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ASYMPTOTIC BEHAVIOR OF BOUNDED MILD SOLUTIONS OF SOME FUNCTIONAL DIFFERENTIAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study further properties of bounded continuous functions which behave like almost automorphic functions at infinity. Then we establish the existence and uniqueness of solutions of some semilinear functional differential and fractional differential equations in Banach spaces.

1. Introduction

The study of the existence of pseudo periodic, pseudo almost periodic and pseudo almost automorphic solutions is one of the most attractive topics in differential equations [16, 17, 10, 11, 20, 28, 29], and recently in fractional differential equations [1, 20, 21]. This is due to the applications in physics, control theory, mathematical biology, and other areas of science.

The purpose of this paper is two-fold. First, we study some properties of almost automorphic functions including the superposition operator $f(t,\varphi(t))$ where $\varphi \in \mathcal{B}$ is a semilinear vector space of functions defined axiomatically, then we use the results to study the existence of asymptotically almost automorphic mild solutions to the functional differential equation with infinite delay

$$\begin{cases} u'(t) &= Au(t) + F(t,u_t), \quad t \geq 0, \\ u_0 &= \phi, \quad t \in (-\infty,0]. \end{cases} \tag{1.1}$$

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This study is motivated by [25, 14] and some recent works [1, 4, 20].

Secondly, we focus on the existence of pseudo periodic, pseudo almost peridic and pseudo almost automorphic mild solutions to the fractional differential equation

$$D_t^\alpha x(t) = Ax(t) + f(t), \quad t \in \mathbb{R} \tag{1.2}$$

and its perturbation

$$D_t^\alpha x(t) = Ax(t) + f(t) + g(t, x(t)), \quad t \in \mathbb{R}, \tag{1.3}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of an α-resolvent family $(S_\alpha(t))_{t\geq 0}$ defined on $\mathbb{X}$, a complex Banach space, $1 \leq \alpha \leq 2$ and D_t^α is the Riemann-Liouville fractional derivative; i.e.,

$$D_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-\tau)^{1-\alpha} f(\tau) d\tau,$$

where Γ is the gamma function. See for instance [2], or [26] page 68.

We use here a definition of mild solution due to C. Lizama [1] (see also [20]). This study completes [6] in the case of pseudo periodic, pseudo almost periodic and pseudo almost automorphic solutions which are more complicated to handle in the context of fractional differential equations. The operator-theoretical approach of Section 4 can be used in many other studies of the existence of wider classes of bounded continuous solutions of various differential and fractional differential equations.

2. Preliminaries

2.1. **Almost Automorphic Functions.** In this paper, $\mathbb{X}$ is a Banach space with norm $\|\cdot\|$. $C_0(\mathbb{R}^+; \mathbb{X})$ will denote the space of all continuous functions $f : \mathbb{R}^+ \to \mathbb{X}$ that vanish at infinity, i.e., such that $\lim_{t\to\infty} \|f(t)\| = 0$. $C_0(\mathbb{R}^+ \times \mathbb{X}; \mathbb{X})$ will represent the class of all continuous functions $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ such that $\lim_{t\to\infty} \|f(t, x)\| = 0$ uniformly for x in any compact subset of $\mathbb{X}$. We denote by $\big((\mathcal{B}(\mathbb{X}), \|.\|_{\mathcal{B}(\mathbb{X})}\big)$ the Banach space of all linear and bounded operators on $\mathbb{X}$.

Definition 2.1. *A continuous function $f : \mathbb{R} \to \mathbb{X}$ is called almost automorphic if, for each real sequence (s_m), there exists a subsequence (s_n) such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. *Denote by* $AA(\mathbb{X})$ *the set of all such functions.*

If the convergence above is uniform in $t \in \mathbb{R}$, then f is an almost periodic function in the sense of Bohr. Thus almost automorphic functions in Bochner's sense constitute a larger class of functions than almost periodic ones. The function $\sin \frac{1}{2+\cos t+\cos \sqrt{2}t}$ is almost automorphic, but not almost periodic.

Note that the function g in the definition above is measurable, but not necessarily continuous. Moreover, $\overline{\{g(t) : t \in \mathbb{R}\}} = \overline{\{f(t) : t \in \mathbb{R}\}}$. The part $\overline{\{g(t) : t \in \mathbb{R}\}} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$ was proved in [23]. The reverse inclusion can be proved similarly.

It is well known that an almost automorphic function is bounded in norm and $AA(\mathbb{X})$ is a Banach space with the sup norm.

Definition 2.2 ([13])**.** *A continuous function* $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ *is called almost automorphic in* t *uniformly for* x *in compact subsets of* $\mathbb{X}$ *if, for every compact subset* K *of* $\mathbb{X}$ *and every real sequence* (s_m)*, there exists a subsequence* (s_n) *such that*

$$g(t,x) = \lim_{n\to\infty} f(t+s_n, x)$$

is well defined for each $t \in \mathbb{R}$, $x \in K$ *and*

$$\lim_{n\to\infty} g(t-s_n, x) = f(t,x)$$

for each $t \in \mathbb{R}$, $x \in K$. *Denote by* $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ *the set of all such functions.*

For more details about almost automorphic functions, see [22, 23].

In what follows $C_0(\mathbb{R}^+,\mathbb{X})$ will denote the space of all continuous functions $h : \mathbb{R}^+ \to \mathbb{X}$ such that $\lim_{t\to\infty} h(t) = 0$ and $C_0(\mathbb{R}^+\times,\mathbb{X})$ will be the space of all continuous functions $h : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ such that $\lim_{t\to\infty} h(t,x) = 0$ uniformly for x in any compact subset of $\mathbb{X}$.

The following definition is slightly different from [22, Definition 2.2.1].

Definition 2.3. *A continuous function* $f : \mathbb{R}^+ \to \mathbb{X}$ $(\mathbb{R}^+ \times \mathbb{X} \to \mathbb{X})$ *is called asymptotically almost automorphic if there exist two functions* $\phi \in AA(\mathbb{X})$ $(AA(\mathbb{R}\times\mathbb{X},\mathbb{X}))$ *and* $h \in C_0(\mathbb{R}^+;\mathbb{X})(C_0(\mathbb{R}^+\times\mathbb{X};\mathbb{X})$ *such that*

$$f(t) = \phi(t) + h(t), \quad t \in \mathbb{R}^+.$$

ϕ *and* h *are called the principal and the corrective terms of* f *respectively.*

Denote by $AAA(\mathbb{X})$ $(AAA(\mathbb{R}^+ \times \mathbb{X},\mathbb{X}))$ *the space of of all asymptotically almost automorphic functions* $f : \mathbb{R}^+ \to \mathbb{X}$ $(f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X})$.

The space $AAA(\mathbb{X})$ of all asymptotically almost automorphic functions $f : \mathbb{R}^+ \to \mathbb{X}$ is a Banach space under the norm

$$|f| = \sup_{t\in\mathbb{R}} \|g(t)\| + \sup_{t\geq 0} \|h(t)\|.$$

Moreover $AAA(\mathbb{X}) = AA(\mathbb{X}) \oplus C_0(\mathbb{R}^+; \mathbb{X})$.

Remark 2.4. Note that $AAA(\mathbb{X})$ can also be equipped with the equivalent norm $\|f\| := \sup_{t\in\mathbb{R}^+} \|f(t)\|$; (cf. Lemma 1.8 [13]). Moreover, the range of any asymptotically almost automorphic function is relatively compact (cf. Lemma 1.9 [13]).

Remark 2.5. If $f \in AAA(\mathbb{X})$ with $f = g + h$, then $\overline{\{g(t) : t \in \mathbb{R}\}} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$ (Lemma 1.7 [13]).

Now denote by $P_\omega(\mathbb{X})$ the space of all ω-periodic functions $f : \mathbb{R} \to \mathbb{X}$ and $AP(\mathbb{X})$ the space of all almost periodic functions $f : \mathbb{R} \to \mathbb{X}$.

Definition 2.6. *A continuous function $x : \mathbb{R} \to \mathbb{X}$ is said to be pseudo periodic* (*respectively pseudo almost periodic, respectively pseudo almost automorphic*) *if x can be decomposed as*

$$x = x_1 + x_2$$

where

$x_1 \in P_\omega(\mathbb{X})$, *(respectively $x_1 \in AP(\mathbb{X})$, respectively $x_1 \in AA(\mathbb{X})$) and*

$$x_2 \in AA_0(\mathbb{X}) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(\sigma)\| d\sigma = 0 \right\}.$$

The spaces of these functions are denoted respectively $PP_\omega(\mathbb{X})$, $PAP(\mathbb{X})$, $PAA(\mathbb{X})$. They turn out to be Banach spaces under the sup norm. We have the inclusions

$$PP_\omega(\mathbb{X}) \subset PAP(\mathbb{X}) \subset PAA(\mathbb{X}).$$

The concept of pseudo almost automorphic functions was suggested by N'Guérékata [23] page 40 and developed by Liang et al. in [16, 17] and generalized to weighted pseudo almost automorphic functions in [3] (see also [18] for a recent study).

2.2. **α-resolvent family.**

Definition 2.7. *A closed linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is said to be the generator of an α-family with $\alpha > 0$ if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(\mathbb{X})$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad Re\lambda > \omega, \quad x \in \mathbb{X}.$$

In this case $S_\alpha(t)$ is called the α-family generated by A.

We recall the following from [1].

Proposition 2.8. *Let $1 \le \alpha \le 2$ and $S_\alpha(t)$ be an α-family on $\mathbb{X}$ generated by A. Then the following hold.*

- *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \ge 0$;*
- *If $x \in D(A)$ and $t \ge 0$, then*
$$S_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-\sigma)AS_\alpha(\sigma)x d\sigma.$$
In particular $\frac{d}{dt}S_\alpha(t)x$ exists.
- *If $x \in D(A)$ and $t \ge 0$, then $\int_0^t g_\alpha(t-\sigma)S_\alpha(\sigma)x d\sigma \in D(A)$ and*
$$S_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-\sigma)S_\alpha(\sigma)x d\sigma.$$
In particular, $S_\alpha(0) = g_\alpha(0)$.

Here $g_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $g_\alpha(t) = 0$ for $t \le 0$, and $\lim\limits_{\alpha\to 0} g_\alpha(t) = \delta(t)$, where δ is the delta function.

For more details about α-resolvent families see for instance [1].

Definition 2.9. ([27]) *A strongly measurable family of operators $\{T(t)\}_{t\ge 0} \subset \mathcal{B}(\mathbb{X})$ is called uniformly integrable if $\int_0^\infty \|T(t)\|_{\mathcal{B}(\mathbb{X})}dt < \infty$.*

In what follows, we will denote
$$|||T||| := \int_0^\infty \|T(t)\|_{\mathcal{B}(\mathbb{X})}dt < \infty$$
for any uniformly integrable family of such operators $\{T(t)\}_{t\ge 0}$.

Note that exponentially stable C_0-semigroups are examples of uniformly integrable families of operators.

2.3. **Phase space.** A phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a semi-normed linear space of functions mapping $]-\infty, 0]$ into $\mathbb{X}$, and satisfying the following fundamental axioms due to Hale and Kato (see for instance in [15]).

- (A_0) If $x : (-\infty, T] \to \mathbb{X}$, is continuous on I and $x_0 \in \mathcal{B}$, then for every $t \in I$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$,
  (ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,

(iii) $\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{0\leq s\leq t} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}$,

where $H \geq 0$ is a constant, $C_1 : [0, +\infty) \to [0, +\infty)$ is continuous, $C_2 : [0, +\infty) \to [0, +\infty)$ is locally bounded and H, C_1, C_2 are independent of $x(\cdot)$.

- (A_1) For the function $x(\cdot)$ in (A_0), x_t is a $\mathcal{B}$-valued continuous function on I.
- (A_2) The space $\mathcal{B}$ is complete.

Remark 2.10. Condition (ii) in (A_0) is equivalent to $\|\phi(0)\| \leq H\|\phi\|_{\mathcal{B}}$, for all $\phi \in \mathcal{B}$.

Definition 2.11. *$\mathcal{B}$ will be called a fading memory if the following holds:*

If $x : \mathbb{R} \to \mathbb{X}$ is a continuous function on $[\sigma, +\infty)$ with $x_\sigma \in \mathcal{B}$ for some $\sigma \in \mathbb{R}$ such that $\|x(t)\| \to 0$ as $t \to +\infty$, then $\|x_t\| \to 0$ as $t \to +\infty$.

For details on fading memories, see for instance [12].

3. Functional Differential Equation

In this Section we will consider the following space of admissible functions:

$$\mathcal{M} := \{f \in BC(\mathbb{R}, \mathbb{X}) : f|_{\mathbb{R}^+} \in AAA(\mathbb{X})\}.$$

This is a Banach space under the sup norm

$$\|f\|_{\mathcal{M}} = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

We consider the functional differential equation with infinite delay

$$\begin{cases} u'(t) &= Au(t) + F(t, u_t), \quad t \geq 0, \\ u_0 &= \phi, \quad t \in (-\infty, 0] \end{cases} \tag{3.1}$$

where $\phi \in BUC((-\infty, 0], \mathbb{X})$, the space of all functions $f : (-\infty, 0] \to \mathbb{X}$ which are bounded and uniformly continuous on $(-\infty, 0]$, and A is the generator of a semigroup $(T(t))_{t\geq 0}$ of type $\omega < 0$; that is,

$$\|T(t)\|_{\mathcal{B}(\mathbb{X})} \leq Me^{\omega t}, \ \forall t \geq 0.$$

Note that the phase space $BUC((-\infty, 0]; \mathbb{X})$ satisfies the axioms (A_0) and (A_1) above.

We make the following assumptions.

- (H_1). $F = g + h \in AAA(\mathbb{X})$ where $g \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ is uniformly continuous on every bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$ and $h \in \mathcal{C}_0(\mathbb{R}_+ \times \mathbb{X}, \mathbb{X})$.

- (H_2). There exists a positive constant μ such that

$$\|F(t,\varphi) - F(t,\psi)\| \leq \mu\|\varphi - \psi\|_{\mathcal{B}} \quad \text{for all } \varphi, \psi \in \mathcal{B}. \tag{3.2}$$

We need the following results.

Lemma 3.1. *If $\mathcal{B}$ is a fading memory and $x \in \mathcal{M}$ such that $x_0 \in \mathcal{B}$, then the function $\mathbb{R}^+ \to \mathcal{B}$, $t \to x_t$ is also in $AAA(\mathbb{X})$.*

Proof. Let $x \in \mathcal{M}$ then $x = y + z$ with $y \in AA(\mathbb{X})$ and $z \in C_0(\mathbb{R}^+, \mathbb{X})$. Hence the translation property of $AA(\mathbb{X})$ allows us to write $y_t \in AA(\mathbb{X})$. On the other hand , $z_t \in C_0(\mathbb{R}^+, \mathbb{X})$ since $\mathcal{B}$ is a fading memory [12]. □

Lemma 3.2. ([13] Lemma 2.2) *Suppose that $g \in AA(\mathbb{R}, \mathbb{X})$ is uniformly continuous on every bounded subset K of $\mathbb{X}$, uniformly in $t \in \mathbb{R}$. Then $u \in AA(\mathbb{R}, \mathbb{X})$ implies that the function $G(\cdot) = g(\cdot, u_\cdot)$ belongs to $AA(\mathbb{R}, \mathbb{X})$.*

Lemma 3.3. *Suppose that $F \in AAA(\mathbb{X})$ with a decomposition $F = g + h$ such that (H_1) is satisfied. Then $u \in AAA(\mathbb{R}, \mathbb{X})$ implies that the function $\phi(\cdot) = F(\cdot, u_\cdot)$ belongs to $AAA(\mathbb{R}, \mathbb{X})$.*

Proof. We proceed as in the proof of Theorem 2.3 in [16]. □

Definition 3.4. *A function $u \in \mathcal{M}$ is said to be a mild solution to Equation* (3.1) *if*

$$u(t) = \begin{cases} \phi(t), \ t \in]-\infty, 0], \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)F(s, u_t)ds, \ \ t \in \mathbb{R}^+. \end{cases} \tag{3.3}$$

Now we state and prove our first main result.

Theorem 3.5. *Suppose that $F \in AAA(\mathbb{X})$ with a decomposition $F = g + h$ such that (H_1) is satisfied.*

Then Equation (3.1) *possesses a mild solution $u \in \mathcal{M}$ provided that*

$$\mu \int_0^\infty C(s)ds < 1. \tag{3.4}$$

Proof. Let $F = g + h$ where $g \in AA(\mathbb{R} \times \mathcal{B}, \mathbb{X})$ and $h \in C_0(\mathbb{R}^+ \times \mathcal{B}, \mathbb{X})$.

Then let u be a mild solution of Equation (3.1) such that $u \in \mathcal{M}$. Then, by Lemma 3.1, $u_t \in AAA(\mathbb{X})$ with $u_t = y_t + z_t$, $y_t \in AA(\mathbb{X})$ and $z_t \in C_0(\mathbb{R}^+, \mathbb{X})$.

Consider

$$\Omega u(t) := T(t)\phi(0) + \int_0^t T(t-s)F(s, u_s)ds.$$

We have

$$F(s,u_s)=F(s,u_s)-F(s,y_s)+g(s,y_s)+h(s,y_s).$$

In view of Lemma 2.11 [13] and (H_1), $g(\cdot,y.)\in AA(\mathbb{R},\mathbb{X})$. Now

$$\|F(s,u_s)-F(s,y_s)\|\le\mu\|u_s-y_s\|=\mu\|z_s\|$$

which tends to 0 as $s\to\infty$.

Also since $\overline{\{y(t):t\in\mathbb{R}\}}$ is compact in $\mathbb{X}$ we have $\|h(s,y_s)\|\to 0$ as $s\to\infty$, uniformly with respect to the second variable. Thus, $F(\cdot,u.)\in AAA(\mathbb{R},\mathbb{X})$.

Now define $\tilde{F}$ by

$$\tilde{F}=\begin{cases} F(t,u_t), & t\ge 0\\ F(0,\phi(0)), & t\le 0.\end{cases}\tag{3.5}$$

Then obviously $\tilde{F}$ is continuous and is in $\mathcal{M}$. Now it is possible using the same argument as in [25] to show that

$$\Omega u(t)=\int_{-\infty}^{t}T(t-s)\tilde{F}(s,u_t)ds,$$

and consequently $(\Omega u)(\cdot)$ is continuous into $\mathbb{R}$ and its restriction on $\mathbb{R}^+$ is in $AAA(\mathbb{X})$. Thus $\Omega:\mathcal{M}\to\mathcal{M}$ is well defined and continuous.

Now, for $u,v\in\mathcal{M}$, using (4) and axiom A_0 (iii) we get

$$\begin{aligned}\|\Omega u(t)-\Omega v(t)\|&\le\mu\int_0^t e^{\omega(t-s)}\|u_s-v_s\|_{\mathcal{B}}ds\\ &\le\mu\int_0^t e^{\omega(t-s)}C(s)\sup_{0\le r\le s}\|u(r)-v(r)\|ds\\ &\le\mu\int_0^\infty C(s)ds\|u-v\|_{\mathcal{M}}\end{aligned}$$

since (3.5) and (A_0)- (iii) hold, and $\omega<0$. Thus, according to (3.4), we get

$$\|\Omega u-\Omega v\|_{\mathcal{M}}\le\mu\int_0^\infty C(s)ds\|u-v\|_{\mathcal{M}}.$$

We deduce the existence and uniqueness of an asymptotically almost automorphic solution in view of Banach's contraction mapping principle. □

4. Pseudo-almost automorphic solutions

Consider now in a Banach space $(\mathbb{X}, \|\cdot\|)$ the fractional differential equation

$$D_t^\alpha x(t) = Ax(t) + f(t), \quad t \in \mathbb{R} \tag{4.1}$$

and its perturbation

$$D_t^\alpha x(t) = Ax(t) + f(t) + g(t, x(t)), \quad t \in \mathbb{R} \tag{4.2}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of an α-resolvent family $(S_\alpha(t))_{t\geq 0}$ defined on $\mathbb{X}$, a complex Banach space, $1 \leq \alpha \leq 2$. We assume also that $(S_\alpha(t))_{t\geq 0}$ is uniformly integrable; i.e.,

$$|||S_\alpha||| := \int_0^\infty \|S_\alpha(\sigma)\|_{\mathcal{B}(\mathbb{X})} d\sigma < \infty,$$

and $f \in BC(\mathbb{R}, \mathbb{X})$.

Remark 4.1. [1] Consider the function

$$x(t) = \int_{-\infty}^t S_\alpha(t-\sigma) f(\sigma) d\sigma.$$

Note that $S_1(t)$ is a C_0-semigroup. Thus, if $\alpha = 1$, the mild solution of Equation (4.1) is given by

$$x(t) = \int_{-\infty}^t S_1(t-\sigma) f(\sigma) d\sigma$$

(see[25]).

If $\alpha = 2$, then $S_2(t)$ is the sine family generated by A and the mild solution of Equation (4.1) is given by

$$x(t) = \int_{-\infty}^t S_2(t-\sigma) f(\sigma) d\sigma.$$

To the best of our knowledge, when $1 < \alpha < 2$, there is no analogue of the semigroup property or cosine functional equation in order to establish, say rigorously, $x(t)$ as a mild solution of Equation (4.1). This is due to the nonlocal character of the fractional differentiation leading always to some presence of memory (see the excellent thesis [2]).

The comments above lead to the following definition.

Definition 4.2. ([1]) *A continuous function $x : \mathbb{R} \to \mathbb{X}$ is said to be a mild solution of Equation* (4.1) (*respectively Equation* (1.3)) *if*

$$x(t) = \int_{-\infty}^t S_\alpha(t-\sigma) f(\sigma) d\sigma$$

(*respectively*

$$x(t) = \int_{-\infty}^{t} S_\alpha(t-\sigma)[f(\sigma) + g(\sigma, x(\sigma))]d\sigma).$$

Lemma 4.3. *Let $(S_\alpha(t))_{t\geq 0}$ be a uniformly integrable α-resolvent family. Let also $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a (jointly) continuous function such that*

$$\|g(t,x) - g(t,y)\| \leq L\|x-y\|, \quad \forall x, y \in \mathbb{X},\ t \in \mathbb{R}$$

and

$$\sup_{t\in\mathbb{R}} \|g(t,0)\| = K < \infty,$$

with $L|||S_\alpha||| < 1$. Then the integral equation

$$x(t) = \int_{-\infty}^{t} S_\alpha(t-\sigma)g(\sigma, x(\sigma))d\sigma, \quad t \in \mathbb{R} \tag{4.3}$$

possesses a unique solution in $BC(\mathbb{R}, \mathbb{X})$

Proof. Let $F : BC(\mathbb{R}, \mathbb{X}) \to BC(\mathbb{R}, \mathbb{X})$ be defined by

$$F\varphi(t) := \int_{-\infty}^{t} S_\alpha(t-\sigma)g(\sigma, \varphi(\sigma))d\sigma, \quad t \in \mathbb{R}.$$

The continuity of F is obvious. Now observe that

$$\begin{aligned}
\|F\varphi(t)\| &\leq \int_{-\infty}^{t} \|S_\alpha(t-\sigma)g(\sigma, \varphi(\sigma))\|d\sigma \\
&\leq \int_{-\infty}^{t} \|S_\alpha(t-\sigma)\|_{\mathcal{B}(\mathbb{X})}\|g(\sigma, \varphi(\sigma)) - g(\sigma, 0)\|d\sigma \\
&+ \int_{-\infty}^{t} \|S_\alpha(t-\sigma)\|_{\mathcal{B}(\mathbb{X})}\|g(\sigma, 0)\|d\sigma \\
&\leq L\int_{-\infty}^{t} \|S_\alpha(t-\sigma)\|_{\mathcal{B}(\mathbb{X})}\|\varphi(\sigma))\|d\sigma + K\int_{-\infty}^{t} \|S_\alpha(t-\sigma)\|_{\mathcal{B}(\mathbb{X})}d\sigma \\
&\leq (L\|\varphi\|_\infty + K)\,|||S_\alpha|||
\end{aligned}$$

which proves that F is well defined.

Now let $\varphi, \psi \in BC(\mathbb{R}, \mathbb{X})$. Then we have

$$\begin{aligned}
\|(F\varphi)(t) - (F\psi)(t)\| &\leq \int_{-\infty}^{t} S_\alpha(t-\sigma)[g(\sigma, \varphi(\sigma)) - g(\sigma, \psi(\sigma))]\|d\sigma \\
&\leq L|||S_\alpha|||\,\|\varphi - \psi\|_\infty.
\end{aligned}$$

Thus,

$$\|F\varphi - F\psi\|_\infty \le L|||S_\alpha|||\,\|\varphi - \psi\|_\infty,$$

which shows that F is a strict contraction since $L|||S||| < 1$. The conclusion follows from the Banach's contraction mapping principle. □

Theorem 4.4. *Let $(S_\alpha(t))_{t\ge 0}$ be a uniformly integrable α-resolvent family. Let $f \in BC(\mathbb{R}, \mathbb{X})$ and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ a (jointly) continuous function such that*

$$\|g(t,x) - g(t,y)\| \le L\|x - y\|, \;\; \forall x, y \in \mathbb{X},\; t \in \mathbb{R}$$

and

$$\sup_{t\in\mathbb{R}} \|g(t,0)\| = K < \infty,$$

with $L|||S_\alpha||| < 1$. Then Equation (1.3) has a unique mild solution in $BC(\mathbb{R}, \mathbb{X})$.

Proof. Let $z(t)$ be the unique bounded mild solution of Equation (4.1) with the representation

$$z(t) = \int_{-\infty}^{t} S_\alpha(t-\sigma) f(\sigma) d\sigma, \;\; t \in \mathbb{R}.$$

Consider the mapping $\Gamma : BC(\mathbb{R}, \mathbb{X}) \to BC(\mathbb{R}, \mathbb{X})$ defined by

$$(\Gamma\varphi)(t) := z(t) + (F\varphi)(t), \;\; t \in \mathbb{R},$$

where the operator F is defined as in Lemma 4.3. Obviously Γ is well defined. Moreover, it is a strict contraction. Thus it has a unique point fixed $w(t)$ in $BC(\mathbb{R}, \mathbb{X})$ which satisfies the equation

$$w(t) = (\Gamma w)(t) = z(t) + (Fw)(t);$$

that is,

$$w(t) = \int_{-\infty}^{t} S_\alpha(t-\sigma)[f(\sigma) + g(\sigma, w(\sigma))] d\sigma,$$

which is a mild solution to Equation (1.3) □

Lemma 4.5. *Let $(S_\alpha(t))_{t\ge 0}$ be a uniformly integrable α-resolvent family and f in $AA(\mathbb{X})$, $(AP(\mathbb{X}))$ and $P_\omega(\mathbb{X})$. Then*

$$\int_{-\infty}^{t} S_\alpha(t-\sigma) f(\sigma) d\sigma$$

belongs to $AA(\mathbb{X})$, $(AP(\mathbb{X}))$ and $P_\omega(\mathbb{X})$.

Proof. Case I: $P_\omega(\mathbb{X})$. Let $f \in P_\omega(\mathbb{X})$. Then we have

$$\begin{aligned} F(t+\omega) &= \int_{-\infty}^{t+\omega} S_\alpha(t+\omega-\sigma)f(\sigma)d\sigma \\ &= \int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma+\omega)d\sigma = \int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma)d\sigma = F(t) \end{aligned}$$

which proves that $F \in P_\omega(\mathbb{X})$.

Case II: $AP(\mathbb{X})$. Suppose $f \in AP(\mathbb{X})$. Let $\epsilon > 0$ be given. Take τ an ϵ-almost period of f. Then we have

$$\begin{aligned} \|F(t+\tau)-F(t)\| &= \left\| \int_{-\infty}^{t+\tau} S_\alpha(t+\tau-\sigma)f(\sigma)d\sigma - \int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma)d\sigma \right\| \\ &= \left\| \int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma+\tau)d\sigma - \int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma)d\sigma \right\| \\ &= \left\| \int_{-\infty}^{t} S_\alpha(t-\sigma)\left(f(\sigma+\tau)d\sigma - f(\sigma)\right)d\sigma \right\| \\ &\leq \int_{-\infty}^{t} \|S_\alpha(t-\sigma)\|_{\mathcal{B}(\mathbb{X})} \|f(\sigma+\tau)-f(\sigma)\| d\sigma \leq |||S_\alpha|||\,\epsilon, \end{aligned}$$

which proves that $F \in AP(\mathbb{X})$.

Case III. $AA(\mathbb{X})$ Let (s'_n) be a sequence of real numbers. Then there exists a subsequence (s_n) of (s'_n) such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

Note that the function g above is measurable and bounded but not necessarily continuous.

Consider

$$F(t) := \int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma)d\sigma$$

and

$$G(t) := \int_{-\infty}^{t} S_\alpha(t-\sigma)g(\sigma)d\sigma.$$

Then we obtain

$$F(t+s_n)=\int_{-\infty}^{t+s_n}S_\alpha(t+s_n-\sigma)f(\sigma)d\sigma=\int_{-\infty}^{t}S_\alpha(t-\xi)f(\xi+s_n)d\xi$$

by letting $\sigma-s_n=\xi$. Thus,

$$\|F(t+s_n)\|\le |||S_\alpha|||\,\|f\|_\infty,\ n=1,2,...$$

Similarly, we get

$$\|G(t-s_n)\|\le |||S_\alpha|||\,\|g\|_\infty,\ n=1,2,...$$

Also continuity of $S_\alpha(\cdot)$ yields $\lim_{n\to\infty}S_\alpha(t-\xi)f(\xi+s_n)=S_\alpha(t-\xi)g(\xi)$. Therefore, in view of Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}F(t+s_n)=G(t)$$

for each $t\in\mathbb{R}$. Similarly, we can prove that

$$\lim_{n\to\infty}G(t-s_n)=F(t),$$

which gives the conclusion of the lemma. □

Lemma 4.6. *Let $(S_\alpha(t))_{t\ge0}$ be a uniformly integrable α-resolvent family and $h\in AA_0(\mathbb{X})$. Then*

$$\int_{-\infty}^{t}S_\alpha(t-\sigma)h(\sigma)d\sigma\in AA_0(\mathbb{X}).$$

Proof. By assumption

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|h(\sigma)\|d\sigma=0.$$

Now let us write

$$\frac{1}{2r}\int_{-r}^{r}\|\int_{-\infty}^{t}S_\alpha(t-\sigma)h(\sigma)d\sigma\|dt\le I_1(r)+I_2(r),$$

where

$$I_1(r)=\frac{1}{2r}\int_{-r}^{r}dt\int_{-r}^{t}\|S_\alpha(t-\sigma)\|_{B(\mathbb{X})}\|h(\sigma)\|d\sigma$$

and

$$I_2(r)=\frac{1}{2r}\int_{-r}^{r}dt\int_{-\infty}^{-r}\|S_\alpha(t-\sigma)\|_{B(\mathbb{X})}\|h(\sigma)\|d\sigma.$$

We have

$$I_1(r)\le\frac{1}{2r}\int_{-r}^{r}\|h(\sigma)\|d\sigma(\int_{\sigma}^{r}\|S_\alpha(t-\sigma)\|_{B(\mathbb{X})}dt)$$

$$= \frac{1}{2r}\int_{-r}^{r}\|h(\sigma)\|d\sigma(\int_{0}^{r-\sigma}\|S_\alpha(s)\|_{B(\mathbb{X})}ds)$$
$$\leq \frac{1}{2r}\int_{-r}^{r}\|h(t)\|dt(\int_{0}^{\infty}\|S_\alpha(s)\|_{B(\mathbb{X})}ds) = \frac{1}{2r}\int_{-r}^{r}\|h(t)\|dt \cdot |||S_\alpha|||,$$

thus, $\lim_{r\to\infty} I_1(r) = 0$.

$$I_2(r) \leq \frac{1}{2r}\int_{-r}^{r}dt\int_{t+r}^{\infty}\|S_\alpha(s)\|_{B(\mathbb{X})}\|h(t-s)\|ds$$
$$\leq \frac{1}{2r}\int_{-r}^{r}dt\int_{2r}^{\infty}\|S_\alpha(s)\|_{B(\mathbb{X})}\|h(t-s)\|ds \leq \|h\|_\infty\int_{2r}^{\infty}\|S_\alpha(s)\|_{B(\mathbb{X})}ds;$$

this proves that $\lim_{r\to\infty} I_2(r) = 0$. The lemma is proved. □

Let $\mathcal{M}(\mathbb{X}) := \{PP_\omega(\mathbb{X}), PAP(\mathbb{X}), PAA(\mathbb{X})\}$.

Theorem 4.7. *Assume that $(S_\alpha(t))_{t\geq 0}$ is a uniformly integrable α-resolvent family and $f \in \Omega \subset \mathcal{M}(\mathbb{X})$. Then*

$$\int_{-\infty}^{t} S_\alpha(t-\sigma)f(\sigma)d\sigma \in \Omega.$$

Proof. Let $f \in \Omega$. Then $f = g + h$, where $h \in AA_0(\mathbb{X})$, and, according to the case, $g \in P_\omega(\mathbb{X})$, $AP(\mathbb{X})$, or $AA(\mathbb{X})$. Consider $F(t) = \Lambda_1(t) + \Lambda_2(t)$, where

$$\Lambda_1(t) := \int_{-\infty}^{t} S_\alpha(t-\sigma)g(\sigma)d\sigma, \qquad \Lambda_2(t) := \int_{-\infty}^{t} S_\alpha(t-\sigma)h(\sigma)d\sigma.$$

We reach the conclusion by using Lemmas 3.4 and 3.5 above. □

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