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DECAY PROPERTY FOR THE LINEAR WAVE EQUATIONS IN TWO-DIMENSIONAL EXTERIOR DOMAINS

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1. Introduction

Let Ω be an exterior domain in $\mathbb{R}^2$ with a compact C^2 boundary $\partial\Omega$. Without loss of generality we may assume $(0,0) \notin \overline{\Omega}$. In this paper, we will consider the Cauchy-Dirichlet problem for the wave equation on exterior domains: For a function $u = u(t,x)$ defined for $(t,x) \in (0,\infty) \times \Omega$,

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad (0,\infty) \times \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega, \tag{1.2}$$

$$u = u_0,\ u_t = u_1 \quad \text{on} \quad \{t = 0\} \times \Omega. \tag{1.3}$$

The total energy $E(t)$ is defined as

$$E(t) = \tfrac{1}{2}\left(\|\nabla u(t)\|^2 + \|u_t(t)\|^2\right), \tag{1.4}$$

where the usual notation is used for $f, g \in L^2(B)$ with a domain $B \subset \mathbb{R}^2$,

$$(f,g)_{L^2(B)} = \int_B fg\,dx, \qquad \|f\|_{L^2(B)} = \sqrt{(f,f)_{L^2(B)}},$$

and their abbreviations are denoted $(f,g) = (f,g)_{L^2(\Omega)}$ and $\|f\| = \|f\|_{L^2(\Omega)}$ for $f, g \in L^2(\Omega)$. For any positive number R such that $\partial\Omega \subset B_R(0) \equiv \{x \in \mathbb{R}^2 : |x| < R\}$, the local energy is defined as

$$E_{\Omega(R)}(t) = \tfrac{1}{2}\left(\|\nabla u(t)\|^2_{L^2(\Omega(R))} + \|u_t(t)\|^2_{L^2(\Omega(R))}\right), \tag{1.5}$$

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which is just the energy localized around the boundary $\Omega(R) \equiv \Omega \cap B_R(0)$. We are concerned with a decay estimate of the local energy for a solution to the wave equations (1.1), (1.2), and (1.3). For decay of a local energy for wave equations on exterior domains, some results are already known. As in the results of Morawetz [12, 13], the boundedness of $\|u(t)\|$ is crucial in deriving the local energy decay of solutions to the wave equations for compactly supported initial data (u_0, u_1). On the other hand, the L^2-boundedness itself holds true without any compactness of the support of initial data. If, in particular, it is assumed that $u_1 = 0$, then the L^2 bound for the solution is obtained without any condition of compact support for initial data u_0 (refer to [8, 9], [12]). Ikehata and Matsuyama [8] were able to show the boundedness in L^2 of solutions to the wave equation if initial data u_1 satisfies $\|du_1\| < \infty$ for $d(x) = |x| \log |x|$. In this paper, under the condition that u_1 belongs to the Hardy space $\mathcal{H}^1(\Omega)$ on Ω (see the definition below), we prove L^2-boundedness and then a local energy decay of rate t^{-1} if the initial data has compact support. Here we notice that any member of the Hardy space on $\mathbb{R}^n$ has an integrability of $L\log(L)$ on $\mathbb{R}^n$ (refer to [10]). We also study decay estimates of total energy and L^2 for the dissipative wave equations if initial data $u_0 + u_1 \in \mathcal{H}^1(\Omega)$. The local energy and L^2-norm of solutions decay like t^{-2} and $t^{-1/2}$, respectively, of that the decay rate may be optimal in comparison with that of the heat kernel and the proof is also based on the boundedness of energy and L^2 of solutions (refer to [14, 15], [6, 8, 9] and also see [11], [2], [7] for the Cauchy problem). Such results as above may be applied to the heat equations

$$u_t - \Delta u = 0 \qquad \text{in} \qquad (0, \infty) \times \mathbb{R}^2, \tag{1.6}$$

$$u = u_0 \qquad \text{on} \qquad \{t = 0\} \times \mathbb{R}^2 \tag{1.7}$$

and the solution u of (1.6) and (1.7) satisfies the estimate

$$\int_0^\infty \|u(s)\|^2_{L^2(\mathbb{R}^2)}\, ds \le C\|u_0\|^2_{\mathcal{H}^1(\mathbb{R}^2)}.$$

However, $\mathcal{H}^1(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$ and the estimate

$$\int_0^\infty \|u(s)\|^2_{L^2(\mathbb{R}^2)}\, ds \le C\|u_0\|^2_{L^1(\mathbb{R}^2)}$$

generally fails. This fact represents a somehow subtle gain to regularity by the Hardy space and motivates our study in this paper.

In Section 2, we treat the linear wave equations (1.1), (1.2), and (1.3). A boundedness in L^2 of solutions is obtained from the Hardy space regularity of initial data, and then a local energy decay is shown in the usual way. Section 3 is devoted to the same estimates as Section 2 for the dissipative wave equations. In Section 4, we show an application of the estimations to a decay in L^2 and total energy for semilinear wave equations with dissipative term.

2. Linear wave equation

We now state our main theorem on the estimate in L^2 of weak solutions to (1.1), (1.2), and (1.3) in the two-space-dimension case, $\Omega \subset \mathbb{R}^2$. The key point is that we choose the initial data u_1 to be in the Hardy space $\mathcal{H}^1(\Omega)$ (refer to [8] and [9], and see the definition below).

We will start with the definition of function spaces (refer to [4]).

Definition 2.1. (Hardy space) *The Hardy space consists of functions f in $L^1(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sup_{r>0} |\phi_r * f(x)| dx$$

is finite, where $\phi_r(x) = r^{-n}\phi(r^{-1}x)$ for $r > 0$ and ϕ is a smooth function on $\mathbb{R}^n$ with compact support in a unit ball with center at the origin $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$. The definition does not depend on choice of a function ϕ.

We will use the Hardy space on the exterior domain Ω.

Definition 2.2. (Hardy space on Ω) *A function $f \in L^1(\Omega)$ is said to be in $\mathcal{H}^1(\Omega)$ if and only if $\widetilde{f} \in \mathcal{H}^1(R^n)$, where f is extended to all of $\mathbb{R}^n$ by letting $f = 0$ outside Ω; this extension of f is denoted by $\widetilde{f}$. A norm on $\mathcal{H}^1(\Omega)$ is defined to be*

$$\|f\|_{\mathcal{H}^1(\Omega)} = \|\widetilde{f}\|_{\mathcal{H}^1(\mathbb{R}^n)}.$$

Definition 2.3. (functions of bounded mean oscillation) *Let f be locally integrable in $\mathbb{R}^n$, that is, $f \in L^1_{loc}(\mathbb{R}^n)$. We say that f is of bounded mean oscillation (abbreviated as BMO) if*

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B |f - (f)_B| dx < \infty,$$

where the supremum ranges over all finite balls $B \subset \mathbb{R}^n$, $|B|$ is the n-dimensional Lebesgue measure of B, and $(f)_B$ denotes the integral mean of f over B, namely $(f)_B = \frac{1}{|B|} \int_B f(x)dx$.

The class of functions of BMO, modulo constants, is a Banach space with the norm $\|\cdot\|_{BMO}$ defined above.

We will prepare the decisive Fefferman-Stein inequality, which means the duality between $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, $(\mathcal{H}^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$. For the proof, see [4].

Theorem 2.4. (Fefferman-Stein inequality) *There is a positive constant C depending only on n such that if $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, then*

$$\left| \int_{\mathbb{R}^n} fg\, dx \right| \le C\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \|g\|_{BMO(\mathbb{R}^n)}.$$

Now we recall the existence of a solution in the energy class, defined in the following. We use the usual Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ with norm

$$\|f\|_{H^1(\Omega)} = \sqrt{\|f\|^2 + \|\nabla f\|^2}, \qquad \|f\|_{H_0^1(\Omega)} = \|\nabla f\|.$$

Definition 2.5. *A function u defined on $(0, \infty) \times \Omega$ is called a weak solution to* (1.1)*,* (1.2)*, and* (1.3) *if u belongs to the energy class $C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ and satisfies the wave equation* (1.1) *in the distribution sense and the initial conditions* (1.2) *and* (1.3) *in the trace sense, respectively.*

Theorem 2.6. *For each $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique solution $u \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ to the problem* (1.1), (1.2), *and* (1.3) *such that*

$$\tfrac{1}{2}\|\nabla u(t)\|^2 + \tfrac{1}{2}\|u_t(t)\|^2 = \tfrac{1}{2}\|\nabla u_0\|^2 + \tfrac{1}{2}\|u_1\|^2 \tag{2.1}$$

holds for any $t \ge 0$.

Our main theorem is the following:

Theorem 2.7. *Suppose that the initial data (u_0, u_1) belongs to $H_0^1(\Omega) \times L^2(\Omega)$ and further $u_1 \in \mathcal{H}^1(\Omega)$. Then, the solution u to the problem* (1.1), (1.2), *and* (1.3) *satisfies*

$$\|u(t)\|^2 \le \|u_0\|^2 + C\|u_1\|^2_{\mathcal{H}^1(\Omega)} \tag{2.2}$$

for all $t \ge 0$ with a certain constant $C > 0$.

We will prove our main theorem, Theorem 2.7.

Proof of Theorem 2.7. Set

$$w(t, x) = \int_0^t u(s, x) ds.$$

Then, since $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ and so $\|\Delta w(t)\| = \|w_{tt}(t)\| = \|u_t(t)\|$, which are locally integrable in $(0,\infty)$, we see from the energy estimate (2.1) that $w \in C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$ satisfies

$$w_{tt} - \Delta w = u_1 \quad \text{in} \quad (0,\infty)\times\Omega, \tag{2.3}$$

$$w = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{2.4}$$

$$w = 0,\ w_t = u_0 \quad \text{on} \quad \{t=0\}\times\Omega, \tag{2.5}$$

so that

$$\tfrac{1}{2}\|w_t(t)\|^2 + \tfrac{1}{2}\|\nabla w(t)\|^2 = \tfrac{1}{2}\|u_0\|^2 + \int_0^t (u_1, w_s(s))ds.$$

Hence, noting again that $w_t = u$, one has

$$\begin{aligned}\tfrac{1}{2}\|u(t)\|^2 + \tfrac{1}{2}\|\nabla w(t)\|^2 &= \tfrac{1}{2}\|u_0\|^2 + \int_0^t \frac{d}{ds}(u_1, w(s))ds \\ &= \tfrac{1}{2}\|u_0\|^2 + (u_1, w(t)).\end{aligned} \tag{2.6}$$

We now make an estimation of the last term of (2.6), which is the main part of our argument. For this purpose, we extend the functions u_1 and w to all of $\mathbb{R}^2$ by letting $u_1 = 0$ and $w(t) = 0$, $0 \le t < \infty$, outside Ω and denote their extensions by $\widetilde{u}_1$ and $\widetilde{w}(t)$, respectively. Then $\widetilde{w} \in C^1([0,\infty); H^1(\mathbb{R}^2))$, since $u \in C([0,\infty); H_0^1(\Omega))$, and thus $w \in C^1([0,\infty); H_0^1(\Omega))$. The last term of (2.6) is estimated by Theorem 2.4 as

$$|(u_1, w(t))| = |(\widetilde{u}_1, \tilde{w}(t))| \le C\|\widetilde{u}_1\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\widetilde{w}(t)\|_{BMO(\mathbb{R}^2)}. \tag{2.7}$$

Employing the Poincaré inequality, Theorem 5.1 in the Appendix, we deduce

$$\|\widetilde{w}(t)\|_{BMO(\mathbb{R}^2)} \le C\,\|\nabla w(t)\|. \tag{2.8}$$

In fact, noting that $\widetilde{w} \in H^1(\mathbb{R}^2)$ on all time-slices, we have by the Hölder inequality

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r} |\widetilde{w}(t) - (\widetilde{w}(t))_{B_r}|dy &\le \Big(\frac{1}{|B_r|}\int_{B_r} |\widetilde{w}(t) - (\widetilde{w}(t))_{B_r}|^2 dy\Big)^{1/2} \\ &\le \Big(\frac{1}{|B_r|}C(\mathrm{diam}(B_r))^2\int_{B_r} |\nabla\widetilde{w}(t)|^2 dy\Big)^{1/2} \\ &\le C\Big(\int_{B_r\cap\Omega} |\nabla\widetilde{w}(t)|^2 dy\Big)^{1/2} \le C\Big(\int_\Omega |\nabla w(t)|^2 dy\Big)^{1/2}\end{aligned}$$

and, by definition of extension and $\mathcal{H}^1(\Omega)$,

$$\|\widetilde{u}_1\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|u_1\|_{\mathcal{H}^1(\Omega)}. \tag{2.9}$$

Gahtering (2.7), (2.8), and (2.9), we have by Cauchy's inequality

$$\begin{aligned} |(u_1, w(t))| &\leq C\|u_1\|_{\mathcal{H}^1(\Omega)}\|\nabla w(t)\| \\ &\leq \frac{C}{2\varepsilon}\|u_1\|^2_{\mathcal{H}^1(\Omega)} + \frac{\varepsilon}{2}\|\nabla w(t)\|^2, \end{aligned} \tag{2.10}$$

where the positive constant C depends on the ones as in Theorem 2.4 and (2.8), and $\varepsilon > 0$ is taken arbitrarily and fixed. Thus, it follows from (2.6) and (2.10) that

$$\tfrac{1}{2}\|u(t)\|^2 + \frac{1-\varepsilon}{2}\|\nabla w(t)\|^2 \leq \tfrac{1}{2}\|u_0\|^2 + \frac{C}{2\varepsilon}\|u_1\|^2_{\mathcal{H}^1(\Omega)}.$$

Choosing $\varepsilon > 0$ to be so small that $\varepsilon < 1$, one has

$$\|u(t)\|^2 \leq \|u_0\|^2 + C\varepsilon^{-1}\|u_1\|^2_{\mathcal{H}^1(\Omega)},$$

which completes the proof of Theorem 2.7. □

As an application of our main theorem, Theorem 2.7, we state a decay estimate of the local energy of a solution to (1.1), (1.2), and (1.3) in the energy class (refer to [8], [9], and [12]).

Theorem 2.8. *Assume that $\partial\Omega$ is star-shaped with respect to the origin. Let $R > 0$ be arbitrarily fixed such that $\partial\Omega \subset B_R(0)$. Then, for each $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ satisfying* $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega(R) = \Omega \cap B_R(0)$ *and furthermore $u_1 \in \mathcal{H}^1(\Omega)$, the weak solution $u(t,x)$ constructed in Theorem* 2.6 *to* (1.1), (1.2), *and* (1.3) *satisfies*

$$E_{\Omega(R)}(t) \leq C\,E(0)\,(t-R)^{-1} \tag{2.11}$$

for all $t > R$, where the positive constant C depends only on the initial data (u_0, u_1).

As is well known, the finite propagation property of the wave equation implies that if the initial data $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ has a compact support, that is, $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega(R)$, then we have

$$\operatorname{supp} u(t) \subset \Omega(R+t) \tag{2.12}$$

for each $t \geq 0$. See Proposition 5.2 in the Appendix.

Proof of Theorem 2.8. In the following, if necessary, as usual, we make a regularization by the mollifier to justify the calculations, and so we will proceed with the formal computation. Multiplying the equation (1.1) by

$tu_t + x \cdot \nabla u$, we calculate

$$\begin{aligned}
&(u_{tt} - \Delta u)(t\,u_t + x \cdot \nabla u) \\
&= \tfrac{1}{2}\left(\partial_t(t\,|u_t|^2) - |u_t|^2\right) - \left(\nabla \cdot (\nabla u\, t\, u_t) - \tfrac{1}{2}\left(\partial_t(t\,|\nabla u|^2) - |\nabla u|^2\right)\right) \\
&\quad + \partial_t(u_t\, x \cdot \nabla u) - \tfrac{1}{2}\left(\nabla \cdot (x\,|u_t|^2) - 2|u_t|^2\right) \\
&\qquad - \left(\nabla \cdot (\nabla u\, x \cdot \nabla u) - |\nabla u|^2 - \tfrac{1}{2}\left(\nabla \cdot (x\,|\nabla u|^2) - 2|\nabla u|^2\right)\right) \\
&= \tfrac{1}{2}\partial_t t\left(|u_t|^2 + |\nabla u|^2\right) - \nabla \cdot (\nabla u\, t\, u_t) + \partial_t(u_t\, x \cdot \nabla u) - \tfrac{1}{2}\nabla \cdot (x\,|u_t|^2) \\
&\quad - \nabla \cdot (\nabla u\, x \cdot \nabla u) + \tfrac{1}{2}\nabla \cdot (x\,|\nabla u|^2) + \tfrac{1}{2}\left(|u_t|^2 - |\nabla u|^2\right) \\
&= \tfrac{1}{2}\partial_t t\left(|u_t|^2 + |\nabla u|^2\right) - \nabla \cdot (\nabla u\, t\, u_t) + \partial_t(u_t\, x \cdot \nabla u) - \tfrac{1}{2}\nabla \cdot (x\,|u_t|^2) \\
&\quad - \nabla \cdot (\nabla u\, x \cdot \nabla u) + \tfrac{1}{2}\nabla \cdot (x\,|\nabla u|^2) \\
&\qquad + \tfrac{1}{2}\left(\partial_t(u_t\, u) - \nabla \cdot (\nabla u\, u) - (u_{tt} - \Delta u)\, u\right). \qquad (2.13)
\end{aligned}$$

Integrating (2.13) over $(0,t) \times \Omega$, we find from Gauss's divergence theorem that the following identity holds:

$$\begin{aligned}
&tE(t) - (x \cdot \nabla u_0, u_1) - \tfrac{1}{2}(u_0, u_1) \qquad (2.14) \\
&= \int_0^t \int_{\partial\Omega} \nu \cdot \nabla u\, x \cdot \nabla u - \tfrac{1}{2}\nu \cdot x |\nabla u|^2 \, dS ds - (x \cdot \nabla u(t), u_t(t)) - \tfrac{1}{2}(u(t), u_t(t)),
\end{aligned}$$

where ν is the unit outward normal vector field to $\partial\Omega$. Since $u(t) = 0$ on $\partial\Omega$ in the trace sense for any t, and thus $\tau \cdot \nabla u = 0$ on $\partial\Omega$ for any t, we find

$$\begin{aligned}
&\int_0^t \int_{\partial\Omega} \nu \cdot \nabla u\, x \cdot \nabla u - \tfrac{1}{2}\nu \cdot x\,|\nabla u|^2 \, dS\, ds \\
&= \int_0^t \int_{\partial\Omega} \nu \cdot \nabla u\,(\nu \cdot x)\,\nu \cdot \nabla u - \tfrac{1}{2}\nu \cdot x\,|\nu \cdot \nabla u|^2 \, dS\, ds \\
&= \int_0^t \int_{\partial\Omega} \tfrac{1}{2}\nu \cdot x\,|\nu \cdot \nabla u|^2 \, dS\, ds, \qquad (2.15)
\end{aligned}$$

where dS in the length element of $\partial\Omega$ and τ is the unit counterclockwise tangent vector field to $\partial\Omega$. Here, since $\partial\Omega$ is star-shaped with respect to the origin, we have

$$\int_0^t \int_{\partial\Omega} (\nu \cdot x)\,|\nu \cdot \nabla u|^2 \, dS\, ds \leq 0$$

so that the identities (2.14) and (2.15) yield

$$tE(t) - (x \cdot \nabla u_0, u_1) - \tfrac{1}{2}(u_0, u_1) \leq -(x \cdot \nabla u(t), u_t(t)) - \tfrac{1}{2}(u(t), u_t(t)). \quad (2.16)$$

Thus, we have only to estimate the first and second terms in the right-hand side of (2.16). Using the finite propagation (2.12) of the support, we have

$$
\begin{aligned}
&|(x\cdot\nabla u(t),u_t(t))| \\
&\le \int_{\Omega(R)} |x||\nabla u(t)||u_t(t)|\,dx + \int_{\Omega(R+t)\setminus\Omega(R)} |x||\nabla u(t)||u_t(t)|\,dx \\
&\le R\int_{\Omega(R)} |\nabla u(t)||u_t(t)|\,dx + (R+t)\int_{\Omega(R+t)\setminus\Omega(R)} |\nabla u(t)||u_t(t)|\,dx \\
&\le \frac{R}{2}\int_{\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx \\
&\quad + \frac{R+t}{2}\int_{\Omega(R+t)\setminus\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx \\
&\le \frac{R}{2}\int_{\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx \\
&\quad + \frac{R}{2}\int_{\Omega} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx + \frac{t}{2}\int_{\Omega(R+t)\setminus\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx \\
&= RE_{\Omega(R)}(t) + RE(t) + \frac{t}{2}\int_{\Omega(R+t)\setminus\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx. \qquad (2.17)
\end{aligned}
$$

Therefore, it follows from (2.16) and (2.17) that

$$
\begin{aligned}
tE(t) &\le (x\cdot\nabla u_0,u_1) + \tfrac{1}{2}(u_0,u_1) + \tfrac{1}{2}|(u(t),u_t(t))| + RE_{\Omega(R)}(t) \\
&\quad + RE(0) + \frac{t}{2}\int_{\Omega(R+t)\setminus\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx, \qquad (2.18)
\end{aligned}
$$

where we have used the energy conservation (2.1), $E(t)=E(0)$. From (2.12) again, it follows that

$$
\begin{aligned}
tE(t) &= \frac{t}{2}\int_{\Omega} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx \\
&= \frac{t}{2}\int_{\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx \\
&\quad + \frac{t}{2}\int_{\Omega(R+t)\setminus\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx \\
&= tE_{\Omega(R)}(t) + \frac{t}{2}\int_{\Omega(R+t)\setminus\Omega(R)} \left(|\nabla u(t)|^2+|u_t(t)|^2\right)\,dx. \qquad (2.19)
\end{aligned}
$$

Hence, combining (2.18) with (2.19), we have

$$(t-R)E_{\Omega(R)}(t) \le (x\cdot\nabla u_0, u_1) + \tfrac{1}{2}(u_0,u_1) + RE(0) + \tfrac{1}{2}|(u(t),u_t(t))|. \quad (2.20)$$

Now, we employ our main theorem Theorem 2.7 to evaluate the last term in the right-hand side of (2.20). We will emphasize that this part is crucial in deriving the local energy decay. By Theorem 2.7 one can estimate

$$|(u(t),u_t(t))| \le \tfrac{1}{2}\|u_t(t)\|^2 + \tfrac{1}{2}\|u(t)\|^2 \le E(0) + \tfrac{1}{2}\big(\|u_0\|^2 + C\|u_1\|^2_{\mathcal{H}^1(\Omega)}\big).$$

Thus, we arrive at

$$\begin{aligned}(t-R)E_{\Omega(R)}(t) &\le (x\cdot\nabla u_0, u_1) + \tfrac{1}{2}(u_0,u_1) + R\,E(0)\\ &\quad + \tfrac{1}{2}\left\{E(0) + \tfrac{1}{2}\left(\|u_0\|^2 + C\|u_1\|^2_{\mathcal{H}^1(\Omega)}\right)\right\}\\ &\equiv C(R)\,E(0),\end{aligned}$$

which completes the proof of Theorem 2.8. □

3. Dissipative wave equation

Our next result is concerned with a decay of solutions for the linear dissipative wave equation

$$u_{tt} - \Delta u + u_t = 0 \qquad \text{in} \quad (0,\infty)\times\Omega, \quad (3.1)$$

$$u = 0 \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \quad (3.2)$$

$$u = u_0, \quad u_t = u_1 \qquad \text{on} \quad \{t=0\}\times\Omega. \quad (3.3)$$

Now we recall the existence of a solution in the energy class to the problem (3.1), (3.2), and (3.3).

Theorem 3.1. *For each* $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, *there exists a unique solution* $u \in C([0,\infty);H_0^1(\Omega))\cap C^1([0,\infty);L^2(\Omega))$ *to the problem* (3.1), (3.2), *and* (3.3) *such that*

$$\tfrac{1}{2}\|\nabla u(t)\|^2 + \tfrac{1}{2}\|u_t(t)\|^2 + \int_0^t \|u_s(s)\|^2 ds = \tfrac{1}{2}\|\nabla u_0\|^2 + \tfrac{1}{2}\|u_1\|^2 \quad (3.4)$$

holds for all positive numbers $t<\infty$, *and*

$$\frac{d}{dt}E(t) = -\|u_t(t)\|^2, \quad (3.5)$$

$$\frac{d}{dt}(u_t(t),u(t)) + \|\nabla u(t)\|^2 + (u_t(t),u(t)) = \|u_t(t)\|^2 \quad (3.6)$$

hold for almost every positive number $t<\infty$.

Proof of Theorem 3.1. The proof is well-known, but we recall the outline for completeness. Of course, the energy equality (3.4) simply follows from integration of (3.5) on $(0, t)$. (3.5) and (3.6) are obtained from the use of test functions u_t and u in the equation (3.1), respectively, and spatial integration by parts, where, if necessary, we use a usual regularization by mollifier to justify the formal computation. □

The first result is on an optimal decay in L^2 of solutions in the energy class to the linear wave equations with dissipative term, (3.1), (3.2), and (3.3). The key point is also to use the Hardy space regularity of initial data as in Theorem 2.7. Compare with [6, Theorem 2.1], where it is assumed that $\|d\,(u_0+u_1)\|<\infty$ for $d(x)=|x|\log B|x|$ and some positive number B depending on Ω.

Theorem 3.2. *Suppose that the initial data (u_0,u_1) belongs to $H_0^1(\Omega)\times L^2(\Omega)$ and further $u_0+u_1\in\mathcal{H}^1(\Omega)$. Then, the solution u to the problem* (3.1)*,* (3.2)*, and* (3.3) *satisfies*

$$(1+t)\|u(t)\|^2\le C\left(\|u_0\|^2_{H^1(\Omega)}+\|u_1\|^2+\|u_0+u_1\|^2_{\mathcal{H}^1(\Omega)}\right) \tag{3.7}$$

for all $t\ge 0$ with a constant $C>0$ independent of $t\in[0,\infty)$.

The proof of Theorem 3.2 consists of a few lemmata.

Lemma 3.3. *Under the same assumptions as in Theorem* 3.2*, it holds that*

$$\tfrac12\|u(t)\|^2+\int_0^t\|u(s)\|^2ds\le\tfrac12\|u_0\|^2+C\|u_0+u_1\|^2_{\mathcal{H}^1(\Omega)}$$

for all $t>0$ with some constant $C>0$.

Proof of Lemma 3.3. Set

$$w(t,x)=\int_0^t u(s,x)ds.$$

Then, $w\in C^1([0,\infty);H_0^1(\Omega))\cap C^2([0,\infty);L^2(\Omega))$ satisfies

$$w_{tt}-\Delta w+w_t=u_1+u_0 \quad \text{in} \quad (0,\infty)\times\Omega, \tag{3.8}$$

$$w=0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{3.9}$$

$$w=0,\quad w_t=u_0 \quad \text{on} \quad \{t=0\}\times\Omega \tag{3.10}$$

and, by a test function w_t in (3.8),

$$\tfrac12\|w_t(t)\|^2+\tfrac12\|\nabla w(t)\|^2+\int_0^t\|w_t(s)\|^2ds=\tfrac12\|u_0\|^2+\int_0^t(u_1+u_0,w_s(s))ds. \tag{3.11}$$

Since

$$\int_0^t (u_1+u_0, w_s(s))ds = \int_0^t \frac{d}{ds}(u_1+u_0, w(s))ds = (u_1+u_0, w(t)),$$

it follows from the identity (3.11) that

$$\tfrac{1}{2}\|w_t(t)\|^2 + \tfrac{1}{2}\|\nabla w(t)\|^2 + \int_0^t \|w_s(s)\|^2 ds = \tfrac{1}{2}\|u_0\|^2 + (u_1+u_0, w(t)). \quad (3.12)$$

On the other hand, by an argument similar to (2.10) in the proof of Theorem 2.7, we find that

$$|(u_1+u_0, w(t))| \le \frac{C}{2\varepsilon}\|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)} + \frac{\varepsilon}{2}\|\nabla w(t)\|^2 \quad (3.13)$$

for $\varepsilon > 0$ with some constant $C > 0$. Thus, (3.12) and (3.13) yield

$$\tfrac{1}{2}\|w_t(t)\|^2 + \frac{1-\varepsilon}{2}\|\nabla w(t)\|^2 + \int_0^t \|w_t(s)\|^2 ds \le \tfrac{1}{2}\|u_0\|^2 + \frac{C}{2\varepsilon}\|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)}.$$

Hence, noting $w_t = u$ and choosing $\varepsilon = 1/2$ above, one has

$$\tfrac{1}{2}\|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds \le \tfrac{1}{2}\|u_0\|^2 + C\|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)}$$

with some constant $C > 0$. □

Now we make a decay estimate of total energy of solutions.

Lemma 3.4. *Under the assumptions as in Theorem* 3.2, *one has*

$$\|u(t)\|^2 \le 12E(0) + 4(u_0, u_1) + 2\|u_0\|^2, \quad (3.14)$$

$$(1+t)E(t) \le E(0) + J_0, \quad (3.15)$$

$$\int_0^t E(s)ds \le J_0, \quad (3.16)$$

where we let $J_0 = 2E(0) + \frac{1}{2}(u_0, u_1) + \frac{1}{4}\|u_0\|^2$ *and* $E(t)$ *is as in* (1.4).

Proof of Lemma 3.4. We find from (3.5) that $E(t)$ is monotone nonincreasing on $t \in (0, \infty)$ and

$$(1+t)\frac{d}{dt}E(t) \le 0,$$

which is integrated by parts over $[0, t]$ and yields

$$(1+t)E(t) - E(0) \le \int_0^t E(s)ds,$$

so that

$$\begin{aligned}(1+t)E(t) &\le \int_0^t E(s)ds + E(0)\\ &= \tfrac{1}{2}\int_0^t \|u_s(s)\|^2 ds + \tfrac{1}{2}\int_0^t \|\nabla u(s)\|^2 ds + E(0). \end{aligned} \tag{3.17}$$

To estimate the second term in the right-hand side of (3.17), we make an integration of the both sides of (3.6) over $(0,t)$ and use the Cauchy inequality

$$\begin{aligned}&\int_0^t \|\nabla u(s)\|^2 ds + \tfrac{1}{2}\|u(t)\|^2\\ &= \int_0^t \|u_s(s)\|^2 ds - (u_t(t), u(t)) + (u_0, u_1) + \tfrac{1}{2}\|u_0\|^2\\ &\le \int_0^t \|u_s(s)\|^2 ds + \left(\|u_t(t)\|^2 + \tfrac{1}{4}\|u(t)\|^2\right) + (u_0, u_1) + \tfrac{1}{2}\|u_0\|^2.\end{aligned}$$

This together with the energy identity (3.4) gives that

$$\int_0^t \|\nabla u(s)\|^2 ds + \frac{1}{4}\|u(t)\|^2 \le 3E(0) + (u_0, u_1) + \tfrac{1}{2}\|u_0\|^2. \tag{3.18}$$

Thus, we see from (3.4), (3.17), and (3.18) that

$$(1+t)E(t) \le \tfrac{1}{2}E(0) + \tfrac{1}{2}\left(3E(0) + (u_0, u_1) + \tfrac{1}{2}\|u_0\|^2\right) + E(0) = E(0) + J_0.$$

Furthermore, as above,

$$\int_0^t E(s)ds \le J_0. \qquad \square$$

Lemma 3.5. *Under the same assumptions as in Theorem* 3.2, *it holds that*

$$(1+t)\|u(t)\|^2 \le C\left(\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2\right) + \|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds$$

for all $t > 0$, *where* $C > 0$ *is a constant.*

Proof of Lemma 3.5. Multiplying both sides of (3.6) by s and integrating it over $(0,t)$, one has

$$\begin{aligned}&\int_0^t s\frac{d}{ds}(u_s(s), u(s))ds + \int_0^t s\|\nabla u(s)\|^2 ds + \tfrac{1}{2}\int_0^t s\frac{d}{ds}\|u(s)\|^2 ds\\ &= \int_0^t s\|u_s(s)\|^2 ds.\end{aligned}$$

Integration by parts gives that

$$\begin{aligned}\int_0^t s\frac{d}{ds}(u_s(s),u(s))ds &= t(u_t(t),u(t)) - \int_0^t \tfrac{1}{2}\frac{d}{ds}\|u(s)\|^2 ds\\ &= t(u_t(t),u(t)) - \tfrac{1}{2}\|u(t)\|^2 + \tfrac{1}{2}\|u_0\|^2\end{aligned}$$

and

$$\int_0^t s\frac{d}{ds}\|u(s)\|^2 ds = t\|u(t)\|^2 - \int_0^t \|u(s)\|^2 ds.$$

Thus, we get

$$\begin{aligned}&t(u_t(t),u(t)) + \tfrac{1}{2}\|u_0\|^2 + \int_0^t s\|\nabla u(s)\|^2 ds + \frac{t}{2}\|u(t)\|^2\\ &= \tfrac{1}{2}\|u(t)\|^2 + \int_0^t s\|u_s(s)\|^2 ds + \tfrac{1}{2}\int_0^t \|u(s)\|^2 ds. \qquad (3.19)\end{aligned}$$

From Hölder's and Cauchy's inequalities in (3.19), it follows that

$$\begin{aligned}&t\|u_t(t)\|^2 + \tfrac{1}{2}\int_0^t \|u(s)\|^2 ds + \int_0^t s\|u_s(s)\|^2 ds + \tfrac{1}{2}\|u(t)\|^2\\ &\geq \frac{t}{4}\|u(t)\|^2 + \int_0^t s\|\nabla u(s)\|^2 ds. \qquad (3.20)\end{aligned}$$

We will estimate the second term in the right-hand side of (3.20). Integrating by parts on t (3.16) of Lemma 3.4, one gets

$$J_0 \geq \int_0^t E(s)ds = tE(t) - \int_0^t s\frac{d}{ds}E(s)ds.$$

Since

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{d}{dt}\|\nabla u(t)\|^2 + \|u_t(t)\|^2 = (\nabla u_t(t),\nabla u(t)) + (u_{tt}(t),u_t(t))\\ &= -(u_t(t),\Delta u(t)) + (u_{tt}(t),u_t(t)) = (u_{tt}(t)-\Delta u(t),u_t(t))\\ &= (-u_t(t),u_t(t)) = -\|u_t(t)\|^2,\end{aligned}$$

we have

$$J_0 \geq tE(t) + \int_0^t s\|u_s(s)\|^2 ds. \qquad (3.21)$$

Noting that (3.21) yields $t\|u_t(t)\|^2 \le 2J_0$ and substituting it and (3.21) into (3.20), we obtain

$$\frac{t}{4}\|u(t)\|^2 \le 3J_0 + \|u(t)\|^2 + \tfrac{1}{2}\int_0^t \|u(s)\|^2 ds,$$

which completes the proof of Lemma 3.5 after choosing an appropriate positive constant C. □

Therefore, Theorem 3.2 is an immediate consequence of Lemmas 3.3 and 3.5.

At the end of this section we obtain the optimal decay of total energy of solutions to the linear wave equations with dissipative term. Compare with [6, Theorem 2.1].

Theorem 3.6. *Suppose that the initial data* (u_0, u_1) *belongs to* $H_0^1(\Omega) \times L^2(\Omega)$ *and further* $u_0 + u_1 \in \mathcal{H}^1(\Omega)$. *Then, the solution* u *to the problem* (3.1), (3.2), *and* (3.3) *satisfies*

$$(1+t)^2 E(t) \le C\left(E(0) + \|u_0\|^2 + \|u_0+u_1\|^2_{\mathcal{H}^1(\Omega)}\right). \tag{3.22}$$

Proof of Theorem 3.6. We have already obtained the key estimation. In fact, combining Lemma 3.3, (3.21), and (3.20) in the proof of Lemma 3.5 we have

$$\begin{aligned} 2\int_0^t sE(s)ds &\le 3J_0 + \|u(t)\|^2 + \tfrac{1}{2}\int_0^t \|u(s)\|^2 ds \\ &\le 3J_0 + \|u_0\|^2 + C\|u_0+u_1\|^2_{\mathcal{H}^1(\Omega)}. \end{aligned} \tag{3.23}$$

From (3.16) in Lemma 3.4 and (3.23) it follows that

$$\int_0^t (1+s)E(s)ds \le 4J_0 + \|u_0\|^2 + C\|u_0+u_1\|^2_{\mathcal{H}^1(\Omega)}. \tag{3.24}$$

By the energy identity (3.5) we find that

$$(1+t)^2 \frac{d}{dt}E(t) \le 0 \tag{3.25}$$

holds for almost every time $t < \infty$; a spatial integration by parts of (3.25) yields

$$0 \ge \int_0^t (1+s)^2 \frac{d}{ds}E(s)ds = (1+s)^2 E(s)\Big|_0^t - 2\int_0^t (1+s)E(s)ds,$$

and thus, by (3.24),

$$(1+t)^2E(t) \leq E(0) + 2\int_0^t (1+s)E(s)ds$$
$$\leq E(0) + 2\left(4J_0 + \|u_0\|^2 + C\|u_0+u_1\|^2_{\mathcal{H}^1(\Omega)}\right),$$

from which the conclusion follows. □

4. Application to the nonlinear equations in $\mathbb{R}^2$

In this final section, we study a decay estimation for a nonlinear dissipative wave equation on the two-dimensional exterior domain Ω, which is an application of our decay results for the linear equations as before. Our nonlinear wave equation of dissipative type considered here is

$$u_{tt} - \Delta u + u_t = \frac{|u|^{p-1}u}{W} \quad \text{on} \quad (0,\infty)\times\Omega, \tag{4.1}$$

$$u = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{4.2}$$

$$u = u_0, \quad u_t = u_1 \quad \text{on} \quad \{t=0\}\times\Omega, \tag{4.3}$$

where $2 < p < \infty$ and the function $W = W(x)$ is defined by

$$W(x) = 1 + |x|\log(B|x|), \qquad x \in \mathbb{R}^2, \tag{4.4}$$

with a positive constant B such that $B|x| \geq 2$ for any $x \in \Omega$. Our result on decay estimations for the nonlinear equations is the following:

Theorem 4.1. *Let $2 < p < \infty$ and initial data $(u_0, u_1) \in H^1(\Omega)\times L^2(\Omega)$ satisfy $u_0 + u_1 \in \mathcal{H}^1(\Omega)$ and*

$$\|u_0\|_{H^1} + \|u_1\|_2 + \|u_0+u_1\|_{\mathcal{H}^1(\Omega)} << 1.$$

Then, there exists a weak solution $u \in C([0,\infty); H^1_0(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ to the problem (4.1), (4.2), *and* (4.3) *such that*

$$\|u(t)\|_2 \leq C(1+t)^{-1/2}, \quad E(t) \leq C(1+t)^{-2} \tag{4.5}$$

hold for any $t \geq 0$ and a positive constant C.

In [6], the same decay estimates are obtained for the semilinear dissipative wave equation without the weight W in the nonlinear term and with initial data of compact support. Instead the equation has the weight and the Hardy space regularity of initial data is assumed here. In two dimensions, a polynomial nonlinearity such as in (4.1) is subcritical for the energy class by the Sobolev-type inequality and the natural critical nonlinearity is an exponential one (see [5] and [1] for the defocusing and focusing cases, respectively). Here we would like to emphasize an effect of the Hardy space regularity of initial data. The nonlinear wave equations on two-dimensional exterior domains will be studied elsewhere, as well as the nonlinear term of critical growth for the energy class (for the Cauchy problem see, for example, [1, 5], [16], and the references therein).

The existence immediately follows from the local-in-time existence result and the usual extension argument, once the decay estimates of L^2 and total energy of solutions are obtained for the nonlinear equation (4.1), (4.2), and (4.3) (see [6, Proposition 2.2] and refer to [17] and [16, Chapter 6]). Thus we will study the decay estimates of L^2 and total energy of solutions. We proceed with our decay estimations along the lines of those in Section 3 for the linear dissipative wave equations (3.1), (3.2), and (3.3). Of course, the main task here is to control the nonlinear terms.

First we recall a special case of the so-called multiplicative inequality and the Hardy inequality. Throughout this section, the usual function space $L^q(\Omega)$ of q-integrable functions on Ω for $q \geq 1$ is also used with its norm

$$\|f\|_q = \|f\|_{L^q(\Omega)} = \Big(\int_\Omega |f|^q dx\Big)^{1/q}.$$

Lemma 4.2. (The Sobolev and Hardy inequalities) *For any $f \in H_0^1(\Omega)$,*

$$\|f\|_q \leq C_s \,\|\nabla f\|^\theta \,\|f\|^{1-\theta} \tag{4.6}$$

holds for any $q \geq 2$ and a positive number $\theta < 1$ such that $\theta = 1 - 2/q$, where a positive constant C_s depends only on q. It holds that

$$\Big\|\frac{f}{W}\Big\| \leq C_h \,\|\nabla f\|, \tag{4.7}$$

where C_h is a uniform positive constant.

We proceed with the continuity method for the proof of our decay estimates. Let $u \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$ be a weak solution to the problem (4.1), (4.2), and (4.3) on a time interval $[0,T)$ for a positive number T. By the local-in-time existence and the continuity in $H_0^1(\Omega)$ of the solution $u(t)$ in $C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$, we suppose that, for some positive number K,

$$\begin{aligned} &\|u(t)\| \leq K\, I_0\, (1+t)^{-1/2}, \\ &\|\nabla u(t)\| + \|u_t(t)\| \leq K\, I_0\, (1+t)^{-1} \end{aligned} \tag{4.8}$$

hold for any $t \in [0,T)$, where we put

$$I_0 = \|u_0\|_{H^1(\Omega)} + \|u_1\| + \|u_0 + u_1\|_{\mathcal{H}^1(\Omega)}. \tag{4.9}$$

We let the total energy $\mathcal{E}(t)$ with an additional term $\mathcal{N}(t)$ from the nonlinear term

$$\mathcal{E}(t) = E(t) - \mathcal{N}(t) = \tfrac{1}{2}\left(\|\nabla u(t)\|^2 + \|u_t(t)\|^2\right) - \frac{1}{p+1}\left\|\frac{u}{W^{1/(p+1)}}\right\|_{p+1}^{p+1}. \tag{4.10}$$

Lemma 4.3.

$$\mathcal{E}(t) + \int_0^t \|u_s(s)\|^2 ds = \mathcal{E}(0) \tag{4.11}$$

and

$$\frac{d}{dt}\mathcal{E}(t) = -\|u_t(t)\|^2, \tag{4.12}$$

$$\frac{d}{dt}(u_t(t), u(t)) + \|\nabla u(t)\|^2 + (u_t(t), u(t)) = \|u_t(t)\|^2 + (p+1)\mathcal{N}(t) \tag{4.13}$$

hold for almost every positive number $t < \infty$.

Proof. As before, if necessary, we make a usual regularization by mollifier to verify the formal calculation. Test functions u_t and u in the equation (4.1) are used to have (4.12) and (4.13). Here we note that the term $\mathcal{N}(t)$ is bounded by the H^1-norm of $u(t)$ (see lemma below). □

First we study a decay estimation in L^2.

Theorem 4.4.

$$\begin{aligned}(1+t)\|u(t)\|^2 \le C &\left(\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2 + \|u_0 + u_1\|^2_{\mathcal{H}^1(\Omega)}\right)\\ &+ C\left(C_h^2 C_s^2 K^{2p} I_0^{2p} + C_s^{p+1} K^{p+1} I_0^{p+1}\right)\end{aligned} \tag{4.14}$$

holds for all $t \ge 0$ with a positive constant C depending only on p.

The proof of Theorem 4.4 consists of a few lemmata.

Lemma 4.5.

$$\begin{aligned}\tfrac{1}{2}\|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds \le \tfrac{1}{2}\|u_0\|^2 + C\|u_0 + u_1\|^2_{\mathcal{H}^1(\Omega)}&\\ + C\left(C_h^2 C_s^2 K^{2p} I_0^{2p} + C_s^{p+1} K^{p+1} I_0^{p+1}\right)&\end{aligned} \tag{4.15}$$

holds for all $t > 0$ with a positive constant C.

Proof of Lemma 4.5. We proceed the proof in exactly same way as in that of Lemma 3.3. Letting

$$w(t,x) = \int_0^t u(s,x) ds,$$

then $w \in C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$ satisfies

$$w_{tt} - \Delta w + w_t = \int_0^t \frac{|u(s)|^{p-1}u(s)}{W} ds \ + u_1 + u_0 \quad \text{in} \quad (0,\infty)\times\Omega, \tag{4.16}$$

$$w = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{4.17}$$

$$w = 0, \quad w_t = u_0 \quad \text{on} \quad \{t=0\}\times\Omega, \tag{4.18}$$

and thus, similarly as in (3.13) in the proof of Lemma 3.3,

$$\begin{aligned} &\tfrac{1}{2}\|u(t)\|^2 + \tfrac{1}{2}\|\nabla w(t)\|^2 + \int_0^t \|u(s)\|^2 ds \\ &= \tfrac{1}{2}\|u_0\|^2 + C\,\|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)} + \Big| \int_0^t \Big(w_s(s), \int_0^s \frac{|u(\tau)|^{p-1}u(\tau)}{W} d\tau \Big) ds \Big|. \end{aligned} \tag{4.19}$$

The last term coming from the nonlinear term is estimated by Fubini's theorem and an integration by parts on s:

$$\begin{aligned} &\int_\Omega dx \int_0^t w_s(s) \int_0^s \frac{|u(\tau)|^{p-1}u(\tau)}{W} d\tau\, ds \\ &= \int_\Omega dx \ w(s) \int_0^s \frac{|u(\tau)|^{p-1}u(\tau)}{W} d\tau \Big|_0^t - \int_0^t w(s) \frac{|u(s)|^{p-1}u(s)}{W} ds \\ &= \Big(w(t), \int_0^t \frac{|u(\tau)|^{p-1}u(\tau)}{W} d\tau \Big) - \int_0^t \Big(w(s), \frac{|u(s)|^{p-1}u(s)}{W} \Big) ds. \end{aligned} \tag{4.20}$$

The first term of (4.20) is bounded by

$$\begin{aligned} \Big\| \int_0^t |u(s)|^p ds \Big\| \Big\| \frac{w(t)}{W} \Big\| &\le C_h \int_0^t \|u(s)\|_{2p}^p ds\, \|\nabla w(t)\| \\ &\le C_h C_s^p \int_0^t \|\nabla u(s)\|^{p-1} \|u(s)\| ds\, \|\nabla w(t)\| \\ &\le C_h C_s^p K^p I_0^p \int_0^t (1+s)^{-p+1/2} ds\, \|\nabla w(t)\| \\ &\le \big(p - \tfrac{3}{2}\big)^{-1} C_h C_s^p K^p I_0^p \|\nabla w(t)\|, \end{aligned} \tag{4.21}$$

where we use Jensen's inequality and Hardy's inequality (4.7) in Lemma 4.2 in the first inequality, and the Sobolev inequality (4.6) in Lemma 4.2 with $\theta = 1 - 1/p$ in the second inequality and, lastly, we substitute (4.8). The second term of (4.20) is estimated as

$$\int_0^t (|u(s)|^p,\, |w(s)|)\, ds \le \int_0^t \|u(s)\|_{p+1}^p \|w(s)\|_{p+1} ds$$
$$\le \int_0^t \|u(s)\|_{p+1}^p \int_0^s \|u(\tau)\|_{p+1} d\tau\, ds$$
$$\le C_s^{p+1} K^{p+1} I_0^{p+1} \int_0^t (1+s)^{-p^2/(p+1)} \int_0^s (1+\tau)^{-p/(p+1)} d\tau\, ds$$
$$\le \frac{p+1}{p-2} C_s^{p+1} K^{p+1} I_0^{p+1}, \tag{4.22}$$

where we firstly use the Hölder inequality, secondly, the Jensen inequality, thirdly, the Sobolev inequality (4.6) in Lemma 4.2 with $\theta = 1 - 2/(p+1)$ and, lastly, (4.8). Finally, gathering (4.21) and (4.22) in (4.19) with (4.20) and using the Cauchy inequality we arrive at our desired estimation. Here we emphasize that we crucially make use of the second term in the left-hand side of (4.19), which is missed in the linear case. □

Lemma 4.6. *It holds that*

$$(1+t)E(t) \le E(0) + J_0 + (3+t)\mathcal{N}(t) + \frac{p-1}{2} \int_0^t \mathcal{N}(s) ds, \tag{4.23}$$

$$\int_0^t E(s) ds \le J_0 + 2\mathcal{N}(t) + \frac{p+1}{2} \int_0^t \mathcal{N}(s) ds, \tag{4.24}$$

where, as before, $J_0 = 2E(0) + \frac{1}{2}(u_0, u_1) + \frac{1}{4}\|u_0\|^2$, $E(t)$ *is as in* (1.4), *and* $\mathcal{N}(t)$ *as in* (4.10).

Proof of Lemma 4.6. The proof is done exactly similarly to that of Lemma 3.4. A multiplication of (4.12) by $(1+t)$ and an integration of the resulting equality by parts over $(0, t)$ give that

$$(1+t)\mathcal{E}(t) \le \int_0^t \mathcal{E}(s) ds + \mathcal{E}(0),$$

and thus

$$(1+t)E(t) \le \mathcal{E}(0) + \int_0^t E(s) ds - \int_0^t \mathcal{N}(s) ds + (1+t)\mathcal{N}(t). \tag{4.25}$$

We now estimate the second term in the right-hand side of (4.25). By (4.11)

$$E(t) + \int_0^t \|u_s(s)\|^2 ds \le \mathcal{N}(t) + E(0). \tag{4.26}$$

Similarly as in (3.18) in the proof of Lemma 3.4, we obtain from integration of both sides of (4.13) over $(0,t)$ and (4.12)

$$\int_0^t \|\nabla u(s)\|^2 ds + \frac{1}{4}\|u(t)\|^2 \\ \leq 3\mathcal{E}(0) + (u_0, u_1) + \tfrac{1}{2}\|u_0\|^2 + 3\mathcal{N}(t) + (p+1)\int_0^t \mathcal{N}(s) ds. \tag{4.27}$$

Thus, (4.26) and (4.27) give (4.24), and a substituition of (4.24) into (4.25) yields (4.23). □

Proof of Theorem 4.4. Multiplying the both sides of (4.13) by s and integrating it over $(0,t)$, we have

$$\int_0^t s\frac{d}{ds}(u_s(s), u(s)) ds + \int_0^t s\|\nabla u(s)\|^2 ds + \tfrac{1}{2}\int_0^t s\frac{d}{ds}\|u(s)\|^2 ds \\ = \int_0^t s\|u_s(s)\|^2 ds + \int_0^t s\left\|\frac{u}{W^{1/(p+1)}}\right\|_{p+1}^{p+1} ds. \tag{4.28}$$

Similarly as in the proof of Lemma 3.5 we obtain from (4.28)

$$\frac{t}{4}\|u(t)\|^2 + \tfrac{1}{2}\|u_0\|^2 + \int_0^t s\|\nabla u(s)\|^2 ds \leq t\|u_t(t)\|^2 \tag{4.29} \\ + \int_0^t s\|u_s(s)\|^2 ds + \tfrac{1}{2}\|u(t)\|^2 + \tfrac{1}{2}\int_0^t \|u(s)\|^2 ds + \int_0^t s\left\|\frac{u}{W^{1/(p+1)}}\right\|_{p+1}^{p+1} ds.$$

We will estimate the second term in the right-hand side of (4.29). By (4.24) of Lemma 4.6,

$$\int_0^t \mathcal{E}(s) ds \leq J_0 + 2\mathcal{N}(t) + \big(\frac{p+1}{2} - 1\big)\int_0^t \mathcal{N}(s) ds,$$

and integration of it by parts on t gives that

$$J_0 + 2\mathcal{N}(t) + \frac{p-1}{2}\int_0^t \mathcal{N}(s) ds \geq \int_0^t \mathcal{E}(s) ds = t\mathcal{E}(t) - \int_0^t s\frac{d}{ds}\mathcal{E}(s) ds \\ = t\mathcal{E}(t) + \int_0^t s\|u_s(s)\|^2 ds, \tag{4.30}$$

where we compute by integration by parts and the equation (4.1)

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt}\left(E(t)-\mathcal{N}(t)\right)\\ &= (\nabla u_t(t), \nabla u(t)) + (u_{tt}(t), u_t(t)) - \Big(\frac{|u(t)|^{p-1}u(t)}{W},\, u_t(t)\Big)\\ &= \Big(u_{tt}(t) - \Delta u(t) - \frac{|u(t)|^{p-1}u(t)}{W},\, u_t(t)\Big) = -\|u_t(t)\|^2.\end{aligned}$$

Therefore, we have by (4.30)

$$tE(t) + \int_0^t s\|u_s(s)\|^2 ds \le J_0 + (2+t)\mathcal{N}(t) + \frac{p-1}{2}\int_0^t \mathcal{N}(s)ds. \quad (4.31)$$

From substitution of (4.31) into (4.29) and (4.15) in Lemma 4.5, it follows that

$$\begin{aligned}&\frac{t}{4}\|u(t)\|^2 + 2\int_0^t sE(s)ds \le \tfrac{1}{2}\|u_0\|^2 + C\,\|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)}\\ &\quad + C\left(C_h^2 C_s^{2p} K^{2p} I_0^{2p} + C_s^{p+1}K^{p+1}I_0^{p+1}\right)\\ &\qquad + 3J_0 + 3(2+t)\mathcal{N}(t) + \frac{3(p-1)}{2}\int_0^t \mathcal{N}(s)ds + (p+1)\int_0^t s\mathcal{N}(s)ds\\ &\le \|u_0\|^2 + C\,\|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)} + 4E(0) + (u_1,u_0)\\ &\qquad + C\,\left(C_h^2 C_s^{2p} K^{2p} I_0^{2p} + C_s^{p+1}K^{p+1}I_0^{p+1}\right), \end{aligned} \quad (4.32)$$

where we compute by the Sobolev inequality (4.6) in Lemma 4.2 with $\theta = 1 - 2/(p+1)$ and (4.8)

$$\begin{aligned}&(2+t)\mathcal{N}(t) \le C\,C_s^{p+1}K^{p+1}I_0^{p+1}(1+t)^{1-p},\\ &\int_0^t \mathcal{N}(s)ds \le C\,C_s^{p+1}K^{p+1}I_0^{p+1}, \\ &\int_0^t s\mathcal{N}(s)ds \le C\,C_s^{p+1}K^{p+1}I_0^{p+1}.\end{aligned} \quad (4.33)$$

□

Next we make a decay estimation of the total energy of solutions to the nonlinear wave equations with dissipative term.

Theorem 4.7. *It holds that*

$$\begin{aligned}(1+t)^2E(t) &\le C\big(\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2 + \|u_0+u_1\|^2_{\mathcal{H}^1(\Omega)}\big)\\ &\qquad + C\big(C_h^2 C_s^{2p} K^{2p} I_0^{2p} + C_s^{p+1}K^{p+1}I_0^{p+1}\big).\end{aligned} \quad (4.34)$$

Proof of Theorem 4.7. From (4.24) in Lemma 4.6 and (4.32) in the proof of Theorem 4.4 it follows that

$$\int_0^t (1+s)E(s)ds \le C\big(\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2 + \|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)}\big) + C\big(C_h^2 C_s^{2p} K^{2p} I_0^{2p} + C_s^{p+1} K^{p+1} I_0^{p+1}\big). \tag{4.35}$$

Multiplying (4.12) by $(1+t)^2$ and integrating the resulting equality on $(0,t)$, we have

$$\begin{aligned}(1+t)^2 E(t) &\le \mathcal{E}(0) + 2\int_0^t (1+s)E(s)ds \\ &\qquad + (1+t)^2\mathcal{N}(t) + 2\int_0^t (1+s)\mathcal{N}(s)ds \\ &\le C\left(\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2 + \|u_1+u_0\|^2_{\mathcal{H}^1(\Omega)}\right) \\ &\qquad + C\left(C_h^2 C_s^{2p} K^{2p} I_0^{2p} + C_s^{p+1} K^{p+1} I_0^{p+1}\right),\end{aligned} \tag{4.36}$$

where we use (4.35) and the same estimations as in (4.33) by Lemma 4.2 and (4.8) and, in particular,

$$(1+t)^2\mathcal{N}(t) \le \frac{1}{p+1} C_s^{p+1} K^{p+1} I_0^{p+1} (1+t)^{2-p}. \qquad \square$$

Proof of (4.5) in Theorem 4.1. From (4.34) in Theorem 4.4 and (4.36) in Theorem 4.7, it follows that

$$\begin{aligned}&(1+t)^{-1/2}\|u(t)\| + (1+t)^{-1}\left(\|\nabla u(t)\| + \|u_t(t)\|\right) \\ &\le C\left(1 + C_h C_s^p K^p I_0^{p-1} + C_s^{(p+1)/2} K^{(p+1)/2} I_0^{(p-1)/2}\right) I_0.\end{aligned} \tag{4.37}$$

We choose the positive number K as $K \ge 4C$ for the positive constant C in (4.37) depending only on p, and I_0 to be so small that

$$C\left(C_h C_s^p K^p I_0^{p-1} + C_s^{(p+1)/2} K^{(p+1)/2} I_0^{(p-1)/2}\right) \le K/4$$

and then, by the local-in-time existence and the continuity in $H_0^1(\Omega)$ of the solution $u(t)$ in $C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega))$, we can take a positive number δ such that (4.8) holds for any $t \in [0, T+\delta]$. By contradiction, we see that the existence time T for which (4.8) holds has to be ∞. $\square$

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5. Appendix

Here we recall some facts.

Theorem 5.1. (Poincaré inequality) *Let D be a ball or cube in $\mathbb{R}^n$ and $u \in H^1(D)$. Then for every measurable set $E \subset D$ of positive measure we have*

$$\int_D |u(x) - (u)_E|^2\,dx \le 2^n(\mathrm{diam}(D))^2 \frac{|D|}{|E|} \int_D |\nabla u(x)|^2\,dx,$$

where $(u)_E = \frac{1}{|E|}\int_E u(x)\,dx$ is the integral mean of u on E.

For the proof see [10].

We recall the finite speed propagation of support of a solution u to the wave equation (1.1). Let $T > 0$ and the cone $C = \{(t,x) : R \le t \le T,\ |x| \le t\}$. See [3] for the proof.

Proposition 5.2. *If $u \equiv u_t \equiv 0$ on $\{t = R\} \times B_R(0)$, then $u \equiv 0$ within the cone C.*

Remark. We see in particular that if u is a solution of (1.1) with the initial conditions

$$u(0) = u_0, \qquad u_t(0) = u_1,$$

then $u(R, x_0)$ depends only upon the values of u_0 and u_1 within $B_R(0)$.

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