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ANALYTIC PROPERTY OF A COUPLED SYSTEM OF WAVE-PLATE TYPE WITH THERMAL EFFECT

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Abstract. In this work, we consider a coupled system of wave-plate type with thermal effect. We show that the associated C_0-semigroup is analytic and exponentially stable.

1. Introduction

In this paper, we study the analytic property and the exponential stability of the C_0-semigroup associated with the following coupled system of wave-plate type with thermal effect:

$$\varrho_1 u_{tt} - \Delta u - \mu\Delta u_t + \alpha\Delta v = 0 \quad \text{in} \quad \Omega\times(0,\infty), \tag{1.1}$$

$$\varrho_2 v_{tt} + \gamma\Delta^2 v + \alpha\Delta u + m\Delta\theta = 0 \quad \text{in} \quad \Omega\times(0,\infty), \tag{1.2}$$

$$\tau\theta_t - k\Delta\theta - m\Delta v_t = 0 \quad \text{in} \quad \Omega\times(0,\infty), \tag{1.3}$$

$$u = v = \frac{\partial v}{\partial\nu} = \theta = 0 \quad \text{on} \quad \Gamma\times(0,\infty), \tag{1.4}$$

$$(u(x,0), v(x,0), \theta(x,0)) = (u_0(x), v_0(x), \theta_0(x)), \quad \text{in} \quad \Omega \tag{1.5}$$

$$(u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) \quad \text{in} \quad \Omega, \tag{1.6}$$

where Ω an open bounded subset of $\mathbb{R}^n$ with smooth boundary Γ; u and v represent the vertical deflections of the membrane and of the plate, respectively. By θ we denote the difference of temperature. Finally, ϱ_1, ϱ_2, μ, γ, k, m, τ and α are positive constants.

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The above model can be used to describe the evolution of a system consisting of an elastic membrane and an elastic plate, subject to an elastic force that attracts the membrane to the plate with coefficient α, subject to a thermal effect (see [7]).

Let us first recall some related work in the literature. The two-dimensional thermoelastic Kirchhoff plate equations

$$v_{tt} - \gamma\Delta v_{tt} + \Delta^2 v + m\Delta\theta = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.7}$$

$$\theta_t - k\Delta\theta - m\Delta v_t = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.8}$$

were studied by different authors, see for example [1, 3, 4, 5, 6, 8]. It is now clear that, when $\gamma = 0$, the semigroup associated with the thermoelastic equation (1.7)-(1.8) is not only exponentially stable, but also analytic (see [1, 6]). However, in the case of $\gamma > 0$, the semigroup is only exponentially stable, neither compact nor differentiable (see [2, 4, 5, 6]).

Our main result is that the semigroup associated to system (1.1)-(1.6) is analytic. This, in particular, implies the exponential stability of the associated energy and also the so-called spectrum determined growth property (SDG-property) of the corresponding semigroup. Our method is based on Theorem 1.3.3 of [6] (see also Theorem 3.1 below). Note that this problem can be solved as a perturbation of an analytic semigroup but in that case we need to take $\alpha << 1$ very small. Following our method depending on the domain the values of α can be large.

The rest of this work is organized as follows. In section 2, we show the well posedness of system (1.1)-(1.6). In section 3 we show that the corresponding semigroup is analytic.

2. Existence and uniqueness

In this section we will study the existence and uniqueness of strong and global solutions for the system (1.1)-(1.6) using semigroup techniques. From now on, for simplicity and without lost of generality we assume that $\varrho_1 = \varrho_2 = \tau = 1$.

Let us denote by $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ the Hilbert space with inner product given by

$$\begin{aligned} < U, V > = & \int_\Omega [\nabla u_1 \cdot \overline{\nabla v_1} + u_2\overline{v_2} - \alpha(\nabla u_1 \overline{\nabla v_3} + \nabla u_3 \overline{\nabla v_1})]\, dx \\ & + \gamma \int_\Omega \Delta u_3 \overline{\Delta v_3}\, dx + \int_\Omega [u_4\overline{v_4} + u_5\overline{v_5}]\, dx, \end{aligned} \tag{2.1}$$

where $U = (u_1, u_2, u_3, u_4, u_5)^T$ and $V = (v_1, v_2, v_3, v_4, v_5)^T$. To get the inner product well defined we impose the condition $\alpha^2 < 4\gamma\lambda_1^2$, where λ_1 is the first eigenvalue of the Laplacian. Therefore, depending on the domain and the value of γ we can take relatively large values of α. Let us define the operator $\mathcal{A}$ as

$$\mathcal{D}(\mathcal{A}) = \Big\{U \in \mathcal{H} : \ \ u_1 + \mu u_2, \ u_5 \in H_0^1(\Omega) \cap H^2(\Omega), \ u_2 \in H_0^1(\Omega), \\ u_3 \in H^4(\Omega), \ \ u_4 \in H_0^2(\Omega)\Big\}$$

and

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ \Delta & \mu\Delta & -\alpha\Delta & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ -\alpha\Delta & 0 & -\gamma\Delta^2 & 0 & -m\Delta \\ 0 & 0 & 0 & m\Delta & k\Delta \end{bmatrix}.$$

Putting $\varphi = u_t$ and $\psi = v_t$, the equations (1.1), (1.2) and (1.3) can be written as the following initial-value problem:

$$\frac{dU}{dt} = \mathcal{A}U, \qquad U(0) = U_0, \tag{2.2}$$

with $U = (u, \varphi, v, \psi, \theta)^T$, $U_0 = (u_0, u_1, v_0, v_1, \theta_0)^T$. Note that

$$\begin{aligned} \langle \mathcal{A}U, U\rangle = & \int_\Omega \nabla\varphi \cdot \overline{\nabla u}\, dx + \int_\Omega (\Delta u + \mu\Delta\varphi - \alpha\Delta v)\overline{\varphi}\, dx \\ & + \gamma \int_\Omega \Delta\psi\overline{\Delta v}\, dx + \int_\Omega (-\gamma\Delta^2 v - \alpha\Delta u - m\Delta\theta)\overline{\psi}\, dx \\ & - \alpha \int_\Omega (\nabla\varphi\overline{\nabla v} + \nabla\psi\overline{\nabla u})]\, dx + \int_\Omega (k\Delta\theta + m\Delta\psi)\overline{\theta}\, dx. \end{aligned}$$

Taking the real part we get

$$\operatorname{Re}\langle \mathcal{A}U, U\rangle = -\mu \int_\Omega |\nabla\varphi|^2\, dx - k \int_\Omega |\nabla\theta|^2\, dx \le 0. \tag{2.3}$$

Thus, $\mathcal{A}$ is a dissipative operator. □

To show the well posedness we use the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be a linear operator with dense domain $\mathcal{D}(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$ (resolvent set of $\mathcal{A}$), then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. See Z. Liu and S. Zheng [6].

Under the above notation we can establish the following theorem.

Theorem 2.2. *The operator $\mathcal{A}$ generates a C_0-semigroup $S(t)$ of contractions in $\mathcal{H}$.*

Proof. We will show that $0 \in \rho(\mathcal{A})$. Let us take $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$; then there exists $U \in \mathcal{H}$ such that

$$\mathcal{A}U = F; \tag{2.4}$$

i.e.,

$$\begin{cases} \varphi = f_1, \\ \Delta u + \mu\Delta\varphi - \alpha\Delta v = f_2 \\ \psi = f_3, \\ -\gamma\Delta^2 v - \alpha\Delta u - m\Delta\theta = f_4, \\ k\Delta\theta + m\Delta\psi = f_5. \end{cases} \tag{2.5}$$

Using standard results on linear elliptic equations it follows that there exists $(u, \varphi, v, \psi, \theta) \in \mathcal{D}(\mathcal{A})$. Moreover we have $||U||_{\mathcal{H}} \leq M||F||_{\mathcal{H}}$, with M being a positive constant. Thus, $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$. From Theorem (2.1), our conclusion follows. □

3. Main result

We will use the following result.

Theorem 3.1. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions in a Hilbert space. Suppose that*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \Re\} \equiv i\Re. \tag{3.1}$$

Then, $S(t)$ is analytic if and only if

$$\overline{\lim_{|\beta|\to\infty}} \|\beta(i\beta - \mathcal{A})^{-1}\| < \infty \tag{3.2}$$

Proof. See Z. Liu and S. Zheng [6].

For any $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, let us denote by $U = (u, \varphi, v, \psi, \theta)^T$ the solution of the system

$$(i\beta I - \mathcal{A})U = F. \tag{3.3}$$

In terms of the component we get

$$i\beta u - \varphi = f_1 \quad \text{in} \quad H_0^1(\Omega), \tag{3.4}$$

$$i\beta\varphi - \Delta u - \mu\Delta\varphi + \alpha\Delta v = f_2 \quad \text{in} \quad L^2(\Omega), \tag{3.5}$$

$$i\beta v-\psi=f_3 \quad \text{in} \quad H_0^2(\Omega), \tag{3.6}$$

$$i\beta\psi+\gamma\Delta^2 v+\alpha\Delta u+m\Delta\theta=f_4 \quad \text{in} \quad L^2(\Omega), \tag{3.7}$$

$$i\beta\theta-k\Delta\theta-m\Delta\psi=f_5 \quad \text{in} \quad L^2(\Omega). \tag{3.8}$$

Taking the inner product of (3.3) with $\overline{U}$ in $\mathcal{H}$ we get

$$\int_\Omega |\nabla\varphi|^2 dx+\int_\Omega |\nabla\theta|^2 dx\leq C||F||_{\mathcal{H}}||U||_{\mathcal{H}}. \tag{3.9}$$

Lemma 3.1. *Under the above conditions for any $\epsilon>0$ there exists a positive constant C_ϵ such that*

$$|\beta|^2\int_\Omega |\theta|^2\, dx\leq \epsilon|\beta|^2\|U\|^2+c_\epsilon\|F\|^2.$$

Proof. From (3.6) we get that

$$\|\psi\|_{H^2}\leq|\beta|\|v\|_{H^2}+\|f_3\|_{H^2}.$$

Using interpolation in the above inequality we get

$$\begin{aligned}\|\psi\|_{H^1}&\leq C\|\psi\|_{H^2}^{1/2}\|\psi\|_{L^2}^{1/2}\leq C\left(|\beta|\|v\|_{H^2}+\|f_3\|_{H^2}\right)^{1/2}\|\psi\|_{L^2}^{1/2}\\&\leq C|\beta|^{1/2}\|U\|+C\|U\|^{1/2}\|F\|^{1/2}.\end{aligned} \tag{3.10}$$

Multiplying equation (3.8) by $\overline{i\beta\theta}$ we get

$$|\beta|^2\int_\Omega|\theta|^2\,dx+k\overline{i\beta}\int_\Omega|\nabla\theta|^2\,dx+m\overline{i\beta}\int_\Omega\nabla\psi\cdot\nabla\overline{\theta}\,dx=\overline{i\beta}\int_\Omega f_5\overline{\theta}\,dx$$

from which we obtain

$$\begin{aligned}|\beta|^2\int_\Omega|\theta|^2\,dx&\leq C|\beta|\|U\|\|F\|+c|\beta|\|\nabla\psi\|\|\nabla\theta\|\\&\leq C_\epsilon|\beta|\|U\|\|F\|+\frac{\epsilon}{2C}|\beta|\|\nabla\psi\|^2.\end{aligned} \tag{3.11}$$

Substitution of (3.10) into (3.11) gives our conclusion. □

Lemma 3.2. *Under the above conditions for any $\epsilon>0$ there exists a positive constant C_ϵ such that*

$$\int_\Omega|\nabla\psi|^2\,dx\leq\epsilon^2|\beta|\|U\|^2+\frac{c_\epsilon}{|\beta|}\|F\|^2.$$

Proof. From Lemma 3.1 we get

$$|\beta|\int_\Omega|\theta|^2\,dx\leq\epsilon^2|\beta|\|U\|^2+\frac{c_\epsilon}{|\beta|}\|F\|^2.$$

Multiplying equation (3.8) by ψ we get

$$i\beta\int_\Omega \theta\overline{\psi}\,dx + k\int_\Omega \nabla\theta\cdot\nabla\overline{\psi}\,dx + m\int_\Omega |\nabla\psi|^2\,dx = \int_\Omega f_5\overline{\psi}\,dx.$$

From this it follows that

$$\int_\Omega |\nabla\psi|^2\,dx \le c|\beta|\|\theta\|\|\psi\| + C\|U\|\|F\|$$
$$\le c_\epsilon|\beta|\|\theta\|^2 + \frac{\epsilon}{3}|\beta|\|\psi\| + C\|U\|\|F\| \le c_\epsilon|\beta|\|\theta\|^2 + \frac{\epsilon}{3}|\beta|\|U\|^2 + C\|U\|\|F\|.$$

Using Lemma 3.1 our conclusion follows. □

Lemma 3.3. *Under the above conditions for any $\epsilon > 0$ there exists a positive constant C_ϵ such that*

$$|\beta|^2\int_\Omega |\psi|^2\,dx \le \epsilon^2|\beta|\|U\|^2 + C_\epsilon\|F\|^2.$$

Proof. Let us consider the following decomposition: $\psi = \psi_1 + \psi_2$, where

$$i\beta\psi_1 - \Delta\psi_1 = f_4, \qquad \psi_1 = 0, \quad \text{in} \quad \partial\Omega$$

and ψ_2 satisfies

$$i\beta\psi_2 + \Delta\psi_1 + \gamma\Delta^2 v + \alpha\Delta u + m\Delta\theta = 0.$$

It is not difficult to see that

$$|\beta|^2\|\psi_1\|^2 + \|\Delta\psi_1\|^2 \le C\|F\|^2. \tag{3.12}$$

On the other hand we have

$$|\beta|\|\psi_2\|_{H^{-2}} \le C\|\psi_1\|_{L^2} + C\|\Delta v\| + C\|u\|_{L^2} + C\|\theta\|_{L^2}.$$

Using (3.12) it follows that

$$|\beta|\|\psi_2\|_{H^{-2}} \le C\|U\| + \frac{C}{|\beta|}\|F\|. \tag{3.13}$$

Using interpolation, inequality (3.12) and Lemma 3.2 we get

$$\|\psi_2\|_{H^{-1}} \le C\|\psi_2\|_{H^{-2}}^{2/3}\|\psi_2\|_{H^1}^{1/3} \le C\|\psi_2\|_{H^{-2}}^{2/3}(\|\psi_1\|_{H^1} + \|\psi\|_{H^1})^{1/3}$$
$$\le C\|\psi_2\|_{H^{-2}}^{2/3}(\epsilon|\beta|^{1/2}\|U\| + \frac{C_\epsilon}{|\beta|^{1/2}}\|F\|)^{1/3}.$$

Now, using (3.13) we get

$$\|\psi_2\|_{H^{-1}} \le \frac{\epsilon C}{|\beta|^{1/2}}\|U\| + \frac{C_\epsilon}{|\beta|^{3/2}}\|F\|.$$

On the other hand, since

$$\|\psi_2\|_{H^1} \leq \|\psi_1 + \psi\|_{H^1} \leq \epsilon|\beta|^{1/2}\|U\| + \frac{c_\epsilon}{|\beta|^{1/2}}\|F\|$$

we have that

$$\begin{aligned}\|\psi_2\|_{L^2} &\leq C\|\psi_2\|_{H^{-1}}^{1/2}\|\psi_2\|_{H^1}^{1/2} \\ &\leq C\Big[\frac{\epsilon C}{|\beta|^{1/2}}\|U\| + \frac{C_\epsilon}{|\beta|^{3/2}}\|F\|\Big]^{1/2}\Big[\epsilon|\beta|^{1/2}\|U\| + \frac{c_\epsilon}{|\beta|^{1/2}}\|F\|\Big]^{1/2} \\ &\leq \epsilon C\|U\| + \frac{C_\epsilon}{|\beta|}\|F\|\end{aligned}$$

which implies that

$$|\beta|\|\psi\| \leq \epsilon C|\beta|\|U\| + C_\epsilon\|F\|.$$

From this our conclusion follows. □

Lemma 3.4. *Under the above conditions for any $\epsilon > 0$ there exists a positive constant C_ϵ such that*

$$|\beta|^2\int_\Omega |\Delta v|^2\, dx \leq \epsilon^2|\beta|^2\|U\|^2 + C_\epsilon\|F\|^2.$$

Proof. Multiplying equation (3.7) by $\overline{\psi}$ we get

$$i\beta\|\psi\|^2 + \int_\Omega \Delta v\Delta\overline{\psi}\, dx + \alpha\int_\Omega \Delta u\overline{\psi}\, dx + m\int_\Omega \Delta\theta\overline{\psi}\, dx = \int_\Omega f_4\overline{\psi}\, dx.$$

Using (3.6) we have

$$|\beta|\int_\Omega |\Delta v|^2\, dx \leq |\beta|\|\psi\|^2 + \alpha\|\nabla u\|\|\nabla\psi\| + \|\nabla\theta\|\|\nabla\psi\| + C\|U\|\|F\|.$$

Then Lemma 3.2 implies

$$\|\nabla\theta\|\|\nabla\psi\| \leq \epsilon|\beta|\|U\|^2 + \frac{c_\epsilon}{|\beta|}\|F\|.$$

On the other hand

$$\begin{aligned}\|\nabla u\|\|\nabla\psi\| &\leq \|\nabla u\|\|i\beta\nabla v - \nabla f_3\| \leq \|\nabla u\|\|i\beta\nabla v\| + C\|U\|\|F\| \\ &\leq \|\nabla\varphi\|\|\nabla v\| + C\|U\|\|F\| \leq C\|\nabla\varphi\|^2 + \frac{1}{2}\|\Delta v\|^2 + C\|U\|\|F\| \\ &\leq \frac{1}{2}\|\Delta v\|^2 + C\|U\|\|F\|.\end{aligned}$$

Using Lemma 3.2 we get

$$|\beta|\int_\Omega |\Delta v|^2\, dx \leq \epsilon|\beta|\|U\|^2 + \frac{c_\epsilon}{|\beta|}\|F\|$$

for $|\beta| > 1/2$. Therefore, our conclusion follows. □

Theorem 3.2. *The semigroup $S(t)$, generated by $\mathcal{A}$, is analytic.*

Proof. To show the analyticity we use Theorem 3.1. From (3.4) we get that

$$|\beta|\|\nabla u\| \leq \|U\|^{1/2}\|F\|^{1/2} + \|F\|.$$

Taking the inner product of (3.4) with $\overline{i\beta\varphi}$ we get

$$|\beta|^2\|\nabla\varphi\|^2 \leq \epsilon C|\beta|^2\|U\|^2 + C_\epsilon\|F\|^2.$$

Using Lemma 3.1, Lemma 3.2, Lemma 3.3 we get that $|\beta|\|U\| \leq \|F\|$ which means that $\mathcal{A}$ is analytic. □

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