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ON SECOND-ORDER QUADRATIC SYSTEMS AND ALGEBRAS

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Abstract. This paper is devoted to a systematic study of quadratic systems of second-order ordinary differential equations which are defined in commutative algebras. Conserved quantities are found and several examples defined in Jordan and Nahm algebras are analyzed. Hamiltonian and gradient systems are considered.

1. Introduction

Let V be a real finite-dimensional vector space. A second-order quadratic differential equation on V is of the form

$$\ddot{X}(t) = Q(X(t)),$$

where $Q : V \to V$ is a second-order quadratic function, (a function homogeneous of degree 2), i.e.,

$$Q(aZ) = a^2 Q(Z),\ \forall a \in \mathbb{R},\ \forall Z \in V.$$

Using Q, an algebra structure $\mathbb{A} = (V, \alpha)$ is induced on V with multiplication α defined by

$$2\alpha(X,Y) = Q(X+Y) - Q(X) - Q(Y),\ \forall X, Y \in V.$$

We shall employ the notation $\alpha(X,Y) = X \times Y = XY,\ \forall X, Y \in \mathbb{A}$. Clearly, the multiplication is commutative, but, in general, not associative.

First-order differential equations in algebras have been studied extensively in recent years (see e.g. [4], [5], [6], [7], and [11]) and hence, by reduction-of-order techniques, second-order equations as well ([11]). However, the theory of ordinary differential equations ([2], [3]) often shows that studying the

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equations as a second-order equation produces special results which cannot immediately be deduced by considering the equation as a system of first-order equations.

The purpose of the present paper is to present some such special features.

A motivating example is given by the equation

$$\ddot{X} = X^2$$

in the algebra $\mathbb{R}$ of real numbers. This equation has first integrals given by

$$3\dot{X}^2 - 2X^3 = k,$$

k a constant, the solution of which may be analyzed using phase-plane methods (or direct integration) ([2]). In case also higher powers (than 2) occur one may use elliptic functions to integrate the analogous conserved quantity (see e.g. [10]). For a commutative algebra $\mathbb{A}$, a similar calculation leads to first-order nonlinear equations whose solutions we shall analyze, in part, below.

Imposing a gradient-like and Hamiltonian structure, we see that the algebra $\mathbb{A}$ should have a nondegenerate, associative, and bilinear form $C : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$, where associative means that $C(XY, Z) = C(X, YZ)$, $\forall X, Y, Z \in \mathbb{A}$. For an algebra $\mathbb{A}$ with such an associative form C, we show that

$$X^2 = -\text{grad}V(X),$$

where $V(X) = -\frac{1}{3}C(X, X^2)$, and our system becomes the gradient system

$$\ddot{X} = -\text{grad}V(X).$$

Furthermore the equivalent first-order system will be a Hamiltonian system with Hamiltonian given by $H(X, Y) = \frac{1}{2}C(Y, Y) - \frac{1}{3}C(X, X^2)$. This also leads us to a discussion of further conservation laws and their relationship to identity elements of the algebra.

All nilpotent (of index 2) elements of the algebra give rise to stationary solutions (equilibrium points). Using a theorem of Markus ([6]) we show that all equilibria are unstable.

In their interval of definition solutions of the quadratic equations are easily seen to be analytic. This leads to a discussion of power series (Taylor series), as well as Laurent series representations of the solutions. Several examples are presented giving algorithms for determining the coefficients of the power series solutions, and it is finally shown that if the Laurent series contains only finitely many terms with negative powers, the highest negative power must be ≥ -2.

2. Basics

Let $\mathbb{A}$ be a commutative algebra over $\mathbb{R}$ and consider the second-order ordinary differential equation

$$\ddot{X} = X^2, \tag{2.1}$$

where $X : (-\infty, \infty) \to \mathbb{A}$, $\dot{X} := \frac{d}{dt}X$, $X^2 := X \times X$ and $\times$ is the multiplication of $\mathbb{A}$, which will henceforth (most often) simply be abbreviated by juxtaposition. If $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$, then, upon multiplying (2.1) by $\dot{X}$, we obtain

$$\dot{X}\ddot{X} = X^2\dot{X},$$

which implies

$$\frac{1}{2}\frac{d}{dt}(\dot{X}^2) - \frac{1}{3}\frac{d}{dt}(X^3) = 0, \tag{2.2}$$

and hence

$$\frac{1}{2}\dot{X}^2 - \frac{1}{3}X^3 = k, \tag{2.3}$$

where k is a constant. This latter equation can be analyzed using phase-plane methods ([3]). We also note that (2.1) has, for any $c \in \mathbb{R}$, $X(t) = \frac{6}{(c-t)^2}$ as a solution. This observation will motivate the use of idempotent elements $E = E^2$ of the algebra to find special solutions of (2.1).

Considering (2.1) in a general commutative algebra $\mathbb{A}$, we shall need various product rules of differentiation there. Letting $X, Y, Z : (-\infty, \infty) \to \mathbb{A}$, we have

$$\frac{d}{dt}(XY) = \dot{X}Y + X\dot{Y},$$

which implies

$$\frac{d}{dt}[(XY)Z] = (\dot{X}Y)Z + (X\dot{Y})Z + (XY)\dot{Z};$$

in particular,

$$\frac{d}{dt}(X^3) = (2X\dot{X})X + (X^2)\dot{X}, \tag{2.4}$$

which implies

$$X^2\dot{X} = \frac{d}{dt}(X^3) - 2X(X\dot{X}), \tag{2.5}$$

because $\mathbb{A}$ is commutative.

Using (2.1) we obtain from (2.4) and (2.5) that

$$\frac{1}{2}\frac{d}{dt}(\dot{X}^2) - \frac{1}{3}\frac{d}{dt}(X^3) = \dot{X}X^2 - \frac{1}{3}[(2X\dot{X})X + X^2\dot{X}] \tag{2.6}$$

$$= \frac{1}{3}[2X^2\dot{X} - 2(X\dot{X})X] = \frac{2}{3}(X, X, \dot{X}),$$

where $(X, Y, Z) = (XY)Z - X(YZ)$ is the associator function in $\mathbb{A}$. Thus

$$3\frac{d}{dt}(\dot{X}^2) - 2\frac{d}{dt}(X^3) = 4(X, X, \dot{X}). \tag{2.7}$$

These considerations imply the following:

Proposition 2.1. *Let X be a solution of* (2.1). *Then* $(X, X, \dot{X}) = 0$ *if, and only if,* $3(\dot{X}^2) - 2(X^3) = k$, *where k is a constant.*

We note that the latter expression has a similar structure to scalar equations.

Definition 2.2. *An algebra $\mathbb{A}$ is called alternative, provided that*

$$(X, X, Y) = 0 = (Y, X, X), \ \forall X, Y \in \mathbb{A}.$$

We note that if $\mathbb{A}$ is commutative and alternative, then

$$(X, X, Y) = 0 \iff (Y, X, X) = 0, \ \forall X, Y \in \mathbb{A}.$$

It is known, cf. [9], that any simple nonassociative, alternative algebra is an 8-dimensional Cayley algebra, whereas if the algebra is both associative and commutative, then it must be either $\mathbb{R}$ or $\mathbb{C}$.

We henceforth restrict to *simple* algebras, since we wish to consider structures which have associative forms. These are related to gradient and Hamiltonian systems (see [11]).

Definition 2.3. *A symmetric bilinear form* $C : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$ *is called an associative form, provided that*

$$C(XY, Z) = C(X, YZ), \ \forall X, Y, Z \in \mathbb{A}.$$

The form is called nondegenerate whenever $C(X, Y) = 0$, $\forall Y \in \mathbb{A}, \Rightarrow X = 0$.

See [9] for more details on such forms. There, one also may find Dieudonné's theorem

Theorem 2.4. *Let $\mathbb{A}$ be an algebra with a nondegenerate associative form. Also assume that every nonzero ideal $\mathbb{B}$ of $\mathbb{A}$ is such that $\mathbb{B}^2 \neq 0$. Then*

- $\mathbb{A} = \bigoplus_{i=1}^{n} \mathbb{A}_i$, *where* $\mathbb{A}_1, \ldots, \mathbb{A}_n$ *are simple ideals of* $\mathbb{A}$.
- *For* $X = X_1 \oplus \cdots \oplus X_n, \ X_i \in \mathbb{A}_i$, *the equation* $\ddot{X} = X^2$ *decouples into the equations* $\ddot{X}_i = X_i^2$, *in the simple algebras* $\mathbb{A}_i, \ i = 1, \ldots, n$.

We hence shall restrict ourselves to the study of (2.1) in simple algebras; cf. [4] and [6].

We shall see later how the existence of an associative form will give rise to gradient or Hamitonian systems. Further, conditions on the associator $(X, X, \dot{X})$ will make the solution of (2.6) simpler. It is known that Jordan and Nahm algebras possess associative forms [5]. Future calculations also depend on the existence of identity elements (simple Jordan algebras do, whereas simple Nahm algebras do not have such).

3. Gradient and Hamiltonian formulation

Let $F : \mathbb{A} \to \mathbb{R}$ be defined by $F(X) := C(X, X^2)$, where C is a symmetric, bilinear, and associative form, which is nondegenerate, as defined in the previous section. The *differential* (dF) or the *gradient* $(\mathrm{grad}F)$ of F is defined by (the Taylor formula)

$$F(X+Y) = F(X) + dF(X)(Y) + R(Y),$$

where R represents higher-order terms and dF is a linear mapping for each X. Using the properties of C, one may quickly compute that

$$dF(X)(Y) = 2C(X, XY) + C(Y, X^2) = 3C(X^2, Y).$$

Hence, we may write

$$dF(X) = 3X^2.$$

It follows that, if we let $V(X) = -\frac{1}{3}F(X)$, then

$$X^2 = -\mathrm{grad}V(X),$$

and our system (2.1) be comes the *gradient* system

$$\ddot{X} = -\mathrm{grad}V(X). \tag{3.1}$$

The *Hamiltonian* (see [3] and [11]) of (3.1) may be obtained by converting (2.1) into a system of first-order equations, by setting

$$\dot{X} = Y, \ \dot{Y} = X^2. \tag{3.2}$$

We now define $H : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$ by

$$H(X, Y) := \tfrac{1}{2}C(Y, Y) - \tfrac{1}{3}C(X, X^2). \tag{3.3}$$

Then, we may, as above, compute

$$\mathrm{grad}H = (H_X, H_Y) = (-X^2, Y) = (-\dot{Y}, \dot{X}).$$

Hence, letting $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$, the above may be rewritten as

$$\dot{Z} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} H_X \\ H_Y \end{pmatrix}, \tag{3.4}$$

which is a system in Hamiltonian form. Let now X be a solution of (2.1) with $\dot{X} = Y$; then, using that C is associative, we obtain

$$\begin{aligned} \dot{H}(X,Y) &= C(Y,\dot{Y}) - \frac{1}{3}[C(\dot{X},X^2) + C(X,2X\dot{X})] \\ &= C(Y,X^2) - \frac{1}{3}[C(Y,X^2) + 2C(X,XY)] = 0. \end{aligned} \tag{3.5}$$

It follows that the Hamiltonian H is conserved along the solution X.

We may summarize all the above in the following:

Proposition 3.1. *Let $\mathbb{A}$ be a commutative algebra with a nondegenerate, associative, bilinear form $C(X,Y)$. Let X be a solution of* (2.1) *in $\mathbb{A}$. Then we have the following:*

- *The potential $V(X) = -\frac{1}{3}C(X,X^2)$ is such that*

$$\ddot{X} = -gradV(X) = X^2.$$

- *The Hamiltonian formulation of* (2.1) *is given by* (3.2) *with Hamiltonian given by* (3.3) *(or any nonzero constant multiple thereof).*
- *If we assume that C is positive definite and S, T_1, T_2, and T_3 are endomorphisms of the simple algebra $\mathbb{A}$, then, if*

$$F(X,Y) = C(SY,Y) + C(T_1X, T_2XT_3X), \ X,Y \in \mathbb{A}, \tag{3.6}$$

is conserved along solutions of (2.1), *there exists a constant μ such that*

$$F(X,Y) = \mu H(X,Y), \ \forall X,Y \in \mathbb{A};$$

i.e., when trying to linearly extend the conservation law H nothing new is obtained.

Proof. The first two parts have been proved above. To see the last part, let X be a solution of (2.1) with $X(t_0) = X_0, \ \dot{X}(t_0) = Y_0$. Then, since

$$\dot{F}(X(t), \dot{X}(t)) = 0,$$

we may compute this expression, and, after some manipulation obtain (S^c, etc., stands for the transpose of the corresponding endomorphism)

$$0 = C(\dot{X}, (S+S^c)X^2 + T_1^c(T_2XT_3X) + T_2^c(T_3XT_1X) + T_3^c(T_1XT_2X)). \tag{3.7}$$

Evaluating this expression at t_0 and using the nondegeneracy of C and further writing X, Y instead of X_0, Y_0, we obtain

$$0 = (S + S^c)X^2 + T_1^c(T_2XT_3X) + T_2^c(T_3XT_1X) + T_3^c(T_1XT_2X). \tag{3.8}$$

Since this expression vanishes identically, its differential must vanish as well. Computing this and denoting by $L(X)$ the left multiplication mapping $L(Z) : \mathbb{A} \to \mathbb{A},\ U \mapsto ZU$, we obtain

$$\begin{aligned} 0 = 2(S + S^c)L(X) &+ T_1^c(L(T_2X)T_3 + L(T_3X)T_2) \\ &+ T_2^c(L(T_3X)T_1 + L(T_1X)T_3) + T_3^c(L(T_1X)T_2 + L(T_2X)T_1). \end{aligned} \tag{3.9}$$

Using the associativity of C we immediately obtain that

$$L(X)^c = L(X),\ \forall X \in \mathbb{A};$$

i.e., $L(X)$ is C-symmetric. Transposing (3.9) and subtracting the result from (3.9), we obtain

$$0 = (S + S^c)L(X) - L(X)(S + S^c). \tag{3.10}$$

Hence, $S + S^c$ commutes with $L(\mathbb{A})$ (the set of all left multiplications). It follows from the positive definiteness of C that $S + S^c$ has a real eigenvalue with eigenspace $\mathbb{A}(\lambda)$. Further, since $S + S^c$ and $L(\mathbb{A})$ commute, $\mathbb{A}(\lambda)$ must be an ideal in $\mathbb{A}$, from which it easily follows that (using that $\mathbb{A}$ is simple) $\mathbb{A}(\lambda) = \mathbb{A}$. Consequently, $(S + S^c) = \lambda I$. Using this, we may rewrite (3.8) as

$$0 = \lambda X^2 + T_1^c(T_2XT_3X) + T_2^c(T_3XT_1X) + T_3^c(T_1XT_2X),\ \forall X \in \mathbb{A}. \tag{3.11}$$

We next determine $2F(X, \dot{X})$ and, after a lengthy computation using (3.11) and the properties of C, obtain

$$2F(X, \dot{X}) = 3\lambda C(\dot{X}, \dot{X}) - 2\lambda C(X, X^2) - 4F(X, \dot{X}); \tag{3.12}$$

i.e.,

$$6F(X, \dot{X}) = 3\lambda C(\dot{X}, \dot{X}) - 2\lambda C(X, X^2) = 6\lambda H(X, \dot{X}). \tag{3.13}$$

3.1. **Remarks.** • Let $\mathbb{A}$ be a simple algebra with identity element E and associative form C. Then H, given by (3.3), is conserved as follows: Denote by g the linear functional $g(X) := C(E, X),\ X \in \mathbb{A}$. We observe that

$$\begin{aligned} g((X, X, \dot{X})) &= C(E, X^2\dot{X} - X(X\dot{X})) \\ &= C(EX^2, \dot{X}) - C((EX)X, \dot{X}) = 0. \end{aligned} \tag{3.14}$$

Using this and (2.6), we obtain

$$g(4(X, X, \dot{X})) = g(3\frac{d}{dt}(\dot{X}^2) - 2\frac{d}{dt}(X^3)) = \frac{d}{dt}g(3\dot{X}^2 - 2XX^2) \tag{3.15}$$

$$= \frac{d}{dt} C(E, 3\dot{X}^2 - 2XX^2) = \frac{d}{dt}(3C(\dot{X}, \dot{X}) - 2C(X, X^2)) = 0$$

(see (3.14) above). Thus the quantity $3C(\dot{X}, \dot{X}) - 2C(X, X^2)$ is conserved along solutions, which, of course, is already given in Proposition 3.1.

Let us assume that C is positive definite and let $\Xi \in \mathbb{A}$. Let us now define the linear functional g by $g(X) := C(\Xi, X),\ X \in \mathbb{A}$, and put

$$\begin{aligned} F(X, \dot{X}) :&= g(3\dot{X}^2 - 2XX^2) = C(\Xi, 3\dot{X}^2) + C(\Xi, -2XX^2) \\ &= C(3\Xi\dot{X}, \dot{X}) + C(-2\Xi X, X^2) = C(S\dot{X}, \dot{X}) + C(T_1 X, T_2 X T_3 X), \end{aligned}$$

where $S = 3L(\Xi) = S^c$, $T_1 = -2L(\Xi)$, and $T_2 = T_3 = I$. Thus, it follows from Proposition 3.1 that, if $F(X, \dot{X})$ is conserved, it must be a constant multiple of $3C(\dot{X}, \dot{X}) - 2C(X, X^2)$ and hence, by Proposition 3.1, again,

$$\lambda I = S + S^c = 2S = 2(3L(\Xi)) = 6L(\Xi).$$

In other words, we must have that the element $\frac{6}{\lambda}\Xi$ be an identity element.

• It would be natural to use an automorphism Φ of $\mathbb{A}$ to obtain a different conservation law

$$F(X, Y) := C(\Phi X, \Phi Y) + C(\phi X, (\Phi X)^2).$$

But, using $S = \Phi^c \Phi$ and $T_i = \Phi$, we see that F is simply a multiple of H.

Summarizing these remarks we see that if $\mathbb{A}$ is a simple algebra with positive definite associative from C, then $\mathbb{A}$ has an identity element if, and only if, there exists $\Xi \in \mathbb{A}$ so that the corresponding F given by g is conserved for the equation (2.1).

4. EQUILIBRIA

As opposed to scalar equations of the form (2.1), where the only stationary solution is the trivial solution, in the higher-dimensional case there may be constant solutions N, which are nontrivial, i.e.,

$$N^2 = 0,$$

namely nilpotent elements of index 2. We also note that such elements have the property that $X(t) := f(t)N$, where $f(t) = at + b,\ a, b \in \mathbb{R}$ are solutions of (2.1) as well.

It also follows easily, in the scalar case, that the trivial solution of (2.1) is unstable. We shall show here that in the higher-dimensional case the trivial solution, as well as other equilibria are unstable. To verify this, we shall make use of a theorem of Markus [6], [4], which we shall state here for the sake of completeness.

Theorem 4.1. *Let $\mathbb{A}$ be a real commutative algebra. Then either $\mathbb{A}$ has a nonzero idempotent element E, or a nonzero nilpotent element N of index* 2.

We now establish the following proposition:

Proposition 4.2. *Let the algebra $\mathbb{A}$ have a positive definite form C and assume that N is an equilibrium solution of* (2.1); *then N is unstable.*

Proof. We first consider the case that $N \neq 0$, then the solution $X(t) = tN$, shows that both the trivial solution, as well as the solution N are both unstable equilibria, as it leaves every neighborhood of both 0 and N. On the other hand, if 0 is the only equilibrium, then by the theorem of Markus, above, there exists a nonzero idempotent E. We define

$$X(t) := \frac{6}{(c-t)^2} E; \tag{4.1}$$

then for any choice of the constant c, $X(\cdot)$ will solve (2.1). Since C is positive definite, it defines a norm $\|\cdot\|$ in the vector space $\mathbb{A}$ and

$$\|X(0)\|^2 = C(X(0), X(0)) = \frac{36}{c^4} C(E, E).$$

Hence, for any given neighborhood of 0, we may choose $\frac{36}{c^4} C(E, E)$ small enough, so $X(0)$ belongs to that neighborhood. Yet $X(t)$ becomes unbounded as t approaches c.

Remark 4.3. If $\mathbb{A}$ has an identity element E and the form C is positive definite as well as associative, then $N^2 = 0$ implies that $N = 0$, viz.

$$C(N, N) = C(EN, N) = C(E, N^2) = 0.$$

Hence, in this case 0 is the only equilibrium. An example for this situation is given by the Jordan algebra $\mathbb{A}$ of symmetric $n \times n$ matrices.

5. Some examples

5.1. **The initial value problem.** Let us consider the initial value problem

$$\ddot{X} = X^2, \ X(0) = X_0, \ \dot{X}(0) = Y_0. \tag{5.1}$$

It follows from the basic existence and uniqueness theorem (Picard-Lindelöf theorem, [2]) that this problem has a unique solution X defined on a maximal interval of existence (a, b) and $\|X(t)\| \to \infty$ as $t \to a$ and $t \to b$, whenever either of these values is finite. We also note that if X is a solution of (5.1) then, for any constant c, $Z(t) := c^2 X(ct)$ solves

$$\ddot{Z} = Z^2, \ Z(0) = c^2 X_0, \ \dot{Z}(0) = c^3 Y_0, \tag{5.2}$$

whereas $Z(t) := X(t-c)$ solves

$$\ddot{Z} = Z^2,\ Z(c) = X_0,\ \dot{Z}(c) = Y_0. \tag{5.3}$$

We hence may restrict ourselves to initial conditions being prescribed at $t = 0$.

To solve the initial value problem (5.1) we look for solutions of the form

$$X(t) = \sum_{n=0}^{\infty} t^n A_n, \tag{5.4}$$

and an easy calculation shows that the coefficients $A_n,\ n = 0, 1, \dots,$ must satisfy

$$A_0 = X_0,\ A_1 = Y_0,\ (n+2)(n+1)A_{n+2} = \sum_{i+j=n} A_i A_j,\ n = 0, 1, \dots; \tag{5.5}$$

hence the infinite series solution (5.4) is uniquely determined and we conclude the analyticity of solutions, as well. Also the solution (5.4) will reside in the subalgebra of $\mathbb{A}$ generated by the coefficients X_0 and Y_0.

5.2. **Periodic solutions.** If it is the case that the algebra contains a nonzero nilpotent element of index 2, then (2.1) has a nontrivial periodic solution, which is a constant solution and hence has minimal period 0. If, on the other hand, X is a periodic solution of (2.1) with positive minimal period, say, T, then

$$Y(t) := c^2 X(ct + d) \tag{5.6}$$

is a nontrivial periodic solution, of minimal period $\frac{T}{c}$, for any positive constant c and arbitrary d. In the case of $\mathbb{A} = \mathbb{R}$ one easily sees that the only periodic solution of (2.1) is the trivial solution. We conjecture the following more general result:

Conjecture 5.1. *Let $\mathbb{A}$ be an algebra of dimension ≥ 2 equipped with a positive definite associative form $C : \mathbb{A} \times \mathbb{A} \to \mathbb{R}$. If X is a solution of* (2.1) *which is periodic, then $X \equiv A_0$, where $A_0^2 = 0$; i.e.,* (2.1) *has nontrivial periodic solutions, if, and only if, it has nilpotent elements of index* 2.

Another motivation for this conjecture is the following. It uses the analyticity of a solution X and the fact that X and all its derivatives will be iT-periodic for every $i = 1, 2, \dots$. This together with the recursive formula (5.5) then will yield an infinite sequence of identities which the subalgebra $\mathbb{R}[X_0, Y_0]$ must satisfy (X_0 and Y_0 are the initial conditions determining X). This is unlikely, unless all the products in the subalgebra are zero, e.g., $X_0^2 = 0$.

6. Laurent series solutions

6.1. **Generalities.** Returning to the fact that for any constant $c \neq 0$, $X(t) = \frac{6}{(t-c)^2}$ solves the scalar initial value problem

$$\ddot{X} = X^2,\ X(0) = \frac{6}{c^2},\ \dot{X}(0) = \frac{12}{c^3}, \tag{6.1}$$

we shall seek conditions in order that

$$X(t) := \frac{A_{-2}}{(t-c)^2} + \frac{A_{-1}}{t-c} + \sum_{n=0}^{\infty}(t-c)^n Y_n \tag{6.2}$$

be a Laurent series solution of (2.1) in the algebra $\mathbb{A}$, where

$$Y(t) := \sum_{n=0}^{\infty}(t-c)^n Y_n$$

is analytic.

Computing $\ddot{X}$ and X^2 and equating like powers of $t-c$, we obtain the following relationships:

$$\begin{array}{rcl} 6A_{-2} & = & (A_{-2})^2 \\ A_{-1} & = & A_{-2}A_{-1} \\ 0 & = & (A_{-1})^2 + 2A_{-2}Y_0 \\ 0 & = & A_{-2}Y_1 + A_{-1}Y_0 \\ 2Y_2 & = & 2A_{-2}Y_2 + 2A_{-1}Y_1 + C_0 \\ 6Y_3 & = & 2A_{-2}Y_3 + 2A_{-1}Y_2 + C_1 \\ 12Y_4 & = & 2A_{-2}Y_4 + 2A_{-1}Y_3 + C_2 \\ & \vdots & \\ n(n-1)Y_n & = & 2A_{-2}Y_n + 2A_{-1}Y_{n-1} + C_{n-2}, \end{array} \tag{6.3}$$

where

$$Y^2(t) = \sum_{n=0}^{\infty}(t-c)^n C_n \quad \text{and} \quad C_n = \sum_{i+j=n} Y_i Y_j.$$

Conversely, a Laurent series which satisfies these relationships will represent a solution of (2.1).

Next, note $E = \frac{1}{6}A_{-2}$ is an idempotent and if we denote by $L(E) : \mathbb{A} \to \mathbb{A}$, the linear transformation which is left multiplication by E,

$$L(E)X = EX,$$

then

$$\begin{array}{rcl} L(E)E & = & E \\ L(E)A_{-1} & = & \frac{1}{6}A_{-1} \\ L(E)Y_0 & = & -\frac{1}{12}(A_{-1})^2 \\ L(E)Y_1 & = & -\frac{1}{6}A_{-1}Y_0 \\ (L(E)-\frac{1}{6}I)Y_2 & = & -\frac{1}{6}A_{-1}Y_1 - \frac{1}{12}C_0 \\ (L(E)-\frac{1}{2}I)Y_3 & = & -\frac{1}{6}A_{-1}Y_2 - \frac{1}{12}C_1 \\ (L(E)-I)Y_4 & = & -\frac{1}{6}A_{-1}Y_3 - \frac{1}{12}C_2 \\ & \vdots & \\ (L(E)-\frac{n(n-1)}{12}I)Y_n & = & -\frac{1}{6}A_{-1}Y_{n-1} - \frac{1}{12}C_{n-2}. \end{array} \tag{6.4}$$

Hence $\frac{1}{6}$ is an eigenvalue of $L(E)$ with eigenvector A_{-1}, and we have formulas for computing Y_n in terms of $E, A_{-1}, Y_0, \ldots, Y_{n-1},\ n = 1, 2, \ldots$ and we may use the eigenspace of $L(E)$ to make these computations. If, on the other hand, $\mathbb{A}$ is power associative, i.e.,

$$X^n X^m = X^{n+m},$$

then the only eigenvalues of $L(E)$ are 0, $\frac{1}{2}$, and 1 (see [9]). In this case we obtain the eigenspace decomposition

$$\mathbb{A} = \mathbb{A}(0) \oplus \mathbb{A}\left(\tfrac{1}{2}\right) \oplus \mathbb{A}(1)$$

and $A_{-1} = 0$. We further have the multiplicative relations (see again [9])

$$\begin{array}{l} \mathbb{A}(0)\mathbb{A}(0) \subset \mathbb{A}(0), \quad \mathbb{A}(1)\mathbb{A}(1) \subset \mathbb{A}(1), \\ \mathbb{A}(\frac{1}{2})\mathbb{A}(\frac{1}{2}) \subset \mathbb{A}(0) \oplus \mathbb{A}(1), \ \mathbb{A}(0)\mathbb{A}(1) = 0, \\ \mathbb{A}(0)\mathbb{A}(\frac{1}{2}) \subset \mathbb{A}(1) \oplus \mathbb{A}(\frac{1}{2}), \quad \mathbb{A}(1)\mathbb{A}(\frac{1}{2}) \subset \mathbb{A}(0) \oplus \mathbb{A}(\frac{1}{2}). \end{array} \tag{6.5}$$

We also have the following matrix representations:

$$L(E) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}I_1 & 0 \\ 0 & 0 & I_2 \end{pmatrix}$$

and, for μ not an eigenvalue,

$$(L(E) - \mu I)^{-1} = \begin{pmatrix} -\frac{1}{\mu}I_0 & 0 & 0 \\ 0 & (\frac{1}{2}-\mu)^{-1}I_1 & 0 \\ 0 & 0 & (1-\mu)^{-1}I_2 \end{pmatrix}.$$

6.2. **Remarks.** • As observed above, solutions of (2.1) are uniquely determined by initial conditions. The question arises how these initial conditions determine the idempotent E used in the idempotent decomposition described? And conversely, starting with an idempotent $E \in \mathbb{A}$, what are the initial conditions which give rise to a Laurent series solution so that $A_{-2} = 6E$?

• The eigenvalues $\lambda = 1, \frac{1}{2}$ should be expected to occur. Thus, for $\lambda = 1$, $E^2 = E$ implies $L(E)E = E^2 = E$. For $\lambda = \frac{1}{2}$, we note that an automorphism Φ of $\mathbb{A}$ is a linear solution preserving mapping

$$(\Phi X)^2 = \Phi(X^2) = \Phi(\ddot{X}) = \frac{d^2}{dt^2}(\Phi X).$$

The automorphisms of $\mathbb{A}$ form a Lie group, Aut$\mathbb{A}$, with Lie algebra, Der$\mathbb{A}$, the derivations of $\mathbb{A}$, [8]. Thus, if $D \in \mathrm{Der}\mathbb{A}$ is such that $DE \neq 0$, then

$$DE = D(E^2) = 2E \times DE.$$

Hence

$$(L(E) - \tfrac{1}{2}I)(DE) = 0,$$

i.e., $DE \in \mathbb{A}\left(\frac{1}{2}\right)$.

6.3. **Example.** A specific example of the above is the Jordan algebra $\mathbb{A}$ of 3×3 symmetric matrices (see [1] and [9]) with multiplication defined by

$$XY := \tfrac{1}{2}(X * Y + Y * X),$$

where $*$ is the associative matrix multiplication. The bilinear form

$$C(X,Y) := \mathrm{trace}XY$$

is a positive definite associative form, and for $X = (x_{ij}) \in \mathbb{A}$

$$C(X,X) = \mathrm{trace}X^2 = \sum_i x_{ii}^2 + 2\sum_{i \neq j} x_{ij}^2.$$

A basis for $\mathbb{A}$ is given by

$$F_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.6)$$
$$F_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

with the symmetric multiplication table

$$\begin{array}{c} \\ F_{11}\\ F_{22}\\ F_{33}\\ F_{12}\\ F_{13}\\ F_{23}\end{array}\begin{array}{c}\begin{array}{cccccc} F_{11} & F_{22} & F_{33} & F_{12} & F_{13} & F_{23}\end{array}\\ \begin{pmatrix} F_{11} & 0 & 0 & \frac{1}{2}F_{12} & \frac{1}{2}F_{13} & 0\\ 0 & F_{22} & 0 & \frac{1}{2}F_{12} & 0 & \frac{1}{2}F_{23}\\ 0 & 0 & F_{33} & 0 & \frac{1}{2}F_{13} & \frac{1}{2}F_{23}\\ \frac{1}{2}F_{12} & \frac{1}{2}F_{12} & 0 & F_{11}+F_{22} & \frac{1}{2}F_{23} & \frac{1}{2}F_{13}\\ \frac{1}{2}F_{13} & 0 & \frac{1}{2}F_{13} & \frac{1}{2}F_{23} & F_{11}+F_{33} & \frac{1}{2}F_{12}\\ 0 & \frac{1}{2}F_{23} & \frac{1}{2}F_{23} & \frac{1}{2}F_{13} & \frac{1}{2}F_{12} & F_{22}+F_{33}\end{pmatrix}\end{array}.$$

In this case we obtain that $E = F_{11}$ and the eigenspace decomposition

$$\mathbb{A} = \mathbb{A}(0) \oplus \mathbb{A}\left(\tfrac{1}{2}\right) \oplus \mathbb{A}(1),$$

where $\mathbb{A}(0) = \operatorname{span}\{F_{22}, F_{33}, F_{23}\}$, $\mathbb{A}(1) = \operatorname{span}\{F_{11}\}$, $\mathbb{A}(\frac{1}{2}) = \operatorname{span}\{F_{12}, F_{13}\}$. We may now use the above multiplication table and compute solutions of the form (6.2) using the relations given in (6.3) and find $A_{-2} = 6E \in \mathbb{A}(1)$, $A_{-1} = 0$, $Y_0, Y_1, Y_2 \in \ker L(E) = \mathbb{A}(0)$. Further, $Y_3 = Z_{\frac{1}{2}} + Z_0$, where $Z_0 \in \mathbb{A}(0)$ and $Z_{\frac{1}{2}} \in \mathbb{A}\left(\frac{1}{2}\right)$ are arbitrary (recall that $\frac{1}{2}$ is an eigenvalue). Choosing $Z_{\frac{1}{2}} = 0$, (6.3), and using an induction argument, we find solutions of the form

$$X(t) := \frac{6E}{(t-c)^2} + \sum_{n=0}^{\infty} (t-c)^n Y_n, \tag{6.7}$$

with $Y_n \in \mathbb{A}(0) \oplus \mathbb{A}(1)$, $n = 0, 1, \ldots$, a subalgebra of $\mathbb{A}$. If, on the other hand, $Z_{\frac{1}{2}} \neq 0$, all eigenspaces will be involved in the computations.

6.4. **Example.** Next, in the Jordan algebra of 3×3 matrices, let us consider the idempotent $E = F_{11} + F_{22}$. This gives the eigenspace decomposition

$$\mathbb{A} = \mathbb{A}(0) \oplus \mathbb{A}\left(\tfrac{1}{2}\right) \oplus \mathbb{A}(1),$$

where (using the previous calculations)

$$\mathbb{A}(0) = \operatorname{span}\{F_{33}\}, \quad \mathbb{A}(1) = \operatorname{span}\{F_{11}, F_{22}, F_{12}\}, \quad \mathbb{A}\left(\tfrac{1}{2}\right) = \operatorname{span}\{F_{13}, F_{23}\}.$$

We may now compute again a symmetric multiplication table and determine solutions of the form (6.2) using the relations given in (6.3) and find $A_{-2} = 6E \in \mathbb{A}(0)$, $A_{-1} = 0$, $Y_0, Y_1, Y_2 \in \ker L(E) = \mathbb{A}(0)$. Using (6.3) and an induction argument, as above, we find that we have solutions of the form

$$X(t) := \frac{6E}{(t-c)^2} + \sum_{n=0}^{\infty} (t-c)^n Y_n, \tag{6.8}$$

with $Y_n \in \mathbb{A}(0) \oplus A(1),\ n = 0, 1, \ldots .$

6.5. **Example.** We next consider an example where $\frac{1}{6}$ is an eigenvalue of $L(E)$. Let $\mathbb{H}$ be the Jordan algebra of 2×2 symmetric matrices; i.e.,

$$\mathbb{H} = \text{span}\{F_1, F_2, F_3\},$$

where

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ F_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.9}$$

with the symmetric multiplication table

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \end{array} \begin{array}{c} \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{pmatrix} F_1 & 0 & \frac{1}{2}F_3 \\ 0 & F_2 & \frac{1}{2}F_3 \\ \frac{1}{2}F_3 & \frac{1}{2}F_3 & F_1 + F_2 \end{pmatrix} \end{array}.$$

Let $E = F_1$ and extend $\mathbb{H}$ to the 4-dimensional algebra

$$\mathbb{A} = \text{span}\{F_1, F_2, F_3, F_4\},$$

where we require $L(E)F_4 = \frac{1}{6}F_4$. Let $\mathbb{A}$ have the multiplication table

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{cccc} F_1 & F_2 & F_3 & F_4 \end{array} \\ \begin{pmatrix} F_1 & 0 & \frac{1}{2}F_3 & \frac{1}{6}F_4 \\ 0 & F_2 & \frac{1}{2}F_3 & 0 \\ \frac{1}{2}F_3 & \frac{1}{2}F_3 & F_1 + F_2 & 0 \\ \frac{1}{6}F_4 & 0 & 0 & \frac{1}{6}F_1 \end{pmatrix} \end{array}.$$

Computations show that $C(X, Y) := L(X)L(Y)$ is a positive definite associative form on $\mathbb{A}$. Using the formulas, derived above, for the Laurent series coefficients, we obtain (for the first few)

$$E = F_1 = \tfrac{1}{6}A_{-2},\ A_{-1} = F_4,$$

$$Y_0 = -\frac{1}{72}F_1 + aF_2,\quad Y_1 = bF_2 + \frac{1}{432}F_4,\quad Y_2 = -\frac{1}{10368}F_1 + \frac{1}{2}a^2F_2 + cF_4,$$

where, a, b, and c are arbitrary constants.

6.6. **Example.** As a final example we consider Nahm algebras $\mathbb{A}$, which are constructed using a Lie algebra $\eth$ ([5]), $\mathbb{A} = \eth \times \eth \times \eth$, with multiplication

$$X^2 = \begin{pmatrix} [x_2x_3] \\ [x_3x_1] \\ [x_1x_2] \end{pmatrix}, \quad \text{for} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{A},$$

and $[uv]$ is the Lie algebra multiplication in $\eth$. If $\eth$ is a simple Lie algebra, then $\mathbb{A}$ is simple and has the nondegenerate associative form

$$C(X, X) = K(x_1, x_1) + K(x_2, x_2) + K(x_3, x_3),$$

where K is the Killing form on $\eth$. Also, C is definite, whenever K is (again, see [5]).

For $\eth = \text{so}(3)$, the 3×3 skew-symmetric matrices, we find that $\mathbb{A}$ is 9-dimensional and has an interesting 6-dimensional subalgebra $\mathbb{B}$ which contains many solutions of equation (2.1). We have that $\eth = \text{span}\{L_1, L_2, L_3\}$, where

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The subalgebra $\mathbb{B}$ is defined by $\mathbb{B} = \text{span}\{U_1, U_2, U_3, V_1, V_2, V_3\}$, where

$$U_1 = \begin{pmatrix} L_1 \\ 0 \\ 0 \end{pmatrix}, \; U_2 = \begin{pmatrix} 0 \\ L_2 \\ 0 \end{pmatrix}, \; U_3 = \begin{pmatrix} 0 \\ 0 \\ L_3 \end{pmatrix},$$

and

$$V_1 = \begin{pmatrix} 0 \\ L_3 \\ L_2 \end{pmatrix}, \; V_2 = \begin{pmatrix} L_3 \\ 0 \\ L_1 \end{pmatrix}, \; V_3 = \begin{pmatrix} L_2 \\ L_1 \\ 0 \end{pmatrix},$$

where, in the above columns, an entry 0 is a 3×3 0-matrix.

After computing the multiplication table we find that $E = U_1 + U_2 + U_3$ is an idempotent, $E^2 = E$, and eigenspaces $\mathbb{B}(1) = \text{span}\{E\}$,

$$\mathbb{B}\left(\tfrac{1}{2}\right) = \text{span}\{V_1, V_2, V_3, -U_1 + U_2, -U_1 + U_3\}.$$

Hence, $\mathbb{B} = \mathbb{B}(1) \oplus \mathbb{B}\left(\tfrac{1}{2}\right)$. Relative to this basis we compute the solution

$$X(t) = \frac{A_{-2}}{(t-c)^2} + \frac{A_{-1}}{t-c} + \sum_{n=0}^{\infty} (t-c)^n Y_n$$

and find $A_{-2} = 6E,\ A_{-1} = 0,\ Y_0 = Y_1 = Y_2 = Y_3 = 0,\ Y_4 = a_4 E,\ a_4 \in \mathbb{R}$, $Y_5 = 0$; further, using (6.3) and (6.4) we find that X is given by (the first few terms)

$$X(t) := \frac{6E}{(t-c)^2} + a_4 E(t-c)^4 + a_{10} E(t-c)^{10} + \cdots ,$$

where $a_4, a_{10} \in \mathbb{R}$.

7. More on Laurent series

Let again X be a solution of (2.1) given by a Laurent series. Since the equation is invariant under time shifts, we may assume that $X(t)$ is given by $X(t) = t^{-k} Y(t)$, where $k \in \{0, 1, 2, \dots\}$ and Y is analytic at $t = 0$, with $Y(0) = Y_0 \neq 0$. We recall that $3C(\dot{X}, \dot{X}) - 2C(X, X^2)$ is a conserved quantity (cf. Proposition 3.1). Computing the latter quantity, we obtain that

$$3k^2 t^{-2k-2} C(Y, Y) - 6kt^{-2k-1} C(Y, \dot{Y}) + 3t^{-2k} C(\dot{Y}, \dot{Y}) - 2t^{-3k} C(Y, Y^2)$$

is a conserved quantity (i.e., independent of time). If it is the case that $k > 2$, then the lowest power of t occurring will be t^{-3k}, whose coefficient is given by $C(Y_0, Y_0^2)$. By the conservation property this coefficient must vanish. Examining the coefficients further (since $3k > 2k + 2$), we also conclude that the coefficient of t^{-2k-2}, given by $3k^2 C(Y_0, Y_0)$, must vanish as well. Since C is positive definite, we conclude that $Y_0 = 0$. Therefore, the lowest possible power of a Laurent series equals t^{-2}.

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