Differential and Integral Equations Volume 24, Numbers 9-10 (2011), 845–860

ON A CLASS OF NONVARIATIONAL ELLIPTIC SYSTEMS WITH NONHOMOGENOUS BOUNDARY CONDITIONS

João Marcos do Ó[1]
Departamento de Matemática, Universidade Federal da Paraíba
58059-900, João Pessoa - PB - Brazil

Sebastián Lorca[2]
Departamento de Matemática, Universidad de Tarapacá
Casilla 7-D, Arica, Chile

Pedro Ubilla[3]
Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile

(Submitted by: Reza Aftabizadeh)

Abstract. Using a fixed–point theorem of cone expansion/compression type, we show the existence of at least three positive radial solutions for the class of quasi–linear elliptic systems

$$\begin{cases} -\Delta_p u &= \lambda k_1(|x|) f(u,v) & \text{in} & \Omega, \\ -\Delta_q v &= \lambda k_2(|x|) g(u,v) & \text{in} & \Omega, \\ (u,v) &= (a,b) & \text{on} & \partial\Omega \end{cases}$$

where the nonlinearities $f, g \in C([0,+\infty)^2; [0,+\infty))$ are superlinear at zero and sublinear at $+\infty$. The parameters λ, a and b are positive, Ω is the ball in $\mathbb{R}^N$, with $N \geq 3$, of radius R_0 which is centered at the origin, $1 < p,\ q \leq 2$, and $k_1, k_2 \in C([0,R_0]; [0,+\infty))$.

1. Introduction

This paper is devoted to the study of a class of systems involving quasi–linear elliptic equations subject to non–homogeneous Dirichlet boundary conditions in a ball of $\mathbb{R}^N$, with $N \geq 3$. More precisely, using a fixed–point theorem of cone expansion/compression type, we establish both existence

Accepted for publication: January 2011.
AMS Subject Classifications: 35J65, 35J55, 53C35.
[1]Research partially supported by the National Institute of Science and Technology of Mathematics INCT-Mat, CAPES and CNPq grants 307400/2009-3 and 620108/2008-8.
[2]Supported by FONDECYT Grant # 1051055.
[3]Supported by FONDECYT Grant # 1080430.

and multiplicity of positive radial solutions for the class of quasi-linear elliptic systems

$$\begin{cases} -\Delta_p u &= \lambda k_1(|x|)f(u,v) & \text{in} & \Omega, \\ -\Delta_q v &= \lambda k_2(|x|)g(u,v) & \text{in} & \Omega, \\ (u,v) &= (a,b) & \text{on} & \partial\Omega. \end{cases} \tag{1.1}$$

Here λ is a positive parameter, a and b are non–negative constants, Ω is the ball of $\mathbb{R}^N$, with $N \geq 3$, of radius R_0 centered at origin, and

$$\Delta_m u = div(|\nabla u|^{m-2}\nabla u)$$

is the $m-$Laplacian operator where $1 < m \leq 2(< N)$. The main difficulties in dealing with these systems lie in the facts that the systems might be *non–variational*, and that they contain those *quasi–linear operators* for which $p \neq q$ might occur. We overcome these difficulties by combining appropriate changes of variables with fixed–point techniques of Krasnosel'skii.

We will assume the following five hypotheses:

(K_0) The functions $k_1, k_2 : [0, R_0] \to [0, +\infty)$ are continuous, and are not identically zero in any subinterval of $[0, R_0]$.

(H_0) The nonlinearities $f, g \in C([0, +\infty)^2, [0, +\infty))$ are increasing, satisfying

$$f(s,t), g(s,t) > 0, \quad \text{if} \quad s + t > 0.$$

(H_1) The non-linear terms $f(s,t)$ and $g(s,t)$ are, respectively, "$p-$superlinear" and "$q-$superlinear" at the origin, or in other words

$$\lim_{|(u,v)|\to 0} \frac{f(u,v)}{|(u,v)|^{p-1}} = 0 \quad \text{and} \quad \lim_{|(u,v)|\to 0} \frac{g(u,v)}{|(u,v)|^{q-1}} = 0.$$

(H_2) The non–linear terms $f(s,t)$ and $g(s,t)$ are, respectively, "$p-$sublinear" and "$q-$sublinear" at infinity, or in other words

$$\lim_{|(u,v)|\to +\infty} \frac{f(u,v)}{|(u,v)|^{p-1}} = 0 \quad \text{and} \quad \lim_{|(u,v)|\to +\infty} \frac{g(u,v)}{|(u,v)|^{q-1}} = 0.$$

Here we use the notation $|(u,v)| = |u| + |v|$.

$(H_3)_{a,b}$ Given $a, b > 0$, we suppose that there exist $\bar{\tau}, \bar{\theta}, M > 0$ such that, for all $\tau > \bar{\tau}$, we have

$$\frac{f(\alpha\tau + a, \beta\tau + b)}{f(\alpha + a, \beta + b)} \leq M\varphi_p(\tau), \quad \text{for all } \alpha, \beta \geq \bar{\theta} \tag{1.2}$$

and

$$\frac{g(\alpha\tau + a, \ \beta\tau + b)}{g(\alpha + a, \beta + b)} \leq M\varphi_q(\tau), \quad \text{for all } \alpha, \beta \geq \bar{\theta} \tag{1.3}$$

where, for $m = p$ or $m = q$, the function φ_m is continuous, satisfying the condition

$$\int_1^{+\infty} \tau^{\frac{N(1-m)}{N-m}} \varphi_m(\tau) d\tau < +\infty.$$

Our three main results are the following.

Theorem 1.1. *Suppose that hypotheses* (K_0), (H_0), (H_2), *and* $(H_3)_{a,b}$ *hold, with* $a > 0$ *or* $b > 0$. *Then System* (1.1) *has at least one positive solution.*

Theorem 1.2. *Suppose that hypotheses* (K_0), (H_0), *and* (H_1)–$(H_3)_{a,b}$ *hold, with* $a > 0$ *or* $b > 0$. *Then there exists* $\tilde{\delta} > 0$ *such that, for all* $0 < a + b \leq \tilde{\delta}$, *there exists a* $\bar{\lambda} > 0$ *with the following property: For every* $\lambda > \bar{\lambda}$, *System (1.1) has at least three positive solutions.*

Theorem 1.3. *Suppose that hypotheses* $(K_0), (H_0)$, *and* (H_1)–$(H_3)_{a,b}$ *hold, with* $a = b = 0$. *Then there exists a* $\bar{\lambda} > 0$ *such that, for all* $\lambda > \bar{\lambda}$, *System* (1.1) *has at least two positive solutions.*

The study of System (1.1) was motivated in part by several recent works on elliptic problems involving radial symmetry. Among others we mention [1], [2], [5], [11], [13], [14], [16], [17], [18], [19], [21], [22], [24] with references therein. For results about a class of second–order elliptic problems of the form $-\Delta u = \lambda k(|x|) f(u)$ in Ω with non–homogeneous boundary condition $u = a$ on $\partial\Omega$, where Ω is the ball of radius R_0, see [12]. For elliptic problems involving $p-$Laplacian equations, see [10], [4], [25] and the references therein. Observe that, in [10], the equation $-\Delta_p u = f(x,u)$ in Ω, with $u \in W_0^{1,p}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$ is studied using the Leray–Schauder continuation principle, the direct method of calculus of variations, and the mountain pass theorem. We mention here the article [15], where the main focus is on systems involving the $p-$Laplacian in an exterior domain. In [4], systems involving the $p-$Laplacian operator and sublinear, non–linear terms at infinity are studied. For a survey on elliptic systems, see [8].

It is well known that problems involving the $p-$Laplacian operator appear in many contexts. Some of these problems come from different areas of applied mathematics and physics. For example, they may be found in the study of non–Newtonian fluids, non–linear elasticity, and reaction–diffusions. For discussions about problems modelled by these boundary-value problems, see for example [9].

Examples. Note that the hypotheses of Theorems 1.1 and 1.2 are for example satisfied by nonlinearities of the following two forms:

(a) Let $f, g : [0, +\infty)^2 \to [0, +\infty)$ be increasing functions such that

$$f(u,v) = \frac{v^{p_1}}{1+(u+v)^{q_1}} \qquad \text{and} \qquad g(u,v) = \frac{u^{p_2}}{1+(u+v)^{q_2}},$$

where $N > p, q,\ p_1 > p-1$, and $p_2 > q-1$. In addition, assume that $0 < p_1 - q_1 < \min\{1, (N(p-2)+p)/(N-p)\}$ and $0 < p_2 - q_2 < \min\{1, (N(q-2)+q)/(N-q)\}$.

(b)

$$f(u,v) = \frac{v^{p_1}}{1+(u+v)^{q_1}} \qquad \text{and} \qquad g(u,v) = (u^{q_2}+1)\arctan(u^{p_2}+v^{p_2})$$

where $N > p,\ p_1 > p-1$, and $p_2 > q-1$. In addition, assume that $0 < p_1 - q_1 < \min\{1, (N(p-2)+p)/(N-p)\}$ and $0 < q_2 < \min\{1, (N(q-2)+q)/(N-q)\}$.

The paper is organized as follows: In Section 2, we show that solving System (1.1) is equivalent to finding a fixed point of a suitable completely continuous operator F defined on an associated cone K_1 consisting of non–negative, concave functions. We introduce a positive real number τ^* associated with the operator F and the initial cone K_1, which will be crucial in our fixed–point argument. (See (2.4).) Section 3 is devoted to proving Theorems 1.1, 1.2 and 1.3. Section 4 applies our results to non–radial, quasi-linear elliptic systems.

2. Preliminary results

In this section, we establish the existence of radial solutions of System (1.1). In fact, we obtain positive solutions $u = u(r),\ v = v(r)$ of the system of ordinary differential equations

$$\begin{cases} -(r^{N-1}\phi_p(u'))' = r^{N-1}\lambda k_1(r) f(u+a, v+b) & \text{in } (0, R_0), \\ -(r^{N-1}\phi_q(v'))' = r^{N-1}\mu k_2(r) g(u+a, v+b) & \text{in } (0, R_0) \\ u(R_0) = v(R_0) = u'(0) = v'(0) = 0, \end{cases} \tag{2.1}$$

where $\phi_m(t) = |t|^{m-2}t$. By applying the change of variables $t = a_m(r)$, we can define $z(t) = u(r_p(t))$ and $w(t) = v(r_q(t)))$, where $a_m : [0, R_0) \longrightarrow [0, +\infty)$ is given by

$$a_m(r) = \frac{m-1}{N-m}\left[r^{(m-N)/(m-1)} - R_0^{(m-N)/(m-1)}\right]$$

and $r_m(t)$ is the inverse function of $a_m(r)$. Thus System (2.1) can be rewritten as

$$\begin{cases} -(\phi_p(z'(t)))' = \lambda r_p^{(N-1)p/(p-1)}(t)h_1(r)f(z(t)+a, w(a_q r_p(t))+b) \\ \qquad\qquad\qquad \text{in} \quad (0,+\infty), \\ -(\phi_q(w'(t)))' = \lambda r_q^{(N-1)q/(q-1)}(t)h_2(r)f(z(a_p r_q(t))+a, w(t)+b) \\ \qquad\qquad\qquad \text{in} \quad (0,+\infty), \\ z(0)=w(0) \;=\; z'(+\infty)=w'(+\infty) \;=\; 0, \end{cases} \tag{2.2}$$

where $h_1(t)$ and $h_2(t)$ are continuous functions which do not vanish identically on any subinterval of $(0,+\infty)$.

Integrating the equations of System (2.2) twice and using the boundary conditions, we see that this system takes the form of the system of integral equations

$$\begin{cases} z(t) = \displaystyle\int_0^t \Big[\int_s^{+\infty} \lambda G_p(\tau)h_1(\tau)f(z(\tau)+a, w(a_q r_p(\tau))+b)d\tau\Big]^{1/(p-1)} ds \\ w(t) = \displaystyle\int_0^t \Big[\int_s^{+\infty} \lambda G_q(\tau)h_2(\tau)g(z(a_p r_q(\tau))+a, w(\tau)+b)d\tau\Big]^{1/(q-1)} ds, \end{cases}$$

where

$$G_m(\tau) = \Big(R_0^{(m-N)/(m-1)} + \frac{N-m}{m-1}\tau\Big)^{m(1-N)/(N-m)}. \tag{2.3}$$

Consequently, we may solve System (2.1) using fixed–point techniques. For this, we will use the following well-known fixed–point theorem, which we state without proof. (See for example [6], [7], [20].)

Lemma 2.1. *Let X be a Banach space endowed with norm $|\cdot|$, and let $C \subset X$ be a cone in X. For $R>0$, define $C_R = C \cap B[0,R]$, where $B[0,R] = \{x \in X : |x| \le R\}$ denotes the closed ball of radius R centered at origin of X. Suppose that $F : C \longrightarrow X$ is a completely continuous map and that there exists $0<r<R$ such that either*

(i) $|Fx| < |x|$, *for* $x \in \partial C_r$, *and* $|Fx| > |x|$, *for* $x \in \partial C_R$, *or*
(ii) $|Fx| > |x|$, *for* $x \in \partial C_r$, *and* $|Fx| < |x|$, *for* $x \in \partial C_R$,

where $\partial C_R = \{x \in C : |x| = R\}$. Then F has a fixed point $u \in C$, with $r < |u| < R$.

Consider the space

$$X = \{(z,w) : \;\; z,\; w \in C([0,+\infty),\mathbb{R}) \text{ are bounded functions}\}$$

endowed with the norm $\|(z,w)\| = |z|_\infty + |w|_\infty$ where $|z|_\infty = \sup\{|z(t)| : t \in [0,+\infty)\}$. Let $F : K_1 \longrightarrow X$ be the operator defined by

$$F(z,w)(t) = (A(z,w)(t), B(z,w)(t)),$$

where

$$A(z,w)(t) = \int_0^t \Big[\int_s^{+\infty} \lambda G_p(\tau)h_1(\tau)f(z(\tau)+a, w(a_q r_p(\tau))+b)d\tau\Big]^{\frac{1}{p-1}} ds,$$

$$B(z,w)(t) = \int_0^t \Big[\int_s^{+\infty} \lambda G_q(\tau)h_2(\tau)g(z(a_p r_q(\tau))+a, w(\tau)+b)d\tau\Big]^{\frac{1}{q-1}} ds,$$

and where K_1 is the cone defined by

$$K_1 = \{(z,w) \in X : z, w \text{ are non–negative, concave, and } z(0) = w(0) = 0\}.$$

Note that each coordinate of the elements of K_1 is an increasing function.

Lemma 2.2. *The operator F is well defined, and the cone K_1 is invariant under F. In addition, F is a completely continuous operator.*

Proof. For all $s \geq 0$, note that

$$\int_s^{+\infty} G_m(\tau)d\tau \;=\; \frac{1}{N}G_m(s)^{N(m-1)/m(N-1)}$$

and thus

$$\int_0^{+\infty} \Big(\int_s^{+\infty} G_m(\tau)d\tau\Big)^{\frac{1}{m-1}} ds < +\infty.$$

The first statement now follows.

Note also that the functions $A(z,w)(t)$ and $B(z,w)(t)$ are of class C^2 whose derivatives are, respectively, given by

$$\frac{d}{dt}A(z,w)(t) = \Big[\int_t^{+\infty} G_p(\tau)f(z(\tau)+a, w(a_q r_p(\tau))+b)d\tau\Big]^{1/(p-1)}$$

$$\frac{d^2}{dt^2}A(z,w)(t) = -\frac{1}{p-1}G_p(t)\Big[\frac{d}{dt}A(z,w)(t)\Big]^{p-2} f(z(t)+a, w(a_q r_p(t))+b)$$

and

$$\frac{d}{dt}B(z,w)(t) = \Big[\int_t^{+\infty} G_q(\tau)g(z(a_p r_q(\tau))+a, w(\tau)+b)d\tau\Big]^{1/(q-1)}$$

$$\frac{d^2}{dt^2}B(z,w)(t) = -\frac{1}{q-1}G_q(t)\Big[\frac{d}{dt}B(z,w)(t)\Big]^{q-2} g(z(a_p r_q(t))+a, w(t)+b).$$

Hence, $A(z,w)(t)$ and $B(z,w)(t)$ are increasing and concave. Therefore, $F(K_1) \subset K_1$. This proves the second statement.

It remains to prove that F is a completely continuous operator. Let $|(z_n, w_n)|_\infty \leq C_0$, and let

$$M_1 = \max\{f(t,s) : t, s \geq 0 \text{ and } t+s \in [0, C_0]\}.$$

It follows that

$$|A(z_n,w_n)(t)| \leq M_1^{1/(p-1)} \int_0^{+\infty} \Big[\int_s^{+\infty} G_p(\tau) d\tau \Big]^{1/(p-1)} ds \qquad \text{and}$$

$$\Big| \frac{d}{dt} A(z_n,w_n)(t) \Big| \leq \Big[M_1 \int_0^{+\infty} G_p(\tau) d\tau \Big]^{1/(p-1)}.$$

According to the Arzelá–Ascoli compactness criterion for uniform convergence, there exists a uniformly convergent subsequence $\{A(z_{n_k}, w_{n_k})\} \subset \{A(z_n, w_n)\}$ on compact subsets of $[0, +\infty)$. To prove that there exists a uniformly convergent subsequence of $\{A(z_n, w_n)\}$, it suffices to recall that, given $\epsilon > 0$, there is a $T = T(\epsilon)$ such that

$$\int_T^{+\infty} \Big[\int_s^{+\infty} G_p(\tau) d\tau \Big]^{1/(p-1)} ds \quad < \quad \epsilon.$$

Similarly, there exists a uniformly convergent subsequence of $\{B(z_n, w_n)\}$, and hence of $\{F(z_n, w_n)\}$. We next verify that F is continuous. Let $\{(z_n, w_n)\} \in X$ be such that $|(z_n, w_n) - (z_0, w_o)|_\infty \longrightarrow 0$ as $n \to \infty$. Thus,

$$|A(z_n,w_n)(t) - A(z_0,w_0)(t)| \ \leq \int_0^{+\infty} |\Gamma_n(s) - \Gamma_0(s)| \, ds,$$

where

$$\begin{cases} \Gamma_n(s) = \displaystyle\int_s^{+\infty} G_p(\tau) f(z_n(\tau) + a, w_n(a_q r_p(\tau)) + b) d\tau \\ \Gamma_0(s) = \displaystyle\int_s^{+\infty} G_p(\tau) f(z_0(\tau) + a, w_0(a_q r_p(\tau)) + b) d\tau. \end{cases}$$

From $|(z_n, w_n) - (z_0, w_0)|_\infty \longrightarrow 0$ we conclude that $\Gamma_n(s) \longrightarrow \Gamma_0(s)$ and that $\Gamma_n(s) \leq C/N G_p(s)^{N(p-1)/p(N-1)}$, for all $s \in [0, +\infty)$. Then, by Lebesgue's dominated convergence theorem, we have $|A(z_n, w_n) - A(z_0, w_0)|_\infty \longrightarrow 0$. Similarly, B is continuous. Therefore, F is continuous. □

Given $(z, w) \in K_1 \setminus \{(0,0)\}$, there clearly exists a unique $\tau_1 = \tau_1(z, w)$ such that $2[z(\tau_1) + w(\tau_1)] = \|(z, w)\|$. Define

$$\tau* \ = \ \sup\{\tau_1(F(z,w)) : (z,w) \in K_1\}. \tag{2.4}$$

The following will be crucial in proving our main results.

Theorem 2.3. *Under hypothesis*(H_3), *the number* $\tau*$ *is a positive real number.*

Proof. Suppose to the contrary that $\tau* = +\infty$. Then there must exist a sequence $\{(z_n, w_n)\} \subset K_1 \setminus \{0\}$ such that the sequence $\{\tau_n\} = \{\tau_1(F(z_n, w_n))\}$ is a strictly increasing sequence of positive real numbers converging to $+\infty$. By the definition of τ_n, we have

$$\begin{aligned}&2\big(A(z_n, w_n)(\tau_n) + B(z_n, w_n)(\tau_n)\big)\\&= \int_0^\infty \Big[\int_s^{+\infty} \lambda\Phi_n(\tau)d\tau\Big]^{1/(p-1)} ds \quad + \int_0^\infty \Big[\int_s^{+\infty} \lambda\Psi_n(\tau)d\tau\Big]^{1/(q-1)} ds,\end{aligned}$$

where

$$\begin{aligned}\Phi_n &= G_p(\tau)h_1(\tau)f(z_n(\tau) + a, w_n(a_q r_p(\tau)) + b)\\ \Psi_n &= G_q(\tau)h_2(\tau)g(z_n(a_p r_q(\tau)) + a, w_n(\tau) + b).\end{aligned}$$

Set

$$\Upsilon_n(s) = \int_s^{+\infty} \lambda\Phi_n(\tau)d\tau \quad \text{and} \quad \Lambda_n(s) = \int_s^{+\infty} \lambda\Psi_n(\tau)d\tau.$$

Then we could write

$$\begin{aligned}&2\int_0^{\tau_n} \Upsilon_n(s)^{1/(p-1)}ds + 2\int_0^{\tau_n} \Lambda_n(s)^{1/(q-1)}ds\\&= \int_0^\infty \Upsilon_n(s)^{1/(p-1)}ds + \int_0^\infty \Lambda_n(s)^{1/(q-1)}ds\end{aligned}$$

and hence

$$\begin{aligned}&\int_0^{\tau_n} \Upsilon_n(s)^{1/(p-1)}\ ds + \int_0^{\tau_n} \Lambda_n(s)^{1/(q-1)}\ ds\\&= \int_{\tau_n}^\infty \Upsilon_n(s)^{1/(p-1)}\ ds + \int_{\tau_n}^\infty \Lambda_n(s)^{1/(q-1)}\ ds. \qquad (2.5)\end{aligned}$$

Integration by parts yields

$$\begin{aligned}\int_0^{\tau_n} \Upsilon_n(s)^{1/(p-1)}ds &= \tau_n\Upsilon_n(s)^{1/(p-1)} + \frac{1}{p-1}\int_0^{\tau_n} s\Phi_n(s)\Upsilon_n(s)^{(2-p)/(p-1)}ds\\ \int_0^{\tau_n} \Lambda_n(s)^{1/(q-1)}ds &= \tau_n\Lambda_n(s)^{1/(q-1)} + \frac{1}{q-1}\int_0^{\tau_n} s\Psi_n(s)\Lambda_n(s)^{(2-q)/(q-1)}ds\\ \int_{\tau_n}^\infty \Upsilon_n(s)^{1/(p-1)}ds &= -\tau_n\Upsilon_n(s)^{1/(p-1)} + \frac{1}{p-1}\int_{\tau_n}^\infty s\Phi_n(s)\Upsilon_n(s)^{(2-p)/(p-1)}ds\\ \int_{\tau_n}^\infty \Lambda_n(s)^{1/(q-1)}ds &= -\tau_n\Lambda_n(s)^{1/(q-1)} + \frac{1}{p-1}\int_{\tau_n}^\infty s\Psi_n(s)\Lambda_n(s)^{(2-q)/(q-1)}ds,\end{aligned}$$

which together with (2.5) imply

$$2\tau_n\{\Upsilon_n(s)^{1/(p-1)}ds+\Lambda_n(s)^{1/(q-1)}ds\}$$
$$+\frac{1}{p-1}\int_0^{\tau_n}s\Phi_n(s)\Upsilon_n(s)^{(2-p)/(p-1)}ds+\frac{1}{q-1}\int_0^{\tau_n}s\Psi_n(s)\Lambda_n(s)^{(2-q)/(q-1)}ds$$
$$=\frac{1}{p-1}\int_{\tau_n}^{\infty}s\Phi_n(s)\Upsilon_n(s)^{(2-p)/(p-1)}ds+\frac{1}{q-1}\int_{\tau_n}^{\infty}s\Psi_n(s)\Lambda_n(s)^{(2-q)/(q-1)}ds.$$

Thus,

$$\frac{1}{p-1}\int_0^{\tau_n}s\Phi_n(s)\Upsilon_n(s)^{(2-p)/(p-1)}ds+\frac{1}{q-1}\int_0^{\tau_n}s\Psi_n(s)\Lambda_n(s)^{(2-p)/(p-1)}ds$$
$$\le\frac{1}{p-1}\int_{\tau_n}^{\infty}s\Phi_n(s)\Upsilon_n(s)^{(2-p)/(p-1)}ds+\frac{1}{q-1}\int_{\tau_n}^{\infty}s\Psi_n(s)\Lambda_n(s)^{(2-p)/(p-1)}ds.$$

Since $1<p,\ q\le 2$, for all $n\in\mathbb{N}$, we have

$$\frac{1}{p-1}\Upsilon_n(s)^{(2-p)/(p-1)}\int_0^{\tau_n}s\Phi_n(s)ds+\frac{1}{q-1}\Lambda_n(s)^{(2-q)/(q-1)}\int_0^{\tau_n}s\Psi_n(s)ds$$
$$\le\frac{1}{p-1}\Upsilon_n(s)^{(2-p)/(p-1)}\int_{\tau_n}^{\infty}s\Phi_n(s)ds+\frac{1}{q-1}\Lambda_n(s)^{(2-q)/(q-1)}\int_{\tau_n}^{\infty}s\Psi_n(s)ds.$$

It follows that the following two possibilities would have to occur.

Case 1. There is a subsequence $\{\tau_{n_k}\}$ such that

$$\int_0^{\tau_{n_k}}s\Phi_{n_k}(s)ds\le\int_{\tau_{n_k}}^{\infty}s\Phi_{n_k}(s)ds.$$

Since the subsequences $\{z_{n_k}\}$ and $\{w_{n_k}\}$ are concave, it follows that

$$\int_0^{\tau_{n_k}}\tau K_{n_k}(\tau)d\tau\ge\int_{\tau_{n_k}}^{+\infty}\tau K_{n_k}(\tau)d\tau, \tag{2.6}$$

where $K_{n_k}(\tau)=G(\tau)h_1(\tau)f(\alpha_{n_k}\tau+a,\beta_{n_k}\tau+b)$ with $\alpha_{n_k}=\frac{z_{n_k}(\tau_{n_k})}{\tau_{n_k}}$ and $\beta_{n_k}=\frac{w_{n_k}(\tau_{n_k})}{\tau_{n_k}}$. According to assumption (H_0), we have

$$f(\alpha_{n_k}\tau+a,\beta_{n_k}\tau+b)\ge f(\alpha_{n_k}+a,\beta_{n_k}+b),\quad\text{for all }\tau\ge 1.$$

It then follows from (2.6) that

$$\int_1^{\tau_{n_k}}\tau K_{n_k}(\tau)d\tau\ge\int_{\tau_{n_k}}^{+\infty}\tau K_{n_k}(\tau)d\tau.$$

Therefore,

$$\int_1^{\tau_{n_k}} \tau \underline{h}(\tau) G_p(\tau) d\tau \geq \int_{\tau_{n_k}}^{+\infty} \tau \overline{h}(\tau) G(\tau) \frac{f(\alpha_{n_k}\tau + a, \beta_{n_k}\tau + b)}{f(\alpha_{n_k} + a, \beta_{n_k} + b)} d\tau.$$

Using assumption (H_3), we see easily that the right-side integral of the preceding inequality converges to zero as $n \to \infty$. But this is impossible, since

$$\int_1^{+\infty} \tau \underline{h}(\tau) G(\tau) d\tau > 0.$$

Case 2. There is a subsequence $\{\tau_{n_k}\}$ such that

$$\int_0^{\tau_{n_k}} s\Phi_{n_k}(s) ds \geq \int_{\tau_{n_k}}^{\infty} s\Phi_{n_k}(s) ds.$$

Note that given non–negative constants a, b, x, y, u, v such that $ax + bu \leq ay + bv$ and $u \geq v$, it follows that $x \leq y$. This simple fact implies

$$\int_0^{\tau_{n_k}} s\Psi_{n_k}(s) ds \leq \int_{\tau_{n_k}}^{\infty} s\Psi_{n_k}(s) ds.$$

Proceeding as in Case 1 again leads to a contradiction. This completes the proof of Theorem 2.3. □

3. Proofs of Main Results

In this section, we prove our main results. For this, we define

$$K = \Big\{(z, w) \in K_1 : \ 2(z(t) + w(t)) \geq \|(z, w)\|, \ \text{ for all } \ t \geq \tau * \Big\}.$$

It is not difficult to see that K is a cone which is invariant under the operator F.

Lemma 3.1. *Suppose that hypotheses (K_0), (H_0) and (H_1) hold, that $\lambda > 0$, and that $a + b > 0$. Then there exists an $r_1 > 0$ sufficiently small such that, for $(z, w) \in K$, we have*

$$||F(z, w)|| \geq ||(z, w)|| \quad \textit{when} \quad ||(z, w)|| = r_1.$$

Proof. By hypothesis (H_0) we know that $f(a, b) > 0$. Let $r_1 > 0$, and let $(z, w) \in \partial K_{r_1}$. Then

$$\begin{aligned} |A(z, w)|_\infty &\geq A(z, w)(\tau^*) \\ &\geq \tau^* \Big[\int_{\tau^*}^{+\infty} \lambda G_p(\tau) h_1(\tau) f(z(\tau) + a, w(a_q r_p(\tau)) + b) d\tau \Big]^{\frac{1}{p-1}} \end{aligned}$$

$$\geq C(a,b,\lambda,\tau^*,p,N)) = \frac{C(a,b,\lambda,\tau^*,p,N))}{\|(z,w)\|}\|(z,w)\|,$$

where

$$C(a,b,\lambda,\tau^*,p,N) = \tau^*(f(a,b)\lambda)^{\frac{1}{p-1}}\Big[\int_{\tau^*}^{+\infty} G_p(\tau)h_1(\tau)d\tau\Big]^{\frac{1}{p-1}}.$$

Choosing $r_1 > 0$ sufficiently small, the result follows. □

Lemma 3.2. *Suppose that hypotheses* (K_0)*,* (H_0) *and* (H_2) *hold, that* $\lambda > 0$*, and that* $a+b>0$*. Then there exists an* $r_2 > 0$ *sufficiently large such that, for* $(z,w)\in K$*, we have*

$$||F(z,w)|| \geq ||(z,w)|| \quad \textit{when} \ \ ||(z,w)|| = r_1.$$

Proof. By hypothesis (H_2) we know that, given $\varepsilon > 0$, there exists an $r_\epsilon > 0$ such that

$$f(s,t) \leq \varepsilon(t+s)^{p-1} \quad \text{ and } \quad g(s,t) \leq \varepsilon(t+s)^{q-1}, \ \text{ for all } t+s \geq r$$

where $r \geq r_\epsilon$. It follows from hypothesis (H_0) that, for $(z,w) \in \partial K_r$, we have

$$\begin{aligned}|A(z,w)|_\infty &= \int_0^{\infty}\Big[\int_s^{+\infty} \lambda G_p(\tau)h_1(\tau)f(z(\tau)+a, w(a_q r_p(\tau))+b)d\tau\Big]^{\frac{1}{p-1}}ds\\ &\leq \varepsilon^{\frac{1}{p-1}}\lambda^{\frac{1}{p-1}}(|z|_\infty+|w|_\infty+a+b)\int_0^{+\infty}\Big[\int_s^{+\infty} G_p(\tau)h_1(\tau)\Big]^{\frac{1}{p-1}}d\tau ds\\ &\leq \varepsilon^{\frac{1}{p-1}}\lambda^{\frac{1}{p-1}}\int_0^{+\infty}\Big[\int_s^{+\infty} G_p(\tau)h_1(\tau)\Big]^{\frac{1}{p-1}}d\tau 2r = I_1\end{aligned}$$

where r_ϵ may be chosen such that $r_\epsilon > a+b$. Similarly,

$$|B(z,w)|_\infty \leq \varepsilon^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}\int_0^{+\infty}\Big[\int_s^{+\infty} G_q(\tau)h_1(\tau)\Big]^{\frac{1}{q-1}}d\tau 2r = I_2.$$

Choose $\varepsilon > 0$ such that $I_1 + I_2 < \frac{1}{2}$. Taking $r_2 \geq r_\epsilon$ completes the proof. □

Lemma 3.3. *Suppose that hypotheses* (K_0)*,* (H_0)*, and* $(H_3)_{a,b}$*, with* $a,b\geq 0$*, hold. Given* R*, there exists a* $\bar\lambda > 0$ *such that, for* $(z,w)\in K$ *and all* $\lambda > \bar\lambda$*, we have*

$$||F(z,w)|| \geq ||(z,w)|| \quad \textit{when} \ \ ||(z,w)|| = R.$$

Proof. It follows easily from hypothesis (H_0) that there exist constants $c(f), c(g) > 0$ such that, for every $s,t \geq 0$ satisfying $2(s+t) = R$, we have

$$\frac{f(s,t)}{(s+t)^{p-1}} \geq c(f) \quad \text{and} \quad \frac{g(s,t)}{(s+t)^{q-1}} \geq c(g).$$

Given (z, w) such that $||(z, w)|| = R$, we obtain

$$\begin{aligned} |A(z,w)|_\infty &\geq A(z,w)(\tau^*) \\ &\geq \tau^* \Big[\int_{\tau^*}^{+\infty} \lambda G_p(\tau) h_1(\tau) f(z(\tau) + a, w(a_q r_p(\tau)) + b) d\tau \Big]^{\frac{1}{p-1}}. \end{aligned}$$

Similarly,

$$\begin{aligned} |B(z,w)|_\infty &\geq B(z,w)(\tau^*) \\ &\geq \tau^* \Big[\int_{\tau^*}^{+\infty} \lambda G_q(\tau) h_1(\tau) f(z(a_p r_q(\tau)) + a, w(\tau) + b) d\tau \Big]^{\frac{1}{q-1}}. \end{aligned}$$

On the other hand, given $(z, w) \in K$, there exists a $\tau_1 = \tau_1(z, w) > 0$ such that $2(z(\tau_1) + w(\tau_1)) = \|(z, w)\|$. Since $\tau^* > \tau_1$, and since z, w are increasing functions, we have

$$2(z(\tau^*) + w(\tau^*)) \geq \|(z, w)\|. \tag{3.1}$$

Note that, given $\tau \geq 0$, either $a_q r_p(\tau) \geq \tau$ or $a_p r_q(\tau) \geq \tau$.

In the case $a_q r_p(\tau^*) \geq \tau^*$, we find

$$\begin{aligned} |F(z,w)|_\infty &= |A(z,w)|_\infty + |B(z,w)|_\infty \\ &\geq A(z,w)(\tau^*) + B(z,w)(\tau^*) \geq A(z,w)(\tau^*) \\ &\geq \tau^* \lambda^{\frac{1}{p-1}} \left[f(z(\tau^*), w(\tau^*)) \right]^{\frac{1}{p-1}} \left[\int_{\tau^*}^{+\infty} G_p(\tau) h_1(\tau) d\tau \right]^{\frac{1}{p-1}} \\ &\geq \tau^* \lambda^{\frac{1}{p-1}} \left[f(z(\tau_1), w(\tau_1)) \right]^{\frac{1}{p-1}} \left[\int_{\tau^*}^{+\infty} G_p(\tau) h_1(\tau) d\tau \right]^{\frac{1}{p-1}} \\ &\geq \tau^* \lambda^{\frac{1}{p-1}} c(f)^{\frac{1}{p-1}} \left[\int_{\tau^*}^{+\infty} G_p(\tau) h_1(\tau) d\tau \right]^{\frac{1}{p-1}} \frac{R}{2}. \end{aligned}$$

In the case $a_q r_p(\tau^*) \leq \tau^*$, it is easy to verify that $a_p r_q(\tau^*) \geq \tau^*$. Hence

$$\begin{aligned} |F(z,w)|_\infty &= |A(z,w)|_\infty + |B(z,w)|_\infty \\ &\geq A(z,w)(\tau^*) + B(z,w)(\tau^*) \geq B(z,w)(\tau^*) \\ &\geq \tau^* \lambda^{\frac{1}{q-1}} c(g)^{\frac{1}{q-1}} \left[\int_{\tau^*}^{+\infty} G_q(\tau) h_2(\tau) d\tau \right]^{\frac{1}{q-1}} \frac{R}{2}. \end{aligned}$$

It is not difficult to show that there exists a $\bar{\lambda} > 0$ such that, for $(z, w) \in K$ and all $\lambda > \bar{\lambda}$, we have

$$||F(z,w)|| \geq ||(z,w)|| \quad \text{when} \quad ||(z,w)|| = R$$

which completes the proof. □

Lemma 3.4. *Suppose that hypotheses* (K_0), (H_0), *and* (H_1) *hold. Given* $R > 0$, *there exist* $\tilde{\delta} > 0$ *and* $R_1 \in (0, R)$ *such that, for* $(z, w) \in K$ *and all* $0 \leq a + b \leq \tilde{\delta}$, *we have*

$$||F(z,w)|| \leq ||(z,w)|| \quad \textit{when} \quad ||(z,w)|| = R_1.$$

Proof. By hypothesis (H_1) we know that, given $\varepsilon > 0$, there exists $0 < R_\epsilon < R$ such that, for all $0 \leq t + s \leq \frac{R_\epsilon}{2}$, we have

$$f(s,t) \leq \varepsilon(t+s)^{p-1} \qquad \text{and} \qquad g(s,t) \leq \varepsilon(t+s)^{q-1}.$$

Hence, for all $0 \leq a + b \leq \frac{R_\epsilon}{2}$ and $(z, w) \in \partial K_{\frac{R_\epsilon}{2}}$, we have

$$\begin{aligned}
|A(z,w)|_\infty &= \int_0^{+\infty} \Big[\int_s^{+\infty} \lambda G_p(\tau) h_1(\tau) f(z(\tau) + a, w(a_q r_p(\tau)) + b) d\tau \Big]^{\frac{1}{p-1}} ds \\
&\leq (\varepsilon\lambda)^{\frac{1}{p-1}} (|z|_\infty + |w|_\infty + a + b) \int_0^{+\infty} \Big[\int_s^{+\infty} G_p(\tau) h_1(\tau) \Big]^{\frac{1}{p-1}} d\tau ds \\
&\leq (\varepsilon\lambda)^{\frac{1}{p-1}} \int_0^{+\infty} \Big[\int_s^{+\infty} G_p(\tau) h_1(\tau) \Big]^{\frac{1}{p-1}} d\tau R_\epsilon = J_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|B(z,w)|_\infty &= \int_0^{+\infty} \Big[\int_s^{+\infty} \lambda G_q(\tau) h_1(\tau) g(z(a_p r_q(\tau)) + a, w(\tau) + b) d\tau \Big]^{\frac{1}{q-1}} ds \\
&\leq (\varepsilon\lambda)^{\frac{1}{q-1}} \int_0^{+\infty} \Big[\int_s^{+\infty} G_q(\tau) h_1(\tau) \Big]^{\frac{1}{q-1}} d\tau R_\epsilon = J_2.
\end{aligned}$$

Choose $\varepsilon > 0$ such that $J_1 + J_2 < \frac{1}{2}$. Taking $R_1 = \delta = \frac{R_\epsilon}{2}$ completes the proof. □

We may now prove the theorems stated in the Introduction.

Proof of Theorem 1.1. From Lemmas 3.1 and 3.2 we conclude that the operator F has a fixed point, namely (z, w) in $K_{r_2} \setminus K_{r_1}$.

Proof of Theorem 1.2. Combining Lemmas 3.1 through 3.4, we have that there exists $\tilde{\delta} > 0$ such that, for all $0 < a + b \leq \tilde{\delta}$, there is a $\bar{\lambda} > 0$ satisfying the following property: For every $\lambda > \bar{\lambda}$, there exist three fixed points (z_1, w_1), (z_2, w_2), (z_3, w_3) of the operator F such that

$$r_1 < ||(z_1,w_1)|| < R_1 < ||(z_2,w_2)|| < R < ||(z_3,w_3)|| < r_2.$$

Proof of Theorem 1.3. The proof of Theorem 1.3 is analogous to the preceding two proofs.

4. Non–Radial Problems

As an application, using the sub–and–super–solutions method, we show the existence of a positive solution for the class of elliptic systems

$$\begin{cases} -\Delta_p u &= \lambda \widetilde{k}_1(x) f(u,v) & \text{in} & \Omega, \\ -\Delta_q v &= \lambda \widetilde{k}_2(x) g(u,v) & \text{in} & \Omega, \\ (u,v) &= (a,b) & \text{on} & \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is the ball of radius R_0 centered at origin, the non–linear terms f, g satisfy assumptions (H_0) through (H_3), and the functions $\widetilde{k}_1$, $\widetilde{k}_2 \in C(\Omega, [0, +\infty))$ are not identically zero in any subinterval of $[0, R_0]$. We will further assume that $\widetilde{k}_1$ and $\widetilde{k}_2$ satisfy the following:

(K_1) $\quad 0 \leq \widetilde{k}_1(x) \leq k_1(|x|) \quad$ and $\quad 0 \leq \widetilde{k}_2(x) \leq k_2(|x|)$, for all $x \in \Omega$.

Note that System (4.1) is equivalent to the system

$$\begin{cases} -\Delta_p u &= \lambda \widetilde{k}_1(x) f(u+a, v+b) & \text{in} & \Omega, \\ -\Delta_q v &= \lambda \widetilde{k}_2(x) g(u+a, v+b) & \text{in} & \Omega, \\ (u,v) &= (0,0) & \text{on} & \partial\Omega. \end{cases} \tag{4.2}$$

The following is an existence result about the preceding non-radial system.

Theorem 4.1. *Suppose that hypotheses* (K_0) *and* (K_1) *hold, that the functions* $f(u,v)$ *and* $g(u,v)$ *satisfy hypotheses* (H_0) *and* (H_2), *and that* $a+b>0$. *Then System* (4.2) *has at least one positive solution.*

Proof. It follows from hypothesis (H_0) that, for all $M > 0$, there exists $\delta > 0$ such that, for all $u \in (0, \delta)$, we have

$$f(u+a, b) \geq M u^{p-1}. \tag{4.3}$$

Consider the eigenvalue problem

$$\begin{cases} -\Delta_p u &= \lambda \widetilde{k}_1(x) |u|^{p-2} u & \text{in} & \Omega, \\ u &= 0 & \text{on} & \partial\Omega. \end{cases}$$

Let λ_1 be the first eigenvalue associated to the first eigenfunction φ_1. Taking $t \geq 0$ sufficiently small, we have

$$\lambda_1 \widetilde{k}_1(x)(t\varphi_1)^{p-1} \leq \lambda_1 \widetilde{k}_1(x) \frac{f(t\varphi_1 + a, b)}{M} \leq \widetilde{k}_1(x) f(t\varphi_1 + a, b),$$

where M is chosen such that $\lambda_1 \leq M$. Therefore, $(t\varphi_1, 0)$ is a lower solution of system (4.2).

On the other hand, according to Theorem 1.1, the system

$$\begin{cases} -\Delta_p u &= \lambda k_1(|x|)f(u+a, v+b) & \text{in} & \Omega, \\ -\Delta_q v &= \lambda k_2(|x|)g(u+a, v+b) & \text{in} & \Omega, \\ (u,v) &= (0,0) & \text{on} & \partial\Omega \end{cases}$$

has a positive solution $(\overline{u}, \overline{v})$, which is an upper solution of System (4.2) by hypothesis (K_1). Taking t sufficiently small, we have $(t\varphi_1, 0) \leq (\overline{u}, \overline{v})$. Using the upper–lower solutions method, we obtain a positive solution of system (4.2). (See [23], [3].) □

References

[1] C. Bandle, C.V. Coffman, and M. Marcus, *Nonlinear elliptic problems in annular domains*, J. Differential Equations, 69 (1987), 322–345.

[2] C. Bandle and M.K. Kwong, *Semilinear elliptic problems in annular domains*, Z. Angew. Math. Phys., 40 (1989), 245–257.

[3] A. Cañada, P. Drábek, and J.L. Gámez, *Existence of positive solutions for some problems with nonlinear diffusion*, Trans. Amer. Math. Soc., 349 (1997), 4231–4249.

[4] M. Chhetri, D.D. Hai, and R. Shivaji, *On positive solutions for classes of $p-$Laplacian semipositone systems*, Discrete Contin. Dyn. Syst., 9 (2003), 1063–1071.

[5] C.V. Coffman and M. Marcus, *Existence and uniqueness results for semi-linear Dirichlet problems in annuli*, Arch. Rational Mech. Anal., 108 (1989), 293–307.

[6] K. Deimling, "Nonlinear Functional Analysis," Springer–Verlag, New York, 1985.

[7] D.G. de Figueiredo, *Positive solutions of semilinear elliptic equations*, Lecture Notes in Mathematics vol. 957, Springer–Verlag, Berlin–Heidelberg–New York, 1982, pp. 34–87.

[8] D. de Figueiredo, *Nonlinear elliptic systems*, An. Acad. Brasil. Cinc., 72 (2000), 453–469.

[9] J.I. Díaz, *Nonlinear partial differential equations and free boundaries. Vol. I. Elliptic equations*, 323 pp. of "Research Notes in Mathematics, "edit. Pitman, 106, Boston, MA (1985).

[10] G. Dinca, P. Jebelean, and J. Mawhin, *Variational and topological methods for Dirichlet problems with $p-$Laplacian*, Portugal. Math. (N.S.) 58 (2001), 339–378.

[11] J.M. do Ó, S. Lorca, and P. Ubilla, *Local superlinearity for elliptic systems involving parameters*, J. Differential Equations, 211 (2005), 1–19.

[12] J.M. do Ó, S. Lorca, and P. Ubilla, *Three positive radial solutions for elliptic equations in a ball*, Appl. Math. Lett., 18 (2005), 1163–1169.

[13] J.M. do Ó, S. Lorca, and P. Ubilla, *Three positive solutions for a class of elliptic systems in annular domains*, Proc. Edinburgh Math. Soc., 48 (2005), 365–373.

[14] J.M. do Ó, S. Lorca, and P. Ubilla, *Multiparameter elliptic equations in annular domains*, Contributions to nonlinear analysis, 233–245, Progr. Nonlinear Diff. Eq. Appl., 66, Birkhüser, Basel, 2006

[15] J.M. do Ó, S. Lorca, and P. Ubilla, *Positive radial solutions for some quasilinear elliptic systems in exterior domains*, Commun. Pure Appl. Anal., 5 (2006), 571–581.

[16] D.R. Dunninger and H. Wang, *Existence and multiplicity of positive solutions for elliptic systems*, Nonlinear Anal., 29 (1997), 1051–1060.
[17] D.R. Dunninger and H. Wang, *Multiplicity of positive radial solutions for an elliptic system on an annulus*, Nonlinear Anal., 42 (2000), 803–811.
[18] G. Ercole and A. Zumpano, *Positive solutions for the p-Laplacian in annuli*, Proc. Roy. Soc. Edinburgh Sect. A, 132 (2002), 595–610.
[19] X. Garaizar, *Existence of positive radial solutions for semilinear elliptic equations in the annulus* J. Diff. Eq., 70 (1987), 69–92.
[20] D. Guo and V. Lakshmikantham, "Nonlinear Problem in Abstract Cones," Academinc Press, Orlando, FL, 1988.
[21] D.D. Hai, *Positive solutions for semilinear elliptic equations in annular domains*, Nonlinear Anal., 37 (1999), 1051–1058.
[22] Y.H. Lee, *Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus*, J. Differential Equations, 174 (2001), 420–441.
[23] J. Schoenenberger–Deuel and P. Hess, *A criterion for the existence of solutions of non-linear elliptic boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A, 74 (1974/75), 49–54 (1976).
[24] H. Wang, *On the existence of positive solutions for semilinear elliptic equations in the annulus*, J. Differential Equations, 109 (1994), 1–7.
[25] H. Wang, *Positive radial solutions for quasilinear equations in the annulus*, Discrete Contin. Dyn. Syst. 2005, suppl., 878–885.