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ON A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS IN 2D

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Abstract. We consider a system of nonlinear Schrödinger equations with quadratic nonlinearities in two space dimensions. Our aim is to show time decay estimates of small solutions and nonexistence of the usual scattering states for a system. Furthermore we prove stability in time of small solutions in the neighborhood of solutions to a suitable approximate equation.

1. Introduction

We consider a system of quadratic nonlinear Schrödinger equations in the two dimensional case

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \lambda \overline{v_1} v_2, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = \mu v_1^2, \end{cases} \tag{1.1}$$

where $\lambda, \mu \in \mathbf{C}$, $\Delta = \sum_{j=1}^{2} \partial_j^2$, $\partial_j = \frac{\partial}{\partial x_j}$, $m_1, m_2 > 0$ are the masses of the particles. We make the scaling $v_1 = \frac{1}{\sqrt{|\lambda\mu|}} u_1$ and $v_2 = \frac{\mu}{|\lambda\mu|} u_2$, to exclude the constants λ and μ from system (1.1)

$$\begin{cases} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = \gamma \overline{u_1} u_2, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2, \end{cases} \tag{1.2}$$

where $\gamma = \frac{\lambda\mu}{|\lambda\mu|} \in \mathbf{C}$, $|\gamma| = 1$.

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From the physical point of view, system (1.1) must satisfy the $\mathbf{L}^2$ - conservation law. By system (1.1) we get

$$\begin{aligned} \partial_t |u_1|^2 - \frac{1}{m_1} \operatorname{Re}\left(i\left(\Delta u_1\right)\overline{u_1}\right) &= -2\operatorname{Re}\left(i\gamma \overline{u_1}^2 u_2\right), \\ \partial_t |u_2|^2 - \frac{1}{m_2} \operatorname{Re}\left(i\left(\Delta u_2\right)\overline{u_2}\right) &= -2\operatorname{Re}\left(i u_1^2 \overline{u_2}\right), \end{aligned}$$

then integration by parts yields

$$\frac{d}{dt}\left(\|u_1\|_{\mathbf{L}^2}^2 + \|u_2\|_{\mathbf{L}^2}^2\right) = -2\operatorname{Re}\int_{\mathbf{R}^2}\left(i\left(\gamma-1\right)\overline{u_1}^2 u_2\right)dx.$$

If we assume that

$$\gamma = 1; \tag{1.3}$$

that is, $\lambda\mu > 0$ in system (1.1), then we find the $\mathbf{L}^2$ - conservation law

$$\frac{d}{dt}\left(\|u_1\|_{\mathbf{L}^2}^2 + \|u_2\|_{\mathbf{L}^2}^2\right) = 0.$$

Under the condition (1.3) system (1.2) becomes

$$\begin{cases} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = \overline{u_1} u_2, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2. \end{cases} \tag{1.4}$$

It is interesting to compare system (1.4) with a system of nonlinear Klein-Gordon equations

$$\begin{cases} \frac{1}{2c^2 m_1}\partial_t^2 v_1 - \frac{1}{2m_1}\Delta v_1 + \frac{m_1 c^2}{2} v_1 = -\overline{v_1} v_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 v_2 - \frac{1}{2m_2}\Delta v_2 + \frac{m_2 c^2}{2} v_2 = -v_1^2, \end{cases} \tag{1.5}$$

where c is the speed of light. Letting $v_j = e^{-itm_j c^2} u_j$, we find for u_1 and u_2 the following equations:

$$\begin{cases} \frac{1}{2c^2 m_1}\partial_t^2 u_1 - i\partial_t u_1 - \frac{1}{2m_1}\Delta u_1 = -e^{itc^2(2m_1 - m_2)}\overline{u_1} u_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 u_2 - i\partial_t u_2 - \frac{1}{2m_2}\Delta u_2 = -e^{itc^2(m_2 - 2m_1)} u_1^2. \end{cases}$$

We assume the mass condition

$$2m_1 = m_2; \tag{1.6}$$

then we get

$$\begin{cases} \frac{1}{2c^2 m_1}\partial_t^2 u_1 - i\partial_t u_1 - \frac{1}{2m_1}\Delta u_1 = -\overline{u_1} u_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 u_2 - i\partial_t u_2 - \frac{1}{2m_2}\Delta u_2 = -u_1^2. \end{cases} \tag{1.7}$$

By letting $c \to \infty$ in (1.7) formally the nonrelativistic version of (1.5) can be obtained, which is the system of quadratic nonlinear Schrödinger equations (1.4).

Equation (1.5) is closely related to a system of nonlinear Klein-Gordon equations studied in [5], [9] and a system of Dirac-Klein-Gordon equations studied in [1], [7]. However the case of three space dimensions was considered in these papers. The nonlinear Schrödinger and Klein-Gordon equations with quadratic nonlinearities in three space dimensions are known as supercritical since small solutions behave like free solutions as $t \to \infty$ (see, e.g. [14], [18]). On the other hand, these systems in two space dimensions are critical in general. For example, small solutions of the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = |u|\, u, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}^2$$

do not behave like free solutions as $t \to \infty$ (see [2]). Furthermore, the exact asymptotic behavior of small solutions was obtained in [8] for the initial-value problem and in [6] for the final state problem. Global existence of large solutions was obtained by virtue of the $\mathbf{L}^2$ - conservation law in [22] (see Proposition 1 below).

In the case of the nonlinear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u = |u|\, u, \ \ (t,x) \in \mathbf{R} \times \mathbf{R}^2$$

the nonexistence of asymptotically free solutions was shown in [16]. Global existence of large solutions was obtained by the conservation of energy. However time decay estimates and the asymptotic behavior of solutions are not known up to now even for small data. Global existence of small solutions to the Cauchy problem for system (1.5) with $2m_1 \neq m_2$ with data in some weighted Sobolev space was proved in [19]. It was shown also that the solutions behave asymptotically like free solutions. In [12] global existence and time decay estimates of small solutions to the Cauchy problem for system (1.5) for the case of $2m_1 = m_2$, when the data have compact support and belong to some Sobolev space, was obtained. As far as we know the large-time asymptotic behavior of solutions is not known.

Our purpose in this paper is to prove global existence and obtain time decay estimates of small solutions to the Cauchy problem for system (1.4) with $2m_1 = m_2$. Also we prove the nonexistence of the asymptotically free solutions. Our proof below is different from that of [16] which depends on the finite propagation speed of solutions since nonlinear Schrödinger equations do not have such properties. The case of $2m_1 \neq m_2$ is still an open problem.

Furthermore we prove stability in time of solutions obtained in Theorem 1 in the neighborhood of solutions to a suitable approximate equation. We let

$$\mathbf{H}^{m,s} = \left\{ f = (f_1, f_2) \in \mathbf{L}^2 : \|f\|_{\mathbf{H}^{m,s}} = \sum_{j=1}^{2} \|f_j\|_{\mathbf{H}^{m,s}} < \infty \right\},$$

where

$$\|f\|_{\mathbf{H}^{m,s}} = \left\| (1-\Delta)^{\frac{m}{2}} \left(1+|x|^2\right)^{\frac{s}{2}} f \right\|_{\mathbf{L}^2}.$$

We write $\mathbf{H}^m = \mathbf{H}^{m,0}$ for simplicity. The Fourier transform is defined by

$$\mathcal{F}\phi = \hat{\phi}(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{-i(x\cdot\xi)} \phi(x)\, dx$$

and the inverse Fourier transformation is defined by

$$\mathcal{F}^{-1}\phi = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{i(x\cdot\xi)} \phi(\xi)\, d\xi.$$

Denote by $\mathcal{U}(t) = \mathcal{F}^{-1} e^{-\frac{it}{2}|\xi|^2} \mathcal{F}$ the free Schrödinger evolution group.

We now state our main results.

Theorem 1. *Assume that $2m_1 = m_2$. Then for some $\varepsilon > 0$ there exists a unique global solution $(u_1, u_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)^2$ to the Cauchy problem for system* (1.4) *with the initial conditions $u_1(0) = \phi_1$ and $u_2(0) = \phi_2$ for any initial data $(\phi_1, \phi_2) \in \left(\mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)^2$ satisfying*

$$\sum_{j=1}^{2} \|\phi_j\|_{\mathbf{H}^2 \cap \mathbf{H}^{0,2}} \le \varepsilon.$$

Moreover, the time decay estimate

$$\sum_{j=1}^{2} \|u_j(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1}$$

is true for all $t \ge 0$. Furthermore the solution (u_1, u_2) (1.4) *satisfies two conservation laws*

$$I_1(u_1, u_2) = I_1(\phi_1, \phi_2), \;\; I_2(u_1, u_2) = I_2(\phi_1, \phi_2),$$

where $I_1(u_1, u_2) = \|u_1\|_{\mathbf{L}^2}^2 + \|u_2\|_{\mathbf{L}^2}^2$ and

$$I_2(u_1, u_2) = \frac{1}{m_1} \|\nabla u_1\|_{\mathbf{L}^2}^2 + \frac{1}{2m_2} \|\nabla u_2\|_{\mathbf{L}^2}^2 + 2 \int_{\mathbf{R}^2} Re\left(\overline{u_1}^2 u_2\right) dx.$$

Theorem 1 will be proved below in Section 2 through a priori estimates of solutions. Our main tool is the factorization technique of the free Schrödinger evolution group used in [10].

Proposition 1. *Let* $(\phi_1, \phi_2) \in \left(\mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)^2$. *Then there exists a unique global solution* (u_1, u_2) *of* (1.4) *such that* $(u_1, u_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)^2$ *satisfying conservation laws stated in Theorem* 1.

Proof. We give the outline of the proof. Existence of local solutions is proved in a standard way applying the contraction mapping principle (see [3]). Then the global existence can be easily obtained by a priori estimates of the local solutions. We have the $\mathbf{L}^2$conservation law (1.2). Multiplying the equations of system (1.4) by $\partial_t \overline{v_1}$ and $\partial_t \overline{v_2}$ respectively, and taking the real part of the result, we obtain

$$\begin{aligned} \frac{1}{m_1} \operatorname{Re}\left(\Delta u_1 \partial_t \overline{u_1}\right) &= \partial_t \operatorname{Re}\left(\overline{u_1}^2 u_2\right) - \gamma \operatorname{Re}\left(\overline{u_1}^2 \partial_t u_2\right) \\ \frac{1}{2m_2} \operatorname{Re}\left(\Delta u_2 \partial_t \overline{u_2}\right) &= \operatorname{Re}\left(u_1^2 \partial_t \overline{u_2}\right) \end{aligned}$$

from which we have

$$-\frac{1}{m_1} \operatorname{Re}\left(\Delta u_1 \partial_t \overline{u_1}\right) - \frac{1}{2m_2} \operatorname{Re}\left(\Delta u_2 \partial_t \overline{u_2}\right) + \partial_t \operatorname{Re}\left(\overline{u_1}^2 u_2\right) = 0.$$

Hence integrating in space we find

$$\frac{d}{dt}\left(\frac{1}{m_1}\left\|\nabla u_1\right\|_{\mathbf{L}^2}^2 + \frac{1}{2m_2}\left\|\nabla u_2\right\|_{\mathbf{L}^2}^2 + 2\int_{\mathbf{R}^2} \operatorname{Re}\left(\overline{u_1}^2 u_2\right) dx\right) = 0. \qquad (1.8)$$

By (1.2) and (1.8), we have the conservation of energy. Therefore we have a priori estimates in $\mathbf{H}^{1,0}$ space. An a priori estimate in $\mathbf{H}^{2,0}$ space is proved by the Strichartz estimate and Gronwall's inequality as in paper [22]. Thus we get the desired a priori estimates of solutions in $\mathbf{H}^{2,0}$space. The a priori estimates in $\mathbf{H}^{0,2}$ space, Proposition 1, is proved. □

Next we state the nonexistence of the usual scattering states.

Theorem 2. *Let* $(u_1, u_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)^2$ *be a global solution obtained in Theorem* 1. *Then there does not exist any nontrivial scattering state* $(u_{1+}, u_{2+}) \in \left(\mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)^2$ *such that* $u_{1+} \neq 0$ *and*

$$\sum_{j=1}^{2} \left\| u_j(t) - \mathcal{U}\left(\frac{t}{m_j}\right) u_{j+} \right\|_{\mathbf{L}^2} \to 0 \quad \text{as } t \to \infty.$$

Under some special conditions we obtain the asymptotic behavior of solutions. Denote $\varphi_j(t) = i\mathcal{D}(\frac{1}{m_j})\mathcal{F}\mathcal{U}(-\frac{t}{m_j})u_j(t)$.

Theorem 3. *Let* $(u_1, u_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)^2$ *be a global solution obtained in Theorem 1. Then there exist initial data* $\Psi_1(\xi) \in \mathbf{L}^\infty$ *and* $\Psi_2(\xi) \in \mathbf{L}^\infty$ *for the Cauchy problem*

$$\begin{cases} \partial_t \psi_1 = t^{-1}\overline{\psi_1}\psi_2, t > 1, \\ \partial_t \psi_2 = -t^{-1}\psi_1^2, t > 1, \\ \psi_1(1,\xi) = \Psi_1(\xi), \psi_2(1,\xi) = \Psi_2(\xi), \end{cases} \tag{1.9}$$

such that the asymptotics for the solution (u_1, u_2)

$$u_j(t,x) = it^{-1}e^{\frac{im_j}{2t}|x|^2}\psi_j\big(t, \frac{x}{t}\big) + O\big(\varepsilon^2 t^{-\frac{3}{2}+\gamma}\big)$$

is true for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^2$, *where* $\gamma \in \left(0, \frac{1}{2}\right)$.

We note that the asymptotic behavior of solutions to the Cauchy problem (1.9) is defined by the amplitude $|\Psi_j(\xi)|$ and the angular $\arg\Psi_j(\xi)$ which depend on the global solution (u_1, u_2) (see the proof of Theorem 3 below). However we cannot control the data $\Psi_1(\xi)$ and $\Psi_2(\xi)$ for the Cauchy problem (1.9) and so it is difficult to find exact asymptotic behavior of solutions to the Cauchy problem (1.9). Below in Section 6 we will show that the asymptotic behavior of solutions to the Cauchy problem (1.9) is divided into three cases (asymptotically free, asymptotically free with a phase modification, and aperiodic in time oscillation of the amplitude of the solution). If we concentrate our attention on the final state problem for equation (1.4), it is possible to find exact asymptotic behavior of solutions to (1.4) in the neighborhood of a special solution to the Cauchy problem (1.9). It is also interesting to consider the problem (1.2) without the condition (1.3). In this case, it is expected that (1.2) has a dissipative property under some angular condition on λ. We will discuss these problems in a future work.

We now prove existence results and conservation laws for the convenience of readers.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. In Section 3 we prove the nonexistence of the usual scattering states. In Section 4, we give a simple example of a system without an $\mathbf{L}^2$ conservation law and we obtain the logarithmic growth estimate in time of the $\mathbf{L}^2$ - norm of solutions. Finally Section 5 is devoted to the proof of Theorem 3. In Section 6, we study the properties of the solutions of the Cauchy problem (1.9).

2. Proof of Theorem 1

Define the norm

$$\|u\|_{\widetilde{\mathbf{X}}} = \sum_{j=1}^{2} \sup_{t>0} \left((1+t) \|u_j(t)\|_{\mathbf{L}^\infty} + \|u_j(t)\|_{\mathbf{L}^2} + (1+t)^{-\gamma} \left\| \mathcal{J}_{m_j}^2 u_j(t) \right\|_{\mathbf{L}^2} \right),$$

where $\mathcal{J}_{m_j} = x + i\frac{t}{m_j}\nabla_x$ and $\gamma > 0$ is small. Since the global existence and the conservation laws were proved in Proposition 1, we need only to get the estimate for the solution

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon(1+t)^{-1}.$$

We use the following factorization formulas for the free Schrödinger evolution group:

$$\mathcal{U}\Big(\frac{t}{m}\Big) = M^m(t)\mathcal{D}\Big(\frac{t}{m}\Big)\mathcal{V}\Big(\frac{t}{m}\Big)\mathcal{F},$$

and

$$\mathcal{F}\mathcal{U}\Big(-\frac{t}{m}\Big) = \mathcal{V}\Big(-\frac{t}{m}\Big)i\overline{E}^{\frac{1}{m}}(t)\mathcal{D}\Big(\frac{m}{t}\Big).$$

Here we denote $M(t) = e^{\frac{i}{2t}|x|^2}$, $E(t) = e^{\frac{it}{2}|\xi|^2}$, a dilation operator

$$(\mathcal{D}(t)\phi)(x) = \frac{1}{it}\phi(\frac{x}{t}) \ \text{ and } \ \mathcal{V}\Big(\frac{t}{m}\Big) = \mathcal{F}M^m(t)\mathcal{F}^{-1}.$$

Note that $\mathcal{D}(\frac{m}{t})M^m(t) = E^{\frac{1}{m}}(t)\mathcal{D}(\frac{m}{t})$.

As in [11] we change the variables $v(t) = \overline{E}^m(t)\mathcal{D}(\frac{1}{t})u(t)$ and $E = e^{\frac{it}{2}|\xi|^2}$, $\xi = \frac{x}{t}$ to get

$$\big(i\partial_t + \frac{1}{2m}\Delta_x\big)u = t^{-1}E^m\mathcal{L}_m v,$$

where $\mathcal{L}_m = i\partial_t + \frac{1}{2mt^2}\Delta_\xi$.

Thus we change $u_j(t,x) = it^{-1}E^{m_j}v_j(t,\xi)$ in system (1.4) to get

$$\begin{cases} \mathcal{L}_{m_1}v_1 = t^{-1}E^{m_2-2m_1}\overline{v_1}v_2, \\ \mathcal{L}_{m_2}v_2 = t^{-1}E^{2m_1-m_2}v_1^2. \end{cases}$$

Since we suppose that $2m_1 = m_2$, we obtain

$$\begin{cases} \mathcal{L}_{m_1}v_1 = t^{-1}\overline{v_1}v_2, \\ \mathcal{L}_{m_2}v_2 = t^{-1}v_1^2. \end{cases} \tag{2.1}$$

The norm $\|u\|_{\widetilde{\mathbf{X}}}$ is related with the following norm:

$$\|v\|_{\mathbf{X}} = \sum_{j=1}^{2} \sup_{t>0} \left(\|v_j(t)\|_{\mathbf{L}^\infty} + \|v_j(t)\|_{\mathbf{L}^2} + (1+t)^{-\gamma} \|\Delta v_j(t)\|_{\mathbf{L}^2} \right)$$

$$= \sum_{j=1}^{2} \sup_{t>0} \Big(t \|u_j(t)\|_{\mathbf{L}^\infty} + \|u_j(t)\|_{\mathbf{L}^2} + (1+t)^{-\gamma} \left\| \mathcal{J}_{m_j}^2 u_j(t) \right\|_{\mathbf{L}^2} \Big),$$

where $\gamma > 0$ is small. Without loss of generality we can consider the initial time to be at $t = 1$. Let us prove that

$$\|v(t)\|_{\mathbf{L}^\infty} = \sum_{j=1}^{2} \|v_j(t)\|_{\mathbf{L}^\infty} < C\varepsilon$$

for all $t \geq 1$. If this estimate is not true, then by the continuity in time we can find the first time $T > 1$ such that $\|v(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon$ for all $t \in [1, T]$. By system (2.1) we get

$$\begin{aligned} \partial_t |v_1|^2 - \frac{1}{m_1 t^2} \operatorname{Re}\big(i(\Delta v_1)\overline{v_1}\big) &= -2t^{-1} \operatorname{Re}(i\overline{v_1}^2 v_2), \\ \partial_t |v_2|^2 - \frac{1}{m_2 t^2} \operatorname{Re}\big(i(\Delta v_2)\overline{v_2}\big) &= -2t^{-1} \operatorname{Re}(i v_1^2 \overline{v_2}), \end{aligned}$$

then integration by parts yields the $\mathbf{L}^2$ conservation law for $v = (v_1, v_2)$

$$\frac{d}{dt} \|v(t)\|_{\mathbf{L}^2} = 0.$$

We next differentiate equations (2.1) two times. Since by the Sobolev imbedding theorem $\|\nabla\phi\|_{\mathbf{L}^4}^2 \leq C \|\phi\|_{\mathbf{L}^\infty} \|\Delta\phi\|_{\mathbf{L}^2}$, applying the energy method we get

$$\frac{d}{dt} \|\Delta v(t)\|_{\mathbf{L}^2} \leq C t^{-1} \|v\|_{\mathbf{L}^\infty} \|\Delta v\|_{\mathbf{L}^2}.$$

Hence we have the estimate

$$\|\Delta v(t)\|_{\mathbf{L}^2} \leq C\varepsilon (1+t)^\gamma$$

for all $t \in [1, T]$, where $\gamma > 0$ is sufficiently small.

Next to estimate the norm $\|v\|_{\mathbf{L}^\infty}$ we use the operator

$$\mathcal{V}\Big(-\frac{t}{m}\Big)\phi = \mathcal{F}^{-1}\overline{M}^m(t)\mathcal{F}\phi = \frac{it}{2\pi m} \int_{\mathbf{R}^2} e^{\frac{it}{2m}|\xi-\eta|^2} \phi(\eta) d\eta,$$

so that

$$\mathcal{F}\mathcal{U}\Big(-\frac{t}{m}\Big)u(t) = -\mathcal{V}\Big(-\frac{t}{m}\Big)\mathcal{D}(m)\overline{E}^m(t)\mathcal{D}\Big(\frac{1}{t}\Big)u(t) = -\mathcal{D}(m)\mathcal{V}(-tm)v(t),$$

where

$$\mathcal{V}(-tm)\phi = \frac{itm}{2\pi} \int_{\mathbf{R}^2} e^{\frac{itm}{2}|\xi-\eta|^2} \phi(\eta) d\eta.$$

Note that $\mathcal{V}(-tm)\mathcal{L}_m v(t) = i\partial_t \mathcal{V}(-tm)v(t)$. Therefore applying the operators $\mathcal{V}(-tm_j)$ to equations (2.1) we get

$$\begin{cases} \partial_t \varphi_1 = t^{-1}\mathcal{V}(-tm_1)(\overline{v_1}v_2), \\ \partial_t \varphi_2 = t^{-1}\mathcal{V}(-tm_2)(v_1^2), \end{cases} \tag{2.2}$$

where we denote $\varphi_j(t) = i\mathcal{V}(-tm_j)v_j(t)$. Since $v_j(t) = -i\mathcal{V}(tm_j)\varphi_j(t)$, we obtain

$$\mathcal{V}(-tm_1)(\overline{v_1}v_2) = \overline{\varphi_1}\varphi_2 + R_1, \qquad \mathcal{V}(-tm_2)(v_1^2) = -\varphi_1^2 - R_2,$$

where

$$\begin{aligned} R_1 &= \mathcal{V}(-tm_1)(\overline{\mathcal{V}(tm_1)\varphi_1}\mathcal{V}(tm_2)\varphi_2) - \overline{\varphi_1}\varphi_2, \\ R_2 &= \mathcal{V}(-tm_2)(\mathcal{V}(tm_1)\varphi_1)^2 - \varphi_1^2. \end{aligned}$$

Note that $\|\mathcal{V}(-tm)\phi\|_{\mathbf{H}^\alpha} = \|\phi\|_{\mathbf{H}^\alpha}$ and

$$\begin{aligned} \|(\mathcal{V}(tm)-1)\phi\|_{\mathbf{L}^2} = \ & ig\|(M^{\frac{1}{m}}-1)\mathcal{F}\phi\big\|_{\mathbf{L}^2} \le Ct^{-\frac{\alpha}{2}} \left\||x|^\alpha \mathcal{F}\phi\right\|_{\mathbf{L}^2} \\ & \le Ct^{-\frac{\alpha}{2}} \|\phi\|_{\mathbf{H}^\alpha}. \end{aligned}$$

Hence,

$$\|(\mathcal{V}(tm)-1)\phi\|_{\mathbf{L}^\infty} \le \|(\mathcal{V}(tm)-1)\phi\|_{\mathbf{L}^2}^{\frac{1}{2}} \|\Delta(\mathcal{V}(tm)-1)\phi\|_{\mathbf{L}^2}^{\frac{1}{2}} \le Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^2}.$$

Therefore,

$$\|\mathcal{V}(tm)\phi\|_{\mathbf{L}^\infty} \le \|\phi\|_{\mathbf{L}^\infty} + \|(\mathcal{V}(tm)-1)\phi\|_{\mathbf{L}^\infty} \le \|\phi\|_{\mathbf{L}^\infty} + Ct^{-\frac{1}{2}} \|\phi\|_{\mathbf{H}^2}.$$

Thus we get

$$\begin{aligned} & \|R_1\|_{\mathbf{L}^2} \le C \left\|(\mathcal{V}(-tm_1)-1)(\overline{\mathcal{V}(tm_1)\varphi_1}\mathcal{V}(tm_2)\varphi_2)\right\|_{\mathbf{L}^2} \\ & + \left\|\overline{(\mathcal{V}(tm_1)-1)\varphi_1}\mathcal{V}(tm_2)\varphi_2\right\|_{\mathbf{L}^2} + \|\overline{\varphi_1}(\mathcal{V}(tm_2)-1)\varphi_2\|_{\mathbf{L}^2} \\ \le \ & Ct^{-1}\|\varphi_1\|_{\mathbf{H}^2}\|\mathcal{V}(tm_2)\varphi_2\|_{\mathbf{L}^\infty} + Ct^{-1}\|\nabla\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^4}\|\nabla\mathcal{V}(tm_2)\varphi_2\|_{\mathbf{L}^4} \\ & + Ct^{-1}\|\varphi_2\|_{\mathbf{H}^2}\|\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^\infty} + Ct^{-1}\|\varphi_1\|_{\mathbf{L}^\infty}\|\varphi_2\|_{\mathbf{H}^2} \\ \le \ & Ct^{\gamma-1}\|v_1\|_{\mathbf{X}}\|v_2\|_{\mathbf{X}} \end{aligned}$$

since by the Sobolev imbedding theorem we get $\|\nabla\phi\|_{\mathbf{L}^4} \le C\|\phi\|_{\mathbf{L}^\infty}^{\frac{1}{2}}\|\Delta\phi\|_{\mathbf{L}^2}^{\frac{1}{2}}$. In the same manner

$$\begin{aligned} & \|R_2\|_{\mathbf{L}^2} \le C \left\|(\mathcal{V}(-tm_2)-1)(\mathcal{V}(tm_1)\varphi_1)^2\right\|_{\mathbf{L}^2} \\ & + \left\|\overline{(\mathcal{V}(tm_1)-1)\varphi_1}\mathcal{V}(tm_1)\varphi_1\right\|_{\mathbf{L}^2} + \|\varphi_1(\mathcal{V}(tm_1)-1)\varphi_1\|_{\mathbf{L}^2} \\ \le \ & Ct^{-1}\|\varphi_1\|_{\mathbf{H}^2}\|\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^\infty} + Ct^{-1}\|\nabla\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^4}\|\nabla\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^4} \end{aligned}$$

$$+Ct^{-1}\|\varphi_1\|_{\mathbf{L}^\infty}\|\varphi_1\|_{\mathbf{H}^2} \le Ct^{\gamma-1}\|v_1\|^2_{\mathbf{X}}.$$

We also find

$$\begin{aligned}
&\|\Delta R_1\|_{\mathbf{L}^2} \le \left\|\Delta(\overline{\mathcal{V}(tm_1)\varphi_1}\mathcal{V}(tm_2)\varphi_2)\right\|_{\mathbf{L}^2} + \|\Delta(\overline{\varphi_1}\varphi_2)\|_{\mathbf{L}^2}\\
&\le C\|\varphi_1\|_{\mathbf{H}^2}\|\mathcal{V}(tm_2)\varphi_2\|_{\mathbf{L}^\infty} + C\|\nabla\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^4}\|\nabla\mathcal{V}(tm_2)\varphi_2\|_{\mathbf{L}^4}\\
&+ C\|\varphi_2\|_{\mathbf{H}^2}\|\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^\infty} + C\|\varphi_1\|_{\mathbf{H}^2}\|\varphi_2\|_{\mathbf{L}^\infty} + C\|\nabla\varphi_1\|_{\mathbf{L}^4}\|\nabla\varphi_2\|_{\mathbf{L}^4}\\
&+ C\|\varphi_2\|_{\mathbf{H}^2}\|\varphi_1\|_{\mathbf{L}^\infty} \le Ct^{\gamma}\|v\|^2_{\mathbf{X}}
\end{aligned}$$

and

$$\begin{aligned}
&\|\Delta R_2\|_{\mathbf{L}^2} \le \left\|\Delta(\mathcal{V}(tm_1)\varphi_1)^2\right\|_{\mathbf{L}^2} + \left\|\Delta(\varphi_1^2)\right\|_{\mathbf{L}^2}\\
\le\ & C\|\varphi_1\|_{\mathbf{H}^2}\|\mathcal{V}(tm_1)\varphi_1\|_{\mathbf{L}^\infty} + C\|\nabla\mathcal{V}(tm_1)\varphi_1\|^2_{\mathbf{L}^4}\\
&+C\|\varphi_1\|_{\mathbf{H}^2}\|\varphi_1\|_{\mathbf{L}^\infty} + C\|\nabla\varphi_1\|^2_{\mathbf{L}^4} \le Ct^{\gamma}\|v\|^2_{\mathbf{X}}.
\end{aligned}$$

Hence by the Sobolev imbedding theorem we obtain

$$\|R_j\|_{\mathbf{L}^\infty} \le C\|R_j\|^{\frac{1}{2}}_{\mathbf{L}^2}\|\Delta R_j\|^{\frac{1}{2}}_{\mathbf{L}^2} \le Ct^{\gamma-\frac{1}{2}}\|v\|_{\mathbf{X}}\|\Delta v\|_{\mathbf{L}^2}$$

Thus from (2.2) we get the system

$$\begin{cases}\partial_t\varphi_1 = t^{-1}\overline{\varphi_1}\varphi_2 + O(t^{\gamma-\frac{3}{2}}\|v\|_{\mathbf{X}}\|\Delta v\|_{\mathbf{L}^2}),\\ \partial_t\varphi_2 = -t^{-1}\varphi_1^2 + O(t^{\gamma-\frac{3}{2}}\|v\|_{\mathbf{X}}\|\Delta v\|_{\mathbf{L}^2}).\end{cases} \tag{2.3}$$

We now multiply the equations of system (2.3) by $\overline{\varphi_1}$ and $\overline{\varphi_2}$ to get

$$\begin{aligned}
\partial_t(|\varphi_1|^2) &= t^{-1}\operatorname{Re}(\overline{\varphi_1}^2\varphi_2) + O(t^{\gamma-\frac{3}{2}}\|v\|^3_{\mathbf{X}}),\\
\partial_t(|\varphi_2|^2) &= -t^{-1}\operatorname{Re}(\varphi_1^2\overline{\varphi_2}) + O(t^{\gamma-\frac{3}{2}}\|v\|^3_{\mathbf{X}}).
\end{aligned}$$

Hence

$$\partial_t(|\varphi_1|^2 + |\varphi_2|^2) = O(t^{\gamma-\frac{3}{2}}\|v\|^3_{\mathbf{X}}).$$

Integration with respect to t yields $|\varphi_1(t,\xi)|^2 + |\varphi_2(t,\xi)|^2 \le C\varepsilon^2$ for all $t \in [1,T]$ and $\xi \in \mathbf{R}^2$. Hence we get $\|v(t)\|_{\mathbf{L}^\infty} < C\varepsilon$ for all $t \in [1,T]$. This contradicts our supposition that $\|v(t)\|_{\mathbf{L}^\infty} = C\varepsilon$. Hence $\|v(t)\|_{\mathbf{L}^\infty} < C\varepsilon$ for all $t > 1$. Now by the relations $u_j(t,x) = it^{-1}E^{m_j}v_j(t,\xi)$ and Proposition 1 we find the estimate $\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon(1+t)^{-1}$ for all $t \ge 0$. This completes the proof of Theorem 1.

3. NONEXISTENCE OF THE USUAL SCATTERING STATES

To the contrary we assume that there exist nontrivial final states (u_{1+},u_{2+}) in $(\mathbf{H}^2 \cap \mathbf{H}^{0,2})^2$ such that $u_{1+} \neq 0$ and

$$\sum_{j=1}^{2} \left\| \mathcal{U}(-\frac{t}{m_j})u_j - u_{j+} \right\|_{\mathbf{L}^2} \to 0$$

as $t \to \infty$. Let $(u_1, u_2) \in \mathbf{C}\big([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2}\big)^2$ be a global solution obtained in Theorem 1 which satisfies the time decay estimate

$$\|u_1(t)\|_{\mathbf{L}^\infty} \leq C(1+t)^{-1}.$$

Multiplying the second equation of system (1.4) by $\mathcal{D}(\frac{1}{m_2})\mathcal{F}\mathcal{U}(-\frac{t}{m_2})$ and integrating the result with respect to time we get

$$\mathcal{D}(\frac{1}{m_2})\mathcal{F}\big(\mathcal{U}(-\frac{t}{m_2})u_2 - \mathcal{U}(-\frac{s}{m_2})u_2\big) = -i\int_s^t \mathcal{D}(\frac{1}{m_2})\mathcal{F}\mathcal{U}(-\frac{\tau}{m2})(u_1^2)d\tau. \tag{3.1}$$

We decompose the nonlinear term as follows:

$$\mathcal{U}(-\frac{t}{m_2})u_1^2 = \mathcal{U}(-\frac{t}{m_2})(u_1^2 - (\mathcal{U}(\frac{t}{m_1})u_{1+})^2) + \mathcal{U}(-\frac{t}{m_2})(\mathcal{U}(\frac{t}{m_1})u_{1+})^2.$$

By the factorization formulas for the free Schrödinger evolution group

$$\mathcal{U}(\frac{t}{m}) = M^m(t)\mathcal{D}(\frac{t}{m})\mathcal{V}(\frac{t}{m})\mathcal{F}$$

and

$$\mathcal{F}\mathcal{U}(-\frac{t}{m}) = -\mathcal{V}(-\frac{t}{m})\mathcal{D}(m)\overline{E}^m(t)\mathcal{D}(\frac{1}{t}),$$

since $\mathcal{D}(\frac{1}{m})\mathcal{V}(-\frac{t}{m})\mathcal{D}(m) = -\mathcal{V}(-tm)$, we find

$$\begin{aligned}
&\mathcal{D}(\frac{1}{m_2})\mathcal{F}\mathcal{U}(-\frac{t}{m_2})(\mathcal{U}(\frac{t}{m_1})u_{1+})^2 \\
= \;& -\mathcal{D}(\frac{1}{m_2})\mathcal{V}(-\frac{t}{m_2})\mathcal{D}(m_2)\overline{E}^{m_2}(t)\mathcal{D}(\frac{1}{t})(M^{m_1}(t)\mathcal{D}(\frac{t}{m_1})\mathcal{V}(\frac{t}{m_1})\widehat{u_{1+}})^2 \\
= \;& it^{-1}\mathcal{D}(\frac{1}{m_2})\mathcal{V}(-\frac{t}{m_2})\mathcal{D}(m_2)(\mathcal{D}(\frac{1}{m_1})\mathcal{V}(\frac{t}{m_1})\widehat{u_{1+}})^2 \\
= \;& it^{-1}\mathcal{V}(-tm_2)(\mathcal{D}(\frac{1}{m_1})\mathcal{V}(\frac{t}{m_1})\widehat{u_{1+}})^2 = it^{-1}(\mathcal{D}(\frac{1}{m_1})\widehat{u_{1+}})^2 + R_3
\end{aligned}$$

since we assume that $2m_1 = m_2$, where

$$R_3 = it^{-1}\mathcal{V}(-tm_2)(\mathcal{D}(\frac{1}{m_1})\mathcal{V}(\frac{t}{m_1})\widehat{u_{1+}})^2 - it^{-1}(\mathcal{D}(\frac{1}{m_1})\widehat{u_{1+}})^2.$$

As in the proof of Theorem 1 we find

$$\|R_3\|_{\mathbf{L}^2} \le Ct^{-2} \|u_{1+}\|^2_{\mathbf{H}^{0,2}}. \tag{3.2}$$

We have by (3.1) and (3.2)

$$\left\|\mathcal{U}(-\frac{t}{m_2})u_2 - \mathcal{U}(-\frac{s}{m_2})u_2\right\|_{\mathbf{L}^2} \ge \left\|(\mathcal{D}(\frac{1}{m_1})\widehat{u_{1+}})^2\right\|_{\mathbf{L}^2}\int_s^t \tau^{-1}d\tau \tag{3.3}$$
$$-C\int_s^t \left\|u_1^2 - (\mathcal{U}(\frac{t}{m_1})u_{1+})^2\right\|_{\mathbf{L}^2} d\tau - C(\|u_{1+}\|_{\mathbf{H}^{0,2}} + \|u_{2+}\|_{\mathbf{H}^{0,2}})^2 \int_s^t \tau^{-2}d\tau.$$

By the Cauchy-Schwarz inequality we get

$$\begin{aligned}&\left\|u_1^2 - (\mathcal{U}(\frac{t}{m_1})u_{1+})^2\right\|_{\mathbf{L}^2}\\ \le\ & Ct^{-1}(\|u_{1+}\|_{\mathbf{L}^1} + t\|u_1\|_{\mathbf{L}^\infty})\left\|u_1 - \mathcal{U}(\frac{t}{m_1})u_{1+}\right\|_{\mathbf{L}^2} \le C\delta t^{-1}.\end{aligned}$$

Here we can choose $\delta > 0$ such that $\left\|\mathcal{D}(\frac{1}{m_1})\widehat{u_{1+}}\right\|^2_{\mathbf{L}^4} - C\delta > 0$. Therefore by (3.3) we obtain

$$\left\|\mathcal{U}(-\frac{t}{m_2})u_2 - \mathcal{U}(-\frac{s}{m_2})u_2\right\|_{\mathbf{L}^2} \ge \left(\left\|\mathcal{D}(\frac{1}{m_1})\widehat{u_{1+}}\right\|^2_{\mathbf{L}^4} - C\delta\right)\int_s^t \tau^{-1}d\tau \to \infty,$$

as $t \to \infty$. This contradicts our assumption

$$\left\|u_2(t) - \mathcal{U}(\frac{t}{m_2})u_{2+}\right\|_{\mathbf{L}^2} \to 0$$

as $t \to \infty$. Theorem 2 is proved.

4. A system without $\mathbf{L}^2$ - conservation

In this section we show that the $\mathbf{L}^2$ - conservation law is important for obtaining the time decay estimates of solutions. Let us consider the system

$$\begin{cases} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = 0, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2, \end{cases}$$

with the initial conditions

$$u_1(0) = \phi_1, u_2(0) = \phi_2. \tag{4.1}$$

Since the first equation of this system can be solved explicitly $u_1 = \mathcal{U}(\frac{t}{m_1})\phi_1$, we get the following Cauchy problem for u_2 :

$$\begin{cases} i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = (\mathcal{U}(\frac{t}{m_1})\phi_1)^2, \\ u_2(0) = \phi_2. \end{cases} \tag{4.2}$$

Equation (4.2) is the linear Schrödinger equation and the solution can be represented explicitly by (ϕ_1, ϕ_2). More precisely, we have the following.

Proposition 2. *Suppose that* $\phi_1 \in \mathbf{H}^{0,2}$, $\phi_2 \in \mathbf{L}^2$. *Let* $u_2 \in \mathbf{C}([0,\infty);\mathbf{L}^2)$ *be a global solution of the Cauchy problem* (4.2). *Then the estimate*

$$\|u_2(t)\|_{\mathbf{L}^2} \geq \left\|\widehat{\phi_1}\right\|_{\mathbf{L}^4}^2 \log t - C\,\|\phi_1\|_{\mathbf{H}^{0,2}}^2 \qquad \text{for } t > 1$$

is true.

This fact was pointed out first in papers [20] and [21] in the case of the Klein-Gordon equation and in remark 3 of [13] in the case of the Schrödinger equation. Therefore our result Proposition 2 is not new. However we give a short proof for the convenience of the readers.

Proof. Using the factorization properties of the free Schrödinger evolution group we obtain

$$\mathcal{D}(\frac{1}{m_2})\mathcal{F}\mathcal{U}(-\frac{t}{m_2})(\mathcal{U}(\frac{t}{m_1})\phi_1)^2 = it^{-1}(\mathcal{D}(\frac{1}{m_1})\widehat{\phi_1})^2 + R_3,$$

where as in the previous section

$$R_3 = it^{-1}\mathcal{V}(-tm_2)(\mathcal{D}(\frac{1}{m_1})\mathcal{V}(\frac{t}{m_1})\widehat{\phi_1})^2 - it^{-1}(\mathcal{D}(\frac{1}{m_1})\widehat{\phi_1})^2$$

can be easily estimated:

$$\|R_3\|_{\mathbf{L}^2} \leq Ct^{-2}\,\|\phi_1\|_{\mathbf{H}^{0,2}}^2\,.$$

Multiplying both sides of equation (4.2) by $\mathcal{F}\mathcal{U}(-\frac{t}{m_2})$ we get

$$i\partial_t(\mathcal{D}(\frac{1}{m_2})\mathcal{F}\mathcal{U}(-\frac{t}{m_2})u_2) = it^{-1}(\mathcal{D}(\frac{1}{m_1})\widehat{\phi_1})^2 + R_3.$$

Integration with respect to t yields

$$\begin{aligned}&\mathcal{D}(\frac{1}{m_2})\mathcal{F}\mathcal{U}(-\frac{t}{m_2})u_2(t)\\ &= \mathcal{D}(\frac{1}{m_2})\mathcal{F}\mathcal{U}(-\frac{1}{m_2})u_2(1) - m_1^2\widehat{\phi_1}^2(m_1x)\log t + \int_1^t R_3 d\tau.\end{aligned}$$

Since $\|u_2(1)\|_{\mathbf{L}^2} \leq C(\|\phi_2\|_{\mathbf{L}^2} + \|\phi_1\|_{\mathbf{H}^{0,2}}^2)$ and

$$\int_1^t \|R_3\|_{\mathbf{L}^2}\,d\tau \leq C\,\|\phi_1\|_{\mathbf{H}^{0,2}}^2\,,$$

we find the desired estimate

$$\|u_2(t)\|_{\mathbf{L}^2} \geq \left\|\widehat{\phi_1}\right\|_{\mathbf{L}^4}^2 \log t - C\,\|\phi_1\|_{\mathbf{H}^{0,2}}^2\,.$$

Theorem 2 is proved. □

5. PROOF OF THEOREM 3

By the conditions of Theorem 3 we can assume that $\|v\|_{\mathbf{X}} \le \varepsilon$, and $\|\Delta v\|_{\mathbf{L}^2} \le \varepsilon$ so system (2.3) has the form

$$\begin{cases} \frac{d}{dt}\varphi_1 = t^{-1}\overline{\varphi_1}\varphi_2 + O(\varepsilon^2 t^{-1-\beta}), t > 1, \\ \frac{d}{dt}\varphi_2 = -t^{-1}\varphi_1^2 + O(\varepsilon^2 t^{-1-\beta}), t > 1, \\ \varphi_1(1) = \Phi_1, \varphi_2(1) = \Phi_2, \end{cases} \tag{5.1}$$

where $\beta = \frac{1}{2} - \gamma$, $\gamma \in (0, \frac{1}{2})$. We now compare the asymptotic behavior of solutions of the Cauchy problem (5.1) with the large-time asymptotics of solutions of the unperturbed system of ordinary differential equations depending on $\xi \in \mathbf{R}^2$ as a parameter

$$\begin{cases} \frac{d}{dt}\psi_1 = t^{-1}\overline{\psi_1}\psi_2, t > 1, \\ \frac{d}{dt}\psi_2 = -t^{-1}\psi_1^2, t > 1, \\ \psi_1(1) = \Psi_1, \psi_2(1) = \Psi_2. \end{cases} \tag{5.2}$$

The unperturbed system (5.2) is studied below in Section 6.

In this section we prove the following result.

Lemma 1. *Let the initial data for system* (5.1) *be such that*

$$\sum_{j=1}^{2} \|\varphi_j(1)\|_{\mathbf{L}^\infty} \le \varepsilon,$$

where $\varepsilon > 0$ *is small enough. Then there exist initial data* $\Psi_1(\xi)$ *and* $\Psi_2(\xi)$ *for the Cauchy problem* (5.2) *such that the asymptotics*

$$\varphi_j(t) = \psi_j(t) + O(\varepsilon t^{-\beta}),\ j = 1, 2$$

is true for large time $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^2$.

Proof. Let $(\varphi_1(t), \varphi_2(t))$ be a given solution of system (5.1). Denote $\psi_j^{(0)}(t) \equiv \varphi_j(t)$ and define $\psi_j^{(n)}(t)$ for $n \ge 1$ as a solution of the linearized version of (5.2)

$$\begin{cases} \frac{d}{dt}\psi_1^{(n)} = t^{-1}\overline{\psi_1^{(n-1)}}\psi_2^{(n-1)},\ t > 1, \\ \frac{d}{dt}\psi_2^{(n)} = -t^{-1}(\psi_1^{(n-1)})^2,\ t > 1, \end{cases} \tag{5.3}$$

with a final state condition

$$\lim_{t\to\infty} \left\|\varphi_j(t) - \psi_j^{(n)}(t)\right\|_{\mathbf{L}^\infty} = 0.$$

Let us prove by induction that

$$\sup_{t\geq 1} t^{\beta} \sum_{j=1}^{2} \left\| \varphi_j(t) - \psi_j^{(n)}(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 \tag{5.4}$$

for all $n \geq 0$. For $n = 0$ estimates (5.4) are true. Next by induction we assume that (5.4) is fulfilled for all $0 \leq n \leq k$ and consider (5.3) for $n = k+1$. We already know by Theorem 1 that

$$\sup_{t\geq 1} \sum_{j=1}^{2} \left\| \varphi_j(t) \right\|_{\mathbf{L}^\infty} \leq \varepsilon.$$

Therefore also we have $\left\| \psi_j^{(n)}(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon$ for all $0 \leq n \leq k$. By (5.1) and (5.3) we find for $n = k+1$

$$\begin{aligned} &\sum_{j=1}^{2} \left\| \varphi_j(t) - \psi_j^{(n)}(t) \right\|_{\mathbf{L}^\infty} \\ &\leq C\varepsilon \sum_{j=1}^{2} \int_t^\infty \left\| \varphi_j(\tau) - \psi_j^{(n-1)}(\tau) \right\|_{\mathbf{L}^\infty} \frac{d\tau}{\tau} + O(\varepsilon^2 t^{-\beta}) \leq C\varepsilon^2 t^{-\beta} \end{aligned}$$

from which estimates (5.4) with $n = k+1$ follows. Thus by induction estimate (5.4) is valid for all $n \geq 0$. In the same way as above we find the estimate

$$\sup_{t\geq 1} t^{\beta} \sum_{j=1}^{2} \left\| \psi_j^{(n)}(t) - \psi_j^{(n-1)}(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon^2$$

which implies by the usual contraction mapping principle that there exists a unique solution $(\psi_1(t), \psi_2(t)) \in (\mathbf{C}^1((1,\infty), \mathbf{L}^\infty) \cap \mathbf{C}([1,\infty); \mathbf{L}^\infty))^2$ of the system

$$\begin{cases} \frac{d}{dt}\psi_1 = t^{-1}\overline{\psi_1}\psi_2, t > 1, \\ \frac{d}{dt}\psi_2 = -t^{-1}(\psi_1)^2, t > 1 \end{cases}$$

satisfying

$$\sup_{t\geq 1} t^{\beta} \sum_{j=1}^{2} \left\| \varphi_j(t) - \psi_j(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon^2.$$

Lemma 1 is proved. □

Application of Lemma 1 to system (2.3) yields the result of Theorem 3.

6. Appendix

We consider the following Cauchy problem:

$$\begin{cases} \frac{d}{dt}\psi_1 = t^{-1}\overline{\psi_1}\psi_2, t > 1, \\ \frac{d}{dt}\psi_2 = -t^{-1}\psi_1^2, t > 1, \\ \psi_1(1) = \Psi_1, \psi_2(1) = \Psi_2. \end{cases} \tag{6.1}$$

We make the change $t = e^{t'}$ to exclude the explicit dependence on t in system (6.1) (the prime we then omit). Then we substitute $\psi_1(t) = r(t)e^{i\phi(t)}$ and $\psi_2(t) = \rho(t)e^{i\theta(t)}$, to get from (6.1)

$$\begin{cases} r' + ir\phi' = r\rho e^{i(\theta-2\phi)}, t > 0, \\ \rho' + i\rho\theta' = -r^2 e^{i(2\phi-\theta)}, t > 0. \end{cases}$$

We also denote $\alpha = \theta - 2\phi$, then

$$\begin{cases} r' = r\rho\cos\alpha, \\ \rho' = -r^2\cos\alpha, \\ \alpha' = \frac{r^2-2\rho^2}{\rho}\sin\alpha, \\ \phi' = \rho\sin\alpha. \end{cases} \tag{6.2}$$

By the first two equations of system (6.2) we have $\frac{d}{dt}(r^2 + \rho^2) = 0$, therefore

$$r^2(t) + \rho^2(t) = |\Psi_1|^2 + |\Psi_2|^2$$

for all $t > 0$. Then by the first three equations of system (6.2) we find $\frac{d}{dt}(\rho r^2 \sin\alpha) = 0$. Therefore

$$\rho(t)r^2(t)\sin\alpha(t) = |\Psi_2|\,|\Psi_1|^2 \sin(\arg\Psi_2 - 2\arg\Psi_1)$$

for all $t > 0$. Denote $z = r^2 - 2\rho^2 = 3r^2 - 2b$, $b = |\Psi_1|^2 + |\Psi_2|^2$. Then by system (6.1) we see that $z' = 4\rho r^2\cos\alpha$ and $z'' = -4r^2 z = -\frac{4}{3}(z + 2b)z$. We can exclude the constant b from the equation if we make a change $z(t) = b\widetilde{z}(\widetilde{t})$, $\sqrt{b}t = \widetilde{t}$. Hence we get (the tilde we then omit)

$$\begin{cases} \frac{d^2z}{dt^2} = -\frac{4}{3}(z+2)z, \\ z(0) = r^2(0) - 2\rho^2(0), \\ z'(0) = 4\rho(0)r^2(0)\cos\alpha(0). \end{cases} \tag{6.3}$$

Note that $-2 \le z(t) \le 1$. Multiplying (6.3) by $\frac{dz}{dt}$ and integrating we get

$$(\frac{dz}{dt})^2 + \frac{8}{3}z^2 + \frac{8}{9}z^3 = C.$$

Since $\rho = \sqrt{\frac{1-z}{3}}$, $r^2 = \frac{z+2}{3}$, and $z' = 4\rho r^2 \cos\alpha$, we have $C = \frac{16}{27}(1-z)(z+2)^2\cos^2\alpha + \frac{8}{3}z^2 + \frac{8}{9}z^3$. Therefore we can see that $0 \le C \le \frac{32}{9}$ for $-2 \le z \le 1$. Denote the function $f(z) = C - \frac{8}{3}z^2 - \frac{8}{9}z^3$ for $-2 \le z \le 1$. Then we have the equation

$$(\frac{dz}{dt})^2 = f(z).$$

Consider the function $f(z) = C - \frac{8}{3}z^2 - \frac{8}{9}z^3$ for $-2 \le z \le 1$. It has the local maximum $f(z) = C$ at $z = 0$ and the local minimum $f(z) = C - \frac{32}{9}$ at $z = -2$. Therefore, when $0 < C < \frac{32}{9}$, then there are three roots $z_1 < -2 < z_2 < 0 < z_3 < 1$ of the equation $f(z) = 0$ such that $f(z) \ge 0$ for all $z_2 \le z \le z_3$. Note that the initial data satisfy the inequality $z_2 \le z(0) \le z_3$ since $f(z(0)) = (\frac{dz}{dt})^2 \ge 0$. Therefore the solution $z(t)$ is always periodic with a period $T = \int_{z_2}^{z_3} \frac{dz}{\sqrt{f(z)}}$ and $z_2 \le z(t) \le z_3$. For the exceptional cases $C = 0$ and $C = \frac{32}{9}$ there are two equilibrium points $z = -2$ for $C = 0$ (i.e., $\psi_1 = \Psi_1 = 0$, $\psi_2 = \Psi_2$ in system (6.1)) and $z = 0$ for $C = \frac{32}{9}$ (i.e., $\psi_1 = \Psi_1 e^{\frac{i}{\sqrt{2}}|\Psi_1|\log(1+t)}$, $\psi_2 = \frac{i}{\sqrt{2}|\Psi_1|}\Psi_1^2 e^{\frac{2i}{\sqrt{2}}|\Psi_1|\log(1+t)}$ in system (6.1)).

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