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SHORT PROOFS OF RESULTS BY LANDESMAN, LAZER, AND LEACH ON PROBLEMS RELATED TO RESONANCE

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Abstract. We study two boundary-value problems originally considered by A.C. Lazer and coauthors. The first is an elliptic problem

$$Lu + \alpha u + g(u) = h(x)$$

in a bounded domain $\Omega \subset R^N$, with $u = 0$ on the boundary $\partial\Omega$. It is assumed that $Lu + \alpha u = 0$ has a one-dimensional set of solutions satisfying the same boundary condition. The second is an ODE problem

$$u'' + n^2 u + g(u) = e(t),$$

where e has period 2π and a 2π-periodic solution is sought. Here, the corresponding linear homogeneous equation, $u'' + n^2 u = 0$, has a two-dimensional set of 2π-periodic solutions. In each case, conditions are sought which guarantee the existence of at least one solution to the original problem. We give short proofs of theorems first proved by E.M. Landesman and Lazer, and by Lazer and D.E. Leach, on these two problems.

1. Introduction and statement of results

In 1970, E.M. Landesman and A.C. Lazer considered a class of elliptic boundary-value problems

$$Lu + \alpha u + g(u) = h(x), \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

in a bounded domain Ω [3]. Their interest was in the case where the linear problem

$$Lu + \alpha u = 0, \quad u = 0 \text{ on } \partial\Omega \tag{1.2}$$

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has a one-dimensional space, say span $\{w\}$, of weak solutions.[1] They assumed that g is continuous, with limits at $\pm\infty$ such that

$$g(-\infty) < g(u) < g(\infty) \tag{1.3}$$

for all u. Assuming also that $h \in L^2(\Omega)$, a condition was given relating h, g, and w which is necessary and sufficient for the problem (1.1) to have a weak solution. This paper has attracted much attention over the years. It was preceded by a similar result for ODEs, by Lazer and D. Leach [4], which is also well known. Subsequently, new proofs have been given for each result, and a number of extensions have been found.

Development and application of the results in these papers has continued for the forty years since their publications, with a steady stream of citations over the years. For this reason we believe that improved proofs are of interest. In this paper we will give very short and elementary proofs of the sufficiency parts of these results. Both of these proofs are shorter than any others we have seen. We believe that our proof of the result in [4] is essentially new, and more elementary than others, while for [3] the method is similar to the original proof, but we have shortened the details.[2]

We will start with the ODE case, which is for an equation of the form

$$u'' + n^2 u + g(u) = e(t), \tag{1.4}$$

where n is a positive integer and e has period 2π. The goal is to determine conditions on g and e which ensure that there is a 2π-periodic solution. If $g = 0$ and $e(t) = \sin nt$, for example, then all solutions are unbounded, which is the phenomenon of resonance. By using the variation-of-parameters formula, one easily establishes that if $g = 0$, then there is a periodic solution if and only if the two Fourier coefficients

$$A = \int_0^{2\pi} e(t) \sin nt \; dt, \qquad B = \int_0^{2\pi} e(t) \cos nt \; dt \tag{1.5}$$

are zero. The following result is proved in [4].

Theorem 1. *If $e(t)$ is continuous and 2π-periodic, and g is continuous and satisfies the condition* (1.3) *above, then* (1.4) *has a periodic solution if and*

[1] The original paper, [3], has a good outline of the needed background in PDE theory, such as the definition of weak solutions and of the function spaces $H_0(\Omega)$ and $H_0^1(\Omega)$ needed for this definition. While there are many such expositions, the one there is particularly tailored to this problem. We will not repeat this material here.

[2] Necessity is easier, and we refer to the original papers in each case. Each paper also has results with weaker hypotheses on g.

only if

$$\sqrt{A^2+B^2} < 2\left(g\left(\infty\right) - g\left(-\infty\right)\right), \tag{1.6}$$

where A and B are defined by (1.5)*.*

We will give our new proof of sufficiency in the next section.

To state the result of Landesman and Lazer for (1.1), suppose that w is a nonzero weak solution of (1.1), and let

$$\Omega_+ = \{x \in \Omega : w\left(x\right) > 0\}, \quad \Omega_- = \{x \in \Omega : w\left(x\right) < 0\}.$$

Theorem 2. *Suppose that all weak solutions of* (1.1) *are of the form cw for some constant c. Suppose also $h \in L^2\left(\Omega\right)$, and that g is continuous and satisfies* (1.3)*. Then the problem* (1.1) *has a weak solution if and only if*

$$g\left(-\infty\right) \int_{\Omega^+} |w| \ dx - g\left(\infty\right) \int_{\Omega^-} |w| \ dx \tag{1.7}$$
$$< \int_\Omega h \ w \ dx < g\left(\infty\right) \int_{\Omega^+} |w| \ dx - g\left(-\infty\right) \int_{\Omega^-} |w| \ dx.$$

Again, a short proof of sufficiency is given below.

In both cases, the years since the original paper have seen many advances inspired by this work. Fortunately, there is a recent survey by J. Mawhin [7] of the ODE aspects, following on from [4], and we refer the reader to this work for updates. We know of no comparable reference for Theorem 2, but its influence has been widespread and continuing. We do not attempt to discuss recent developments here.

2. Proof of the result of Lazer and Leach

Our new proof is essentially a winding-number argument. We begin by assuming that g satisfies a local Lipschitz condition. More precisely, we assume

Condition I: *For any u_0 there is an $\varepsilon > 0$ and an $L > 0$ such that if u_1 and u_2 are in the interval $\left[u_0 - \varepsilon, u_0 + \varepsilon\right]$, then $\left|g\left(u_1\right) - g\left(u_2\right)\right| \leq L\left|u_1 - u_2\right|$.*

This implies that solutions to initial-value problems for (1.4) are unique and depend continuously on the initial conditions. We then have the following lemma.

Lemma 2.1. *Suppose that g satisfies* (1.3) *and Condition I. Then for given $r > 0$ and β, consider the unique solution of* (1.4) *such that*

$$u\left(0\right) = r \sin n\beta, \quad u'\left(0\right) = nr \cos n\beta.$$

If r is sufficiently large, then for every β this solution satisfies

$$(u(2\pi) - u(0))\, n\cos n\beta - \big(u'(2\pi) - u'(0)\big)\sin n\beta > 0. \tag{2.1}$$

We will prove this lemma below. To apply it, take r large enough that (2.1) applies for every β. Thus, it applies for all initial conditions $(u(0), u'(0))$ on a circle C_{r_0} of sufficiently large radius r_0. For any β, there are $R > 0$ and γ, both depending continuously on β, such that

$$u(2\pi) - u(0) = R\sin n\gamma,\ \ u'(2\pi) - u'(0) = nR\cos n\gamma, \tag{2.2}$$

and from this and (2.1) we get

$$\sin n(\gamma - \beta) > 0.$$

It follows that as β increases from 0 to $\frac{2\pi}{n}$, γ must also increase by $\frac{2\pi}{n}$, since the initial conditions at $\beta = 0$ and $\beta = \frac{2\pi}{n}$ are the same.

Now assume that there is no periodic solution. In this case, as r is decreased from r_0, R remains positive and γ continues to be well defined, by (2.2) and the requirement that γ vary continuously with (r, β), for all $\beta \in \left[0, \frac{2\pi}{n}\right]$ and $r > 0$. As r decreases, γ must continue to increase by $\frac{2\pi}{n}$ as β goes from 0 to $\frac{2\pi}{n}$. However, for small r, the total change in $\mathbf{v}(r,\beta) = (u(2\pi) - u(0), \frac{u'(2\pi) - u'(2\pi)}{n})$ as β goes around the circle is small, because the solution depends continuously on its initial conditions. On the other hand, as $r \to 0$, R in (2.2) is bounded away from zero, because we are assuming that the solution with $r = 0$ is not periodic. This implies that γ could not increase continuously by a total amount of $\frac{2\pi}{n}$. This contradiction implies that there must be a periodic solution. There remains to prove Lemma 2.1 and to show that Condition I can be replaced by the assumption that g is continuous.

2.0.1. *Proof of Lemma 2.1.*

Proof. First, multiply (1.4) by $\sin(nt + n\beta)$ and integrate from 0 to 2π. With two integrations by parts, this gives

$$\begin{aligned}&\big(u'(2\pi) - u'(0)\big)\sin n\beta - (u(2\pi) - u(0))\, n\cos n\beta \\ &+ \int_0^{2\pi} g(u)\sin(nt + n\beta)\, dt = A\cos n\beta + B\sin n\beta,\end{aligned} \tag{2.3}$$

where A and B are given by (1.5). Also,

$$\int_0^{2\pi} g(u(t))\sin(nt + n\beta)\, dt = \int_\beta^{2\pi+\beta} g(u(s-\beta))\sin ns\ ds. \tag{2.4}$$

The idea now is that for large r, $u\,(s-\beta)$ is large and positive for most of any interval where $\sin ns > 0$, and large and negative for most of any interval where $\sin ns < 0$. To make this precise, use the variation-of-parameters formula to give

$$\begin{aligned} u\,(s) &= r \sin n\beta \cos ns + r \cos n\beta \sin ns \\ &\quad + \frac{1}{n}\int_0^s \sin\,(ns - n\tau)\,(e\,(\tau) - g\,(u\,(\tau)))\,d\tau, \\ u(s-\beta) &= r\big(\sin n\beta(\cos ns \cos n\beta + \sin ns \sin n\beta) \qquad (2.5) \\ &\quad + \cos n\beta(\sin ns \cos n\beta - \sin n\beta \cos ns)\big) + f(s) = r \sin ns + f\,(s)\,, \end{aligned}$$

where f is bounded independent of s, r, and β. Hence for any $\delta \in (0,1]$,

$$\lim_{r\to\infty} g\,(u\,(s-\beta)) = g\,(\infty)$$

uniformly on $\{s : \sin ns \geq \delta\}$ and

$$\lim_{r\to\infty} g\,(u\,(s-\beta)) = g\,(-\infty)$$

uniformly on $\{s : \sin ns \leq -\delta\}$. It follows that

$$\lim_{r\to\infty} \int_0^{2\pi} g\,(u\,(t))\sin\,(nt + n\beta)\,dt = 2\,(g\,(\infty) - g\,(-\infty))\,.$$

Then, (2.3) and (1.6) imply (2.1) for sufficiently large r. This proves Lemma 2.1 and Theorem 1. □

This proves Theorem 1 in the case that g satisfies a local Lipschitz condition. If this is not the case, then we note that the bound on $f\,(s)$ in the proof of Lemma 2.1 depends only on the bounds on e and g. Hence, $r = \bar{r}$ can be chosen so that Lemma 2.1 holds for any continuous function g_1 such that

$$g\,(-\infty) < g_1 < g\,(\infty) \qquad (2.6)$$

on $(-\infty, \infty)$.

We can then find a sequence of functions g_n which all satisfy (2.6) such that each g_n satisfies a local Lipschitz condition, $g_n\,(\pm\infty) = g\,(\pm\infty)$, and g_n converges to g uniformly on $[-\bar{r}, \bar{r}]$. Corresponding to each of these there is a periodic solution u_n of (1.4) (with g_n instead of g). These all take values in $[-\bar{r}, \bar{r}]$, so they are uniformly bounded, and they have bounded derivatives since u_n'' is uniformly bounded. Hence some subsequence of $\{u_n\}$ converges uniformly on $[0, 2\pi]$ to a 2π-periodic function u, and by using an equivalent integral equation, this function is a solution of (1.4).

3. Proof of the result of Landesman and Lazer

A short proof of Theorem 2 was given by Hess [2] in 1974. The proof below is a little shorter, and more elementary. It is assumed that L is a second-order, symmetric, uniformly elliptic operator on a bounded domain D in R^n. Some readers may prefer to consider just the Laplacian.

Assume that the weak solution w in the hypotheses of Theorem 2 has norm $\|w\| = 1$ in $L^2(\Omega)$. Given $v_1 \in w^\perp$ (the subspace of $L^2(\Omega)$ orthogonal to w), and a real number c_1, observe first that the function

$$q = h - g(v_1 + c_1 w) - w \int_\Omega [h - g(v_1 + c_1 w)]\, w\, dx$$

is in $w^\perp$. We define a mapping $T : w^\perp \times R \to w^\perp \times R$ by letting v_2 be the unique solution to

$$\begin{aligned} Lv + \alpha v &= q \text{ in } \Omega \\ v &= 0 \text{ on } \partial\Omega, \end{aligned}$$

and setting

$$c_2 = c_1 + \int_\Omega [h - g(v_1 + c_1 w)]\, w\, dx.$$

The mapping T is compact, and because $h \in L^2(\Omega)$ and g is bounded, there is an M independent of v_1 and c_1 such that $\|v_2\| \le M$.

Further, (1.3) implies that

$$\lim_{c_1 \to \infty} \int_\Omega g(v_1 + c_1 w)\, w\, dx = g(\infty) \int_{\Omega_+} |w|\, dx - g(-\infty) \int_{\Omega_-} |w|\, dx$$
$$\lim_{c_1 \to -\infty} \int_\Omega g(v_1 + c_1 w)\, w\, dx = g(-\infty) \int_{\Omega_+} |w|\, dx - g(\infty) \int_{\Omega_-} |w|\, dx.$$

As a result, (1.7) implies that if c_1 is sufficiently large and positive, then $c_2 < c_1$, while if c_1 is sufficiently large and negative, then $c_2 > c_1$. In addition, g is bounded, so there is an $R > 0$ such that if $c_1 \in [-R, R]$, then $c_2 \in [-R, R]$. If $T(v_1, c_1) = (v_2, c_2)$, then the mapping T is continuous and maps the closed convex set $w^\perp \times [-R, R]$ into a subset of itself with compact closure. By Schauder's fixed-point theorem, T has a fixed point.

If $c_2 = c_1$, then

$$\int_\Omega [h - g(v_1 + c_1 w)]\, w\, dx = 0,$$

and so $v_1 = v_2$ implies that

$$Lv_1 + \alpha v_1 = h - g(v_1 + c_1 w).$$

From the definition of w we obtain

$$(L+\alpha)(v_1+c_1w)=h-g(v_1+c_1w).$$

This completes the proof of sufficiency in Theorem 2.

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