Differential and Integral Equations Volume 24, Numbers 9-10 (2011), 909–922

CLASSIFICATION OF SOLUTIONS OF POROUS MEDIUM EQUATION WITH LOCALIZED REACTION IN HIGHER SPACE DIMENSIONS

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(Submitted by: Takashi Suzuki)

Abstract. We consider the behavior of nonnegative solutions to the Cauchy problem of the porous medium equation with localized reaction term:

$$\begin{cases} u_t = \Delta(u^m) + a(x)u^p, & (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases}$$

where $m > 1$, $p > 0$, $a(x) \geq 0$ is a compactly supported function, and $u_0(x)$ is continuous, nonnegative but not identical with zero, and has compact support as well. We show the relationship between the occurrence of blow-ups and the exponents m and p: in two-dimensional space, all the solutions are globally defined if $0 < p \leq \frac{m+1}{2}$, and the solutions may blow up in finite time if $p \geq m$; in spaces higher than two-dimensional, all the solutions are global if $0 < p < m$, and there exist both global solutions and blow-up solutions if $p \geq m$. We also show that, for any solution, the intersection of its support and the support of $a(x)$ will be non-empty at some time.

1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta(u^m) + a(x)u^p, & (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

Accepted for publication: February 2011.
AMS Subject Classifications: 35K57, 35B44.

where integers $m > 1, p > 0$, the cut-off function $a(x) \geq 0$, the initial function $u_0(x)$ is continuous and nonnegative but not identical with zero, and both $a(x)$ and $u_0(x)$ are compactly supported.

The motivation for the following study lies in [2], which discovered the relationship between the behavior of nonnegative solutions to the problem and the exponents m and p when the space dimension $n = 1$. The result is that for

$$\begin{cases} u_t = (u^m)_{xx} + a(x)u^p, & (x,t) \in \mathbf{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.2}$$

whether the solutions blow up or not depends on m and p, as shown below:
(i) If $0 < p \leq \frac{m+1}{2}$, then all the solutions to (1.2) are globally defined;
(ii) If $\frac{m+1}{2} < p \leq m+1$, then all the solutions to (1.2) blow up in finite time;
(iii) If $p > m+1$, then both global solutions and blow-up solutions to (1.2) exist.

We call $p_0 = \frac{m+1}{2}$ and $p_C = m+1$ the *critical exponent for global existence* and the *Fujita exponent*, respectively. Based on these results, we hope to understand the two exponents for the multi-dimensional case.

For the Cauchy problem of the porous medium equation (PME, for short in the following) possessing a reaction term without the cut-off function $a(x)$ in space dimension higher than one, namely the case $a(x) \equiv 1$, Deng-Levine [1], Galaktionov-Vazquez [3] and Levine [5] have studied the role of exponents in blow-up problems: it has been discovered that $p_0 = 1$ and $p_C = m + \frac{2}{n}$. Levine [6] has discussed the relationship between the reaction term and the behavior of solutions more generally; Pinsky [7] has made a relevant study of the semilinear heat equation with localized reaction. However, just as is mentioned in [2], the relevant multidimensional problem for (1.1) is still "a subject of a future work." This paper has partially solved the problem, and obtained the following results.

When the space dimension is not less than three, there are similar results to the one-dimensional case.

Theorem 1.1. *When $n \geq 3$:*

(i) *If $0 < p < m$, then all the solutions to* (1.1) *are globally defined;*
(ii) *If $p = m$, then the solutions to* (1.1) *blow up in finite time or not depending on the form of the cut-off function $a(x)$ and the size of the initial data $u_0(x)$;*
(iii) *If $p > m$, then the solutions to* (1.1) *blow up in finite time or not depending on the size of the initial data $u_0(x)$.*

For the two-dimensional case, what can be discovered presently is the following.

Theorem 1.2. *When $n = 2$:*

(i) *If $0 < p \leq \frac{m+1}{2}$, then all the solutions to* (1.1) *are globally defined;*
(ii) *If $p = m$, then the solutions to* (1.1) *blow up in finite time, provided the cut-off function $a(x)$ and the size of the initial data $u_0(x)$satisfy proper conditions;*
(iii) *If $p > m$, then the solutions to* (1.1) *blow up in finite time, provided the size of the initial data $u_0(x)$ satisfies a proper condition.*

For the case $\frac{m+1}{2} < p < m$, we have not obtained any result yet and whether there exist global solutions for the case $p \geq m$ is still unknown.

Additionally, in numerical computations of PME, the localization method is often employed. For the purpose of the numerical simulation on *supp* a, we must consider whether the solutions blow up on *supp* a.

At present, we have obtained a result concerning the support of a solution and the support of the cut-off function. In particular, we prove that whether a solution blows up or not, the intersection of its support and the support of the cut-off function will be non-empty at some time.

Theorem 1.3. *There exists $t \in (0, \infty)$ such that supp $u(\cdot, t) \cap$ supp $a \neq \emptyset$.*

2. Construction of the proper solution

As a basis for the proof, we must properly define the weak solution to problem (1.1) that exists locally and is unique. The most usual way is to define it using integration by parts, and then prove its existence and uniqueness. This is convenient for standard PME, i.e. $u_t = \Delta(u^m)$, whereas for the PME with localized reaction, it is difficult to prove the existence and uniqueness. Another feasible method is the application of analytical semigroup and interpolation spaces to semilinear PDEs, as in [8]. However, in this section, we realize the construction of the proper solution by employing the extension of monotonic semigroups, exactly the same method used by Galaktionov-Vazquez in [4].

To start with, let X be an ordered topological space of functions $\Omega \to \bar{\mathbf{R}}_+$, where Ω is an open subset of $\mathbf{R}^n$, $\bar{\mathbf{R}}_+ = [0, \infty) \cup \{\infty\}$; B be a subspace of X which approximates X in a certain way, as explained below; and S_t be a semigroup acting in B. Now we need to extend S_t to act on X. For this purpose, we have to make the following assumptions:

(S1) S_t is order-preserving;

(S2) S_t is continuous and X-closed with respect to monotonic, increasing convergence (m.i.c. for short in the following).

In the second place, we consider a family of "approximation" operators $\{P_n : X \to B\}_{n\in\mathbf{N}}$ satisfying the following conditions:

(P1) $\{P_n\}$ is ordered: for every $u \in X$ and $n > m$, $P_n u \geq P_m u$ holds;
(P2) P_n is continuous under m.i.c.;
(P3) As $n \to \infty$, we have $P_n u \to u, u \in X$.

Next, we define the *extension* of S_t: for every $u \in X$ and $t > 0$, we put

$$T_t u = \lim_{n\to\infty} S_t P_n u.$$

Proposition 2.1. *T_t is a semigroup in X that extends S_t and is continuous under m.i.c.. The limit in the above expression is independent of the approximation sequence $\{P_n\}$ satisfying conditions* (P1)-(P3).

For our application, we assume X to be the space of nonnegative, measurable functions $\mathbf{R}^n \to \bar{\mathbf{R}}_+$, and B is chosen so that the equation

$$u_t = \Delta(u^m) + f(u), \quad m > 0$$

generates a semigroup S_t in B that satisfies (S1) and (S2). We have to assume the function f to be Lipschitz continuous so that S_t will be well defined in B. Finally, the operator P_n can be any of the usual cut-off operators that produce bounded functions.

This construction possesses generality and applies to the case when the reaction term involves the space invariable, i.e., $f(x,u)$, if only it is Lipschitz continuous. For $f(x,u) = a(x)u^p$, which corresponds to problem (1.1), let $g(u) = u^p$, and, without loss of generality, let the cut-off function $a(x)$ be the characteristic function of the closed ball $\bar{B}(0,L)$: $a(x) = \chi_{\bar{B}(0,L)}(x), L > 0$. Perform the following approximation to f: Define

$$a_j(x) = \begin{cases} 1, & |x| \leq L - 2^{-j}L, \\ 2^j L^{-1}(L - |x|), & L - 2^{-j}L < |x| < L, \\ 0, & |x| \geq L; \end{cases}$$

$$g_j(u) = \begin{cases} 2^j u, & 0 \leq u < 2^{-\frac{j}{1-p}}, \\ u^p, & 2^{-\frac{j}{1-p}} \leq u < j, \\ j^p, & u \geq j; \end{cases} \quad (if \;\; 0 < p < 1),$$

$$g_j(u) = \begin{cases} u^p, & 0 \leq u < j \\ j^p, & u \geq j \end{cases} \quad (if \quad p \geq 1);$$

then $\{a_j\}$ and $\{g_j\}$ are both nonnegative, monotonically increasing, and Lipschitz continuous sequences, and meanwhile $a_j(x) \to a(x)$ for any $x \in \mathbf{R}^n$ while $g_j(u) \to g(u)$ for any $u \in [0, \infty)$. Then let

$$f_j(x,u) = a_j(x)g_j(u), \quad (x,u) \in \mathbf{R}^n \times [0,\infty).$$

We can easily prove that $\{f_j\}$ is a sequence of increasing, nonnegative, and Lipschitz continuous functions that satisfy

$$f_j(x,u) \to f(x,u), \quad (x,u) \in \mathbf{R}^n \times [0,\infty).$$

Now let S_t^j be the semigroup generated by the equation

$$u_t = \Delta(u^m) + f_j(x,u)$$

acting on B, and T_t^j be its extension to X as constructed above. Thus, for every $u \in X$, we define

$$T_t u = \lim_{j\to\infty} T_t^j u = \lim_{j\to\infty} \lim_{k\to\infty} S_t^j(P_k u).$$

Another natural definition performs the two approximation processes at the same time:

$$U_t u = \lim_{j\to\infty} S_t^j(P_j u).$$

Proposition 2.2. *The above two definitions are equivalent and provide a semigroup in X that is continuous under m.i.c.. The limit is independent of the approximation sequences $\{P_j\}$ and $\{f_j\}$. Furthermore, we have the equivalent definition*

$$T_t u = \lim_{j,k\to\infty} S_t^j P_k u.$$

This extended semigroup is called the *limit semigroup*. For the initial function $u_0 \in X$, the function $u : \Omega \times [0,\infty) \to \bar{\mathbf{R}}_+$ defined by $u(x,t) = T_t u_0(x)$ is called the *proper solution* of problem (1.1).

Proposition 2.3. *The proper solutions satisfy the standard comparison theorem with respect to the data. In addition, the proper solution is minimal with respect to any kind of weak solution of the problem that satisfies the maximum theorem with respect to bounded weak solutions.*

3. Proof of Theorems 1.1 and 1.2

The above discussion has ensured the local existence and uniqueness of the proper solution to (1.1), and the practicality of the weak comparison theorem for $u \in L^\infty(\mathbf{R}^n) \cap H_0^1(\mathbf{R}^n)$. The essential method in the proofs of this section is the method of comparison, namely, to construct a globally defined supersolution in order to prove the global existence, or to construct a blow-up subsolution to prove that the solutions blow up. In the following we assume that the space dimension $n > 1$, unless otherwise stated.

Now we begin with a simple proof.

Lemma 3.1. *If $0 < p \leq 1$, all the solutions to problem (1.1) are global.*

Proof. From the conditions concerning $a(x)$ and $u_0(x)$, we naturally assume that $|a(x)| \leq M, 0 \leq u_0(x) \leq C$, where M and C are positive numbers. Thus the solution to the Cauchy problem

$$\begin{cases} u_t = Mu^p, & (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = C, & x \in \mathbf{R}^n, \end{cases}$$

namely,

$$u(x,t) = \begin{cases} [C^{1-p} + (1-p)Mt]^{\frac{1}{1-p}}, & 0 < p < 1, \\ Ce^{Mt}, & p = 1 \end{cases}$$

is a supersolution to problem (1.1). Hence when $0 < p \leq 1$, the solution of (1.1) is defined for all $t \in (0, \infty)$, and therefore (1.1) has only global solutions. This proves Lemma 3.1. □

Put another way, there may exist blow-up solutions to (1.1) only if $p > 1$.

In the following, we shall prove two lemmas regarding the existence of global solutions to (1.1) when $n \geq 3$. We shall employ a result regarding stationary solutions([9], page 223, 1A).

Proposition 3.1. *When $n \geq 3$, if $q \geq 2^* - 1 = \frac{n+2}{n-2}$, then the equation*

$$\Delta U + U^q = 0, \quad x \in \mathbf{R}^n, \tag{3.1}$$

has a solution $U \in C^\infty(\mathbf{R}^n)$ with $U(x) > 0, x \in \mathbf{R}^n$.

Thus we have: when $n \geq 3$, if $p > m$ and $q \geq max\{\frac{p}{m}, \frac{n+2}{n-2}\}$, then there exists a positive solution $U(x)$ to equation (3.1). Since $a(x)$ is compactly supported, we can take a constant $\lambda > 0$ large enough to guarantee that

$$a(x) \leq \lambda^{p-m} U^{q-\frac{p}{m}}, \quad x \in \mathbf{R}^n. \tag{3.2}$$

Lemma 3.2. *Let $n \geq 3$. Assume $p > m$ and $q \geq max\{\frac{p}{m}, \frac{n+2}{n-2}\}$. If the initial data satisfies*

$$u_0(x) \leq \lambda^{-1} U^{\frac{1}{m}}, \quad x \in \mathbf{R}^n, \tag{3.3}$$

where $U \in C^\infty(\mathbf{R}^n)$ is a positive solution to (3.1) and λ is a large enough positive constant satisfying (3.2), then the solutions of (1.1) are global-in-time.

Proof. Let $\phi_\lambda(x) = \lambda^{-1}[U(x)]^{\frac{1}{m}}$. We shall prove that the stationary solution $\phi_\lambda(x)$ is a supersolution to (1.1). Substituting its expression into the equation in (1.1), we have

$$\Delta(\phi_\lambda^m) + \lambda^{qm-m}\phi_\lambda^{qm} = 0.$$

It follows from (3.2) that

$$a(x) \le \lambda^{p-m}(\lambda^m\phi_\lambda^m)^{q-\frac{p}{m}} = \lambda^{qm-m}\phi_\lambda^{qm-p}, \quad x \in \mathbf{R}^n. \tag{3.4}$$

Thus,

$$(\phi_\lambda)_t = 0 = \Delta(\phi_\lambda^m) + \lambda^{qm-m}\phi_\lambda^{qm-p}\phi_\lambda^p \ge \Delta(\phi_\lambda^m) + a(x)\phi_\lambda^p.$$

Moreover, by (3.3),

$$\phi_\lambda(x) = \lambda^{-1}U^{\frac{1}{m}} \ge u_0(x), \quad x \in \mathbf{R}^n.$$

Therefore, $\phi_\lambda(x)$ is a supersolution to (1.1) and thus all the solutions to (1.1) are globally defined. This proves Lemma 3.2. □

Lemma 3.3. *When $n \ge 3$, if $p < m$, then all the solutions to* (1.1) *are global.*

Proof. At this time $\frac{n+2}{n-2} > 1 > \frac{p}{m}$, and thus, by Proposition 3.1, (3.1) has a positive C^∞ solution $U(x)$ if $q \ge \frac{n+2}{n-2}$. Since $p - m < 0$, for any continuous and compactly supported initial function $u_0(x)$ and compactly supported cut-off function $a(x)$, there exits a small enough positive constant λ satisfying both (3.2) and (3.3). Therefore, it is proved exactly the same way as before that $\phi_\lambda(x) = \lambda^{-1}[U(x)]^{\frac{1}{m}}$ is a supersolution to (1.1). This completes the proof of Lemma 3.3. □

In the above, we have proved, by constructing the global supersolution to problem (1.1), that when the space dimension $n \ge 3$, if $p > m$ and the size of the initial data is "small," then (1.1) has only global solutions, while if $p < m$, then all the solutions are globally defined, disregarding the choice of the initial function.

Next, we shall explain that all the solutions to problem (1.1) are global if $1 < p \le \frac{m+1}{2}$, again by constructing a global supersolution. This construction is realized with the help of a result in [2].

Proposition 3.2. *If $0 < p \le \frac{m+1}{2}$, then all the solutions to* (1.2) *are global.*

Lemma 3.4. *If $1 < p \le \frac{m+1}{2}$, then every solution to* (1.1) *is global.*

Proof. Since $a(x)$ is a nonnegative and compactly supported function, we assume, without loss of generality, that $0 \le a(x) \le 1$. Proposition 3.2 shows that in the case when the space dimension is one, the solutions to the Cauchy problem

$$\begin{cases} w_t = (w^m)_{xx} + \chi_{[-L,L]} w^p, & (x,t) \in \mathbf{R} \times (0,T), \\ w(x,0) = \phi(x), & x \in \mathbf{R}, \end{cases}$$

are globally defined. When $n > 1$, we define

$$\begin{aligned} &\tilde{u}(x,t) = w(x_1,t), && x = (x_1, ..., x_n), t \in (0,T), \\ &\tilde{\chi}_{[-L,L]}(x) = \chi_{[-L,L]}(x_1), \tilde{\phi}(x) = \phi(x_1), && x = (x_1, ..., x_n), \end{aligned}$$

then $\tilde{u}$ is a global solution to the following problem:

$$\begin{cases} \tilde{u}_t = \Delta(\tilde{u}^m) + \tilde{\chi}_{[-L,L]} \tilde{u}^p, & (x,t) \in \mathbf{R}^n \times (0,T), \\ \tilde{u}(x,0) = \tilde{\phi}(x), & x \in \mathbf{R}^n. \end{cases} \tag{3.5}$$

Since $u_0(x)$ is compactly supported, it also has compact support in x_1. Thus we can choose ϕ, such that *supp* $u_0 \subset$ *supp* $\tilde{\phi}$ and $\sup u_0 \le \tilde{\phi}(x)$ for $x \in$ *supp* u_0, which implies that $\tilde{u}$ is a supersolution to (1.1). Consequently, problem (1.1) has only global solutions. This concludes the proof of Lemma 3.1. □

Then, we shall use the energy method to demonstrate that, when $p > m$, problem (1.1) has blow-up solutions if the initial function $u_0(x)$ satisfies some given condition.

Define the energy function

$$E(t) = \frac{1}{2} \int_{\mathbf{R}^n} |\nabla(u^m)|^2 dx - \frac{m}{p+m} \int_{\mathbf{R}^n} a(x) u^{p+m} dx.$$

Lemma 3.5. *When $p > m$, if there exists $t_0 > 0$ such that $E(t_0) < 0$, then all the solutions to (1.1) blow up in finite time.*

Proof. Since $u \in L^{p+m}(\mathbf{R}^n) \cap H_0^1(\mathbf{R}^n)$, by the definition of $E(t)$ we have

$$\begin{aligned} E'(t) &= \int_{\mathbf{R}^n} \langle \nabla(u^m), \nabla(u^m)_t \rangle dx - m \int_{\mathbf{R}^n} a(x) u^{p+m-1} u_t dx \\ &= -\int_{\mathbf{R}^n} \Delta(u^m) m u^{m-1} u_t dx - m \int_{\mathbf{R}^n} a(x) u^{p+m-1} u_t dx \\ &= m \int_{\mathbf{R}^n} u^{m-1} u_t (-\Delta(u^m) - a(x) u^p) dx = -m \int_{\mathbf{R}^n} u^{m-1} |u_t|^2 dx \le 0, \end{aligned}$$

which shows that $E(t)$ is decreasing. Thus $E(t) \le E(t_0) < 0$ for any $t \ge t_0$.

Now we define another energy function

$$M(t)=\frac{1}{m+1}\int_0^1\int_{\mathbf{R}^n}u^{m+1}(x,s)dxds.$$

For any $t\geq t_0$, we have $M(t)>0$,

$$M'(t)=\frac{1}{m+1}\int_{\mathbf{R}^n}u^{m+1}(x,t)dx>0,$$

and

$$\begin{aligned}M''(t)&=\int_{\mathbf{R}^n}u^{m+1}u_tdx=\int_{\mathbf{R}^n}u^m(\Delta(u^m)+a(x)u^p)dx\\&=-\int_{\mathbf{R}^n}|\nabla(u^m)|^2dx+\int_{\mathbf{R}^n}a(x)u^{p+m}dx\\&=\frac{p+m}{m}\Big[-\frac{1}{2}\int_{\mathbf{R}^n}|\nabla(u^m)|^2dx+\frac{m}{p+m}\int_{\mathbf{R}^n}a(x)u^{p+m}dx\Big]\\&\quad-\Big(1-\frac{p+m}{2m}\Big)\int_{\mathbf{R}^n}|\nabla(u^m)|^2dx.\end{aligned}$$

From $p>m$ it follows that $-\big(1-\frac{p+m}{2m}\big)>0$, thus the inequality

$$M''(t)\geq\frac{p+m}{m}(-E(t))>\frac{p+m}{m}(-E(t_0))>0 \tag{3.6}$$

holds; namely, for any $t\geq t_0$, we have $M''(t)>C>0$, where C is a certain positive constant. Since $M'(t_0)>0$ and $M(t_0)>0$, we obtain, after integrating the above expression over the interval $[t_0,t]$, that $M'(t)\geq Ct+C_1$, where C_1 is a constant. Therefore, $M'(t)\to\infty$ as $t\to\infty$.

In the following we shall prove that there exists $T<\infty$ such that $M(t)\to\infty$ as $t\to T$. Notice that $E'(t)$ can also be written as

$$E'(t)=-\frac{4m}{(m+1)^2}\int_{\mathbf{R}^n}|(u^{\frac{m+1}{2}})_t|^2dx,$$

and thus

$$E(t)-E(t_0)=\int_{t_0}^tE'(s)ds=-\frac{4m}{(m+1)^2}\int_{t_0}^t\int_{\mathbf{R}^n}|(u^{\frac{m+1}{2}})_s|^2dxds.$$

Since $E(t_0)<0$, we have

$$E(t)<-\frac{4m}{(m+1)^2}\int_{t_0}^t\int_{\mathbf{R}^n}|(u^{\frac{m+1}{2}})_s|^2dxds.$$

Thus, by (3.6),

$$M''(t) > \frac{4(p+m)}{(m+1)^2} \int_{t_0}^{t} \int_{\mathbf{R}^n} |(u^{\frac{m+1}{2}})_s|^2 dxds.$$

Multiplying the above expression by the defining expression of $M(t)$, we have

$$M(t)M''(t) > \frac{4(p+m)}{(m+1)^3} \int_{t_0}^{t} \int_{\mathbf{R}^n} (u^{\frac{m+1}{2}})^2 dxds \int_{t_0}^{t} \int_{\mathbf{R}^n} |(u^{\frac{m+1}{2}})_s|^2 dxds.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} M(t)M''(t) &> \frac{4(p+m)}{(m+1)^3} \Big[\int_{t_0}^{t} \int_{\mathbf{R}^n} u^{\frac{m+1}{2}} (u^{\frac{m+1}{2}})_s dxds \Big]^2 \\ &= \frac{p+m}{m+1} \Big[\frac{1}{m+1} \int_{\mathbf{R}^n} u^{m+1}(x,t)dx - \frac{1}{m+1} \int_{\mathbf{R}^n} u^{m+1}(x,t_0)dx \Big]^2 \\ &= \frac{p+m}{m+1} [M'(t) - M'(t_0)]^2. \end{aligned}$$

Due to the fact that $M'(t) \to \infty$ in the limit $t \to \infty$ and since $\frac{p+m}{m+1} > 1$ for $p > m$, there exists a constant $\alpha > 0$ such that

$$M(t)M''(t) \geq (1+\alpha)M'^2(t), \tag{3.7}$$

when t is large enough. This is equivalent to the proposition that $M^{-\alpha}$ is a concave function. (In fact, $M^{-\alpha}$ is concave if and only if

$$\begin{aligned} (M^{-\alpha})'' &= (-\alpha M^{-\alpha-1} M')' \\ &= (-\alpha)(-\alpha-1)M^{-\alpha-2}(M')^2 + (-\alpha)M^{-\alpha-1}M'' \leq 0, \end{aligned}$$

which is equivalent to (3.7).)

Since $M(t) \geq 0$, there exists $0 < T < \infty$ such that $M^{-\alpha}(T) = 0$, and therefore $M(t) \to \infty$ as $t \to T$.

Furthermore, we claim that $M''(t) \to \infty$ as $t \to T$. Assume the opposite, namely, that $M''(t)$ is bounded in $[0,T]$. Let $M''(t) \leq 2C_1$ for $t \in [0,T]$, where C_1 is a positive constant. Integrating the inequality twice over the interval $[0,T]$, we obtain that $M(t) \leq C_1 t^2 + C_2 t + C_3, t \in [0,T]$, where C_2 and C_3 are constants. The right side is bounded in $[0,T]$, and so is $M(t)$, which contradicts the fact that $M(t) \to \infty$ as $t \to T$. Hence, $M''(t) \to \infty$ in the limit $t \to T$.

At the last stage we prove that $u(x,t)$ blows up. By (1.1),

$$\frac{d}{dt}\Big(\frac{1}{m+1} u^{m+1}\Big) = u^m u_t = u^m \Delta(u^m) + a(x)u^{p+m},$$

and thus

$$M''(t) = \frac{d}{dt}\Big(\frac{1}{m+1}\int_{\mathbf{R}^n} u^{m+1}(x,t)dx\Big) = \int_{\mathbf{R}^n} \frac{d}{dt}\Big(\frac{1}{m+1}u^{m+1}(x,t)\Big)dx$$
$$= \int_{\mathbf{R}^n} [u^m\Delta(u^m) + a(x)u^{p+m}]dx = -\int_{\mathbf{R}^n} |\nabla(u^m)|^2 dx + \int_{\mathbf{R}^n} a(x)u^{p+m}dx$$
$$\le \int_{\mathbf{R}^n} a(x)u^{p+m}dx \le meas(supp\ a)\|u(\cdot,t)\|_\infty^{p+m},$$

and consequently $u(x,t)$ blows up in the sense of the L^∞-norm. This proves Lemma 3.2. □

From Lemmas 3.2 and 3.5, we can conclude that when the space dimension $n \ge 3$ and $p > m$, whether the solutions to problem (1.1) blow up or not depends on the initial data: when the size is "small," all the solutions are global; on the contrary, when it is "large," the solutions blow up.

Finally, we consider the case $p = m$. When $n \ge 3$, Proposition 3.1 and the proof of Lemma 3.2 apply in this case. Let $q \ge \frac{n+2}{n-2}$ and $U(x)$ be a positive solution to equation (3.1). If $a(x)$ and $u_0(x)$ satisfy

$$a(x) \le U^{q-1}, \quad x \in \mathbf{R}^n, \tag{3.8}$$

$$u_0(x) \le U^{\frac{1}{m}}, \quad x \in \mathbf{R}^n \tag{3.9}$$

then all the solutions to (1.1) are global. Hence we have the following lemma.

Lemma 3.6. *When $n \ge 3$ and $p = m$, all the solutions to* (1.1) *are globally defined if $a(x)$ and $u_0(x)$ satisfy* (3.8) *and* (3.9) *respectively.*

At last, we shall prove that when the cut-off function $a(x)$ satisfies some certain condition, (1.1) has only blow-up solutions.

Lemma 3.7. *When $n \ge 2$ and $p = m$, all the solutions to* (1.1) *blow up if $a(x)$ satisfies*

$$a(x) \ge \delta > 0, \quad x \in B_R(0)(R \gg 1),$$

where δ is a constant such that $\delta > \lambda_R$, and λ_R is the first eigenvalue of $-\Delta$ in ball $B_R(0)$.

Proof. By the assumption,

$$\begin{cases} -\Delta\phi = \lambda_R\phi, & x \in B_R(0), \\ \phi(x) = 0, & x \in \partial B_R(0), \end{cases} \tag{3.10}$$

where ϕ is the first eigenfunction corresponding to λ_R such that $\|\phi\| = 1$.

Let

$$E(t) = \int_{B_R(0)} u(x,t)\phi(x)dx,$$

so that

$$E'(t) = \int_{B_R(0)} (\Delta(u^m) + a(x)u^m)\phi(x)dx$$
$$= \int_{B_R(0)} \Delta\phi u^m dx + \int_{\partial B_R(0)} \frac{\partial u^m}{\partial n}\phi dS - \int_{\partial B_R(0)} u^m \frac{\partial \phi}{\partial n} dS + \int_{B_R(0)} au^m\phi dx.$$

Consider the right side. By Hopf's lemma, $\int_{\partial B_R(0)} u^m \frac{\partial\phi}{\partial n} dS < 0$. Combined with $a(x) \geq \delta$ and (3.10), we have

$$E'(t) > (\delta - \lambda_R) \int_{B_R(0)} \phi u^m dx.$$

After integrating it over $[0, t]$,

$$E(t) \geq E(0) + (\delta - \lambda_R) \int_0^t \int_{B_R(0)} \phi u^m dxds.$$

Notice that

$$E(t) = \int_{B_R(0)} u\phi dx = \int_{B_R(0)} u\phi^{\frac{1}{m}}\phi^{\frac{m-1}{m}} dx$$
$$\leq \Big(\int_{B_R(0)} u^m \phi dx\Big)^{\frac{1}{m}} \Big(\int_{B_R(0)} \phi dx\Big)^{\frac{m-1}{m}},$$

or

$$E^m(t) \leq \Big(\int_{B_R(0)} u^m\phi dx\Big) \|\phi\|_1^{m-1} = \int_{B_R(0)} u^m\phi dx.$$

Thus, we have

$$E(t) \geq E(0) + (\delta - \lambda_R) \int_0^t E^m(s)ds.$$

Since $E(0) > 0$ and $\delta - \lambda_R > 0$, $E(t)$ blows up in finite time since $m > 1$, while, according to the definition,

$$E(t) = \int_{B_R(0)} u\phi dx \leq \|u\|_\infty \|\phi\|_1 = \|u\|_\infty.$$

Consequently, there exists $T < \infty$ such that $\|u(\cdot, t)\|_\infty \to \infty$ as $t \to T$. This proves Lemma 3.7. □

The above discussion has explained that the case $p = m$ differs from the case $p > m$ ($n \geq 3$), in that whether the solutions to (1.1) blow up or not depends not only on the size of the initial data, but also on the form of the cut-off function.

These lemmas complete the proof of Theorems 1.1 and 1.2.

4. Proof of Theorem 1.3

Case 1. Let the solution to (1.1) be globally defined; that is, $T=\infty$. The Cauchy problem

$$\begin{cases} u_t = \Delta(u^m), & (x,t) \in \mathbf{R}^n \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{4.1}$$

has a self-similar solution $u_S(x,t)$ with constant energy (i.e. the Barenblatt solution, [9], pages 19-21):

$$u_S(x,t) = t^{-\frac{n}{n\sigma+2}} \left[\frac{\sigma}{2(n\sigma+2)} \left(\eta_0^2 - |x|^2 t^{-\beta} \right)_+ \right]^{\frac{1}{\sigma}},$$

where $\sigma = m-1, \beta = \frac{2}{n\sigma+2}$,

$$\eta_0 = \eta_0(E_0) = \left\{ \pi^{-\frac{n}{2}} \left[\frac{2(n\sigma+2)}{\sigma} \right]^{\frac{1}{\sigma}} \frac{\Gamma(\frac{n}{2}+1+\frac{1}{\sigma})}{\Gamma(\frac{1}{\sigma}+1)} E_0 \right\}^{\frac{\sigma}{n\sigma+2}}$$

and $E_0 = \int_{\mathbf{R}^n} u(x,t)dx$ is a fixed positive constant chosen beforehand.

Since the maximum point and the size of support of $u_S(x,t)$ are proportional to $t^{-\frac{n}{n\sigma+2}} \eta_0^{\frac{2}{m-1}}$ and $\eta_0 t^{\frac{\beta}{2}}$ respectively, we can appropriately choose η_0 such that $u_S(x,t_0) < u_0(x)$ for some $t_0 > 0$. Then let t_0 be the initial time, and $\tilde{u}_S(x,t)$ be a Barenblatt solution to the Cauchy problem corresponding to (4.1) starting at t_0. We can choose $\tilde{u}_S(x,t)$ as a subsolution to (1.1).

Since $\eta_0 t^{\frac{\beta}{2}} \to \infty$ as $t \to \infty$, *supp* $\tilde{u}_S(\cdot,t)$ expands as time passes. While for the support of the solution to (1.1), we have *supp* $u(\cdot,t) \supset$ *supp* $\tilde{u}_S(\cdot,t)$. Hence there exists $t \in (0,\infty)$ such that *supp* $u(\cdot,t) \cap$ *supp* $a \neq \emptyset$.

Case 2. Let the solution blow up in finite time $T<\infty$. By the definition of blow-up set: $B(u) = \left\{ x : \exists x_n \to x, t_n \to T^-, s.t. \lim_{n\to\infty} u(x_n,t_n) = \infty \right\}$, there exists some t in the neighborhood of T such that $B(u) \subset$ *supp* $u(\cdot,t)$. Thus it suffices to prove *supp* $a \cap B(u) \neq \emptyset$ in the following.

Assume the opposite, namely, that $u(x,t)$ does not blow up on *supp* a. Since *supp* a is compact and $u \in H_0^1(\mathbf{R}^n)$, $u(x,t)$ is uniformly bounded on *supp* a within its time of existence; that is, there exists a constant $M>0$ such that

$$\sup_{x \in supp\ a} |u(x,t)| < M$$

for any $t \in (0,T)$. Additionally, since $u(x,t)$ is a blow-up solution, by Lemma 3.1, we have $p>1$, and thus u^{p-1} is also uniformly bounded on *supp* a. Now let $|au^{p-1}| \leq M_1$ (M_1 is a positive constant), and we obtain

$$u_t \leq \Delta(u^m) + M_1 u, \quad (x,t) \in \mathbf{R}^n \times (0,T). \tag{4.2}$$

After the transformation $v(x,t) = e^{-M_1 t}u(x,t)$, inequality (4.2) takes the form $v_t \le e^{(m-1)M_1 t}\Delta(v^m)$. For $t \in (0,T), e^{(m-1)M_1 t}$ is bounded: $e^{(m-1)M_1 t} \le C$ (C is a positive constant), and thus the above inequality can be further written as $v_t \le C\Delta(v^m)$. Again perform the transformation $\tilde{v}(x,t) = v(x, \frac{t}{C})$, and we finally obtain

$$\tilde{v}_t \le \Delta(\tilde{v}^m), \quad (x,t) \in \mathbf{R}^n \times (0,T).$$

Notice that $\tilde{v}(x,t)$ has the initial data $\tilde{v}(x,0) = u_0(0)$.

It follows that $\tilde{v}$ is a subsolution to problem (4.1). Therefore, as in Case 1, we can choose an appropriate Barenblatt solution $u_S(x,t)$ such that $u_S(x,0) \ge u_0(x), x \in \mathbf{R}^n$. By comparison, we have

$$\tilde{v}(x,t) \le u_S(x,t), \quad (x,t) \in \mathbf{R}^n \times (0,T),$$

which contradicts the assumption that $\tilde{v}(x,t)$ blows up as $t \to T^-$. This concludes the proof of Theorem 1.3. □

Acknowledgments. Kang and Zhou were supported in part by NNSF grant of China 10871152. The authors would like to thank the referee for his valuable comments. This work is part of Wang's master thesis supervised by Zhou in Wuhan University.

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