x|<2R)}.$$

Since $s'_{1-\gamma-2\varepsilon} < 2$, it holds for each fixed t that

$$\begin{aligned} &\sum_{|\alpha|\le 2} \|\Gamma^\alpha F_p(u_{k-1}(t,\,\cdot))\|_{L^{s'_{1-\gamma-2\varepsilon}}(x\in\Omega:|x|<2R)} \\ &\lesssim \sum_{|\alpha|\le 2} \|\Gamma^\alpha F_p(u_{k-1}(t,\,\cdot))\|_{L^2(x\in\Omega:|x|<2R)} \\ &\lesssim \sum_{|\alpha|\le 2} \|\Gamma^\alpha u_{k-1}(t,\,\cdot)\|^p_{L^{s_\gamma}(x\in\Omega:|x|<2R)}. \end{aligned} \tag{4.18}$$

The second summand in the right side of (4.17) is thus dominated by $C_1 M^p_{k-1}$, and we conclude that $M_k \le C_0\varepsilon' + 2C_0\, C_1(A(S)M_{k-1})^p$. For ε' sufficiently small, by the definition of $A(S)$, we obtain

$$M_k \le 2\, C_0\varepsilon', \quad k = 1, 2, 3, \dots \tag{4.19}$$

To finish the proof of Theorem 0.19 we need to show that u_k converges to a solution of the equation (4.1). For this it suffices to show that

$$\begin{aligned} A_k = (A(S))^{-1} \big\| \, |x|^{\frac{-\frac{n}{2}+1-\gamma}{p}} (u_k - u_{k-1}) \, \big\|_{L^p_t L^p_r L^2_\omega([0,S]\times\{|x|>2R\})} \\ + \|u_k - u_{k-1}\|_{L^p_t L^{s_\gamma}_x([0,S]\times\{x\in\Omega:\,|x|<2R\})} \end{aligned}$$

tends geometrically to zero as $k \to \infty$. Since $|F_p(v) - F_p(w)| \lesssim |v - w|(\,|v|^{p-1} + |w|^{p-1}\,)$ when v and w are small, the proof of (4.19) can be adapted to show that, for small $\varepsilon' > 0$, there is a uniform constant C so that

$$A_k \leq C(A(S))^p A_{k-1}(M_{k-1} + M_{k-2})^{p-1},$$

which, by (4.19), implies that $A_k \leq \frac{1}{2}A_{k-1}$ for small ε'. Since A_1 is finite, the claim follows, which finishes the proof of Theorem 0.19. □

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