Differential and Integral Equations Volume 24, Numbers 9-10 (2011), 829–844

EXISTENCE OF POSITIVE SOLUTIONS TO DIRICHLET BOUNDARY-VALUE PROBLEMS FOR SOME RECURSIVE DIFFERENTIAL SYSTEMS

Robert Dalmasso
Laboratoire Jean Kuntzmann, Equipe EDP,
51 rue des Mathématiques, Domaine universitaire, BP 53
38041 Grenoble Cedex 9, France

(Submitted by: Reza Aftabizadeh)

Abstract. In this paper we establish the existence of positive solutions for two kinds of nonlinear differential systems.

1. Introduction

In this paper we first consider the nonlinear system

$$\begin{aligned} &u_j'' + f_{j+1}(u_{j+1}) = 0 \quad \text{in } (-R,R), \quad j = 1, \cdots, m-1,\\ &u_m'' + f_1(u_1) = 0 \quad \text{in } (-R,R),\\ &u_j(\pm R) = 0, \quad j = 1, \cdots, m, \end{aligned} \tag{1.1}$$

where $R > 0$, $m \geq 2$ is an integer and f_j, $j = 1, \cdots, m$ satisfy the following hypotheses:

(H_1) $f_j : [0,\infty) \to [0,\infty)$ is a continuous function;

(H_2) f_j is nondecreasing;

(H_3) for all $c > 0$

$$\lim_{t\to\infty} (f_1 \circ cf_2 \circ \cdots \circ cf_m)(t)/t = \infty, \ \lim_{t\to\infty} (f_m \circ cf_1 \circ \cdots \circ cf_{m-1})(t)/t = \infty,$$

and for $k \in \{2, \cdots, m-1\}$ when $m \geq 3$

$$\lim_{t\to\infty} (f_k \circ cf_{k+1} \circ \cdots \circ cf_m \circ cf_1 \circ \cdots \circ cf_{k-1})(t)/t = \infty;$$

(H_4) for all $c > 0$

$$\lim_{t\to 0} (f_1 \circ cf_2 \circ \cdots \circ cf_m)(t)/t = 0, \ \lim_{t\to 0} (f_m \circ cf_1 \circ \cdots \circ cf_{m-1})(t)/t = 0,$$

and for $k \in \{2, \cdots, m-1\}$ when $m \geq 3$

$$\lim_{t\to 0} (f_k \circ cf_{k+1} \circ \cdots \circ cf_m \circ cf_1 \circ \cdots \circ cf_{k-1})(t)/t = 0.$$

Accepted for publication: January 2011.
AMS Subject Classifications: 34B18, 35J57.

$u = (u_1, \cdots, u_m) \in (C^2[-R, R])^m$ is a positive solution of (1.1) if the functions u_j satisfy (1.1) and $u_j > 0$ on $(-R, R)$ for $j = 1, \cdots, m$.

(H_3) (respectively $(H4)$) means that the system is superlinear at ∞ (respectively 0). The case of a sublinear system has been studied for a semilinear elliptic system in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$; see [4] when $m = 2$ and also [14] for a particular case, but with f_j depending on $u_1, \cdots u_m$ and $m \geq 2$. Notice that the result in [4] obviously extends to $m \geq 2$. When $m = 2$ and f_1, f_2 are both superlinear at 0 and ∞ this type of system has been extensively studied in balls and smooth bounded domains $\Omega \subset \mathbb{R}^n$; see for instance [2], [11], [16]. The case of an annulus with $m \geq 2$ and f_j superlinear at 0 and ∞ for $j = 1, \cdots, m$ was treated in [5]. More general operators are studied in [15] and [18], but the superlinearity (respectively sublinearity) conditions are different and in a sense stronger. Finally, the following type of system:

$$\begin{aligned} -\Delta u_1 &= a_1(x)u_1^{\alpha_{11}} + b_1(x)u_2^{\alpha_{12}} \quad \text{in } \Omega, \\ -\Delta u_2 &= a_2(x)u_1^{\alpha_{21}} + b_2(x)u_2^{\alpha_{22}} \quad \text{in } \Omega, \\ u_1 &= u_2 = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has been considered in [6], [9] and [19]. $\Omega \subset \mathbb{R}^n$ is a ball or a smooth bounded domain, the exponents α_{ij} are nonnegative real numbers and a_j, b_j are nonnegative continuous functions on $\overline{\Omega}$. The superlinearity assumptions on the exponents α_{ij} are

$$\text{either} \quad \alpha_{11} > 1, \quad \text{or} \quad \alpha_{22} > 1, \quad \text{or} \quad \alpha_{12}\alpha_{21} > 1.$$

Notice that $\alpha_{ii} > 1$ means that the Emden-Fowler equation $-\Delta u_i = u_i^{\alpha_{ii}}$ is superlinear, while $\alpha_{12}\alpha_{21} > 1$ is the usual superlinearity assumption for the Lane-Emden system

$$\begin{aligned} -\Delta u_1 &= u_2^{\alpha_{12}}, \\ -\Delta u_2 &= u_1^{\alpha_{21}}. \end{aligned}$$

This latter condition corresponds to (H_3), (H_4) above, (H_5) below and (H_6) in Theorem 3.

Our first result is the following theorem.

Theorem 1 *Let f_j, $j = 1, \cdots, m$ satisfy (H_1)-(H_4). Then problem* (1.1) *has at least one positive solution $u = (u_1, \cdots, u_m) \in (C^2[-R, R])^m$ which is symmetric with respect to the origin.*

Next we study the existence of positive solutions of the nonlinear elliptic system

$$\begin{aligned} &\Delta u_j + f_{j+1}(u_{j+1}) = 0 \quad \text{in } \Omega(a,b), \quad j = 1, \cdots, m-1, \\ &\Delta u_m + f_1(u_1) = 0 \quad \text{in } \Omega(a,b), \\ &u_j = 0 \quad \text{on } \partial\Omega(a,b), \quad j = 1, \cdots, m, \end{aligned} \tag{1.2}$$

where $0 < a < b < \infty$, $\Omega(a,b)$ denotes the annulus $\{x \in \mathbb{R}^n : a < |x| < b\}$ $(n \geq 2)$, $m \geq 2$ is an integer and f_j, $j = 1, \cdots, m$ satisfy (H_1) and

(H_5) There exist a_j, b_j, p_j, p'_j, q_j such that $0 < a_j \leq b_j$, $0 < p'_j \leq p_j \leq q_j$ for $j = 1, \cdots, m$ with $p'_1 \cdots p'_m > 1$ and

$$a_j t^{p_j} \leq f_j(t) \leq b_j g_j(t) \quad \forall\, t \geq 0,\ j = 1, \cdots, m,$$

where $g_j(t) = t^{p'_j}$ for $0 \leq t \leq 1$ and $g_j(t) = t^{q_j}$ for $t \geq 1$.

If $2 \leq m \leq 3$ we assume that there exists at least one $j \in \{1, \cdots, 3\}$ such that $p_j \leq 1$.

If $m \geq 4$ we assume that there exists $k \in \{2, \cdots, m\}$ such that $p_j \leq 1$ for $j \in \{2, \cdots, k\}$, $p_1 > 1$ and $p_j > 1$ for $j \in \{k+1, \cdots, m\}$ if $m > k$ (notice that when $k = m$, necessarily $p_1 > 1$).

We have the following theorem.

Theorem 2 *Let f_j, $j = 1, \cdots, m$ satisfy (H_1) and (H_5). Then problem* (1.2) *has at least one positive radial solution $u = (u_1, \cdots, u_m) \in (C^2(\overline{\Omega(a,b)}))^m$.*

Recently, there have been significant studies of semilinear parabolic systems of m equations; see [8], [13] and the references therein. The particular case $m = 3$ is considered in [8]. Our results correspond to the existence of stationary solutions on $(-R, R)$ and $\Omega(a,b)$. Moreover Theorem 1 (respectively Theorem 2) complements [18] (respectively [15]). Our results also complement [3] and [14] in some cases.

The proof of Theorem 1 and Theorem 2 consists of first obtaining a priori estimates on the positive solutions and then applying well-known properties of compact mappings taking a cone in a Banach space into itself [7].

In Section 2 we prove Theorem 1. We also give another existence result. Theorem 2 is proved in Section 3. Finally, in Section 4 we give some examples.

2. Proof of Theorem 1

We first note that if $u = (u_1, \cdots, u_m) \in (C^2[-R, R])^m$ is a positive solution of (1.1), then $u_j(-x) = u_j(x)$ for $x \in [-R, R]$ and $u'_j < 0$ on $(0, R)$

for $j = 1, \cdots, m$. This follows from [17] using Remark 1 of [10, page 219]. Therefore positive solutions of (1.1) can be treated as positive solutions of

$$\begin{aligned} & u_j'' + f_{j+1}(u_{j+1}) = 0 \quad \text{in } [0, R), \quad j = 1, \cdots, m-1, \\ & u_m'' + f_1(u_1) = 0 \quad \text{in } [0, R), \\ & u_j(R) = u_j'(0) = 0, \quad j = 1, \cdots, m. \end{aligned} \tag{2.1}$$

Now we prove that there exists $C > 0$ such that

$$||u_j||_\infty \leq C, \quad j = 1, \cdots, m, \tag{2.2}$$

for all positive solutions $u = (u_1, \cdots, u_m) \in (C^2[0, R])^m$ of (2.1). The Green's function of the operator $-d^2/dx^2$ for the Dirichlet problem in $[0, R)$ is given by

$$G(t, s) = \begin{cases} R - t, & 0 \leq s \leq t \leq R, \\ R - s, & 0 \leq t \leq s \leq R. \end{cases}$$

Let $A > 9/R^2$. (H_3) with $c = R^2/9$ implies that there exists $t(A, R) > 0$ such that

$$(f_1 \circ cf_2 \circ \cdots \circ cf_m)(t) \geq At, \quad \forall\, t \geq t(A, R), \tag{2.3}$$

$$(f_m \circ cf_1 \circ \cdots \circ cf_{m-1})(t) \geq At, \quad \forall\, t \geq t(A, R), \tag{2.4}$$

and for $k \in \{2, \cdots, m-1\}$ (if $m \geq 3$)

$$(f_k \circ cf_{k+1} \circ \cdots \circ cf_m \circ cf_1 \circ \cdots \circ cf_{k-1})(t) \geq At, \quad \forall\, t \geq t(A, R). \tag{2.5}$$

Let $t \in [\frac{R}{3}, \frac{2R}{3}]$. If $m = 2$, using (H_2), we have

$$\begin{aligned} u_1(t) &= \int_0^R G(t, s) f_2(u_2(s))\, ds \geq (R - t) \int_0^t f_2(u_2(s))\, ds \\ &\geq (R - t) t f_2(u_2(t)) \geq \frac{R^2}{9} f_2(u_2(t)) \geq \frac{R^2}{9} (f_2 \circ \frac{R^2}{9} f_1)(u_1(t)), \end{aligned}$$

and (2.4) implies that $u_1(t) \leq t(A, R)$ for $t \in [\frac{R}{3}, \frac{2R}{3}]$. Since $u_1' < 0$ on $(0, R)$ we deduce that

$$|u_1(t)| \leq t(A, R) \quad \forall\, t \in [\tfrac{R}{3}, R].$$

In the same way, using (2.3) we obtain

$$|u_2(t)| \leq t(A, R) \quad \forall\, t \in [\tfrac{R}{3}, R].$$

Now if $m \geq 3$ we write

$$\begin{aligned} u_1(t) &= \int_0^R G(t, s) f_2(u_2(s))\, ds \geq (R - t) \int_0^t f_2(u_2(s))\, ds \\ &\geq (R - t) t f_2(u_2(t)) \geq \frac{R^2}{9} f_2(u_2(t)) \end{aligned}$$

$$\geq \frac{R^2}{9}(f_2 \circ \frac{R^2}{9} f_3 \circ \cdots \circ \frac{R^2}{9} f_m \circ \frac{R^2}{9} f_1)(u_1(t)),$$

and (2.5) implies that $u_1(t) \leq t(A,R)$ for $t \in [\frac{R}{3}, \frac{2R}{3}]$. Since $u_1' < 0$ on $(0,R)$ we deduce that

$$|u_1(t)| \leq t(A,R) \quad \forall\, t \in [\tfrac{R}{3}, R]. \tag{2.6}$$

In the same way, using (2.3)-(2.5) we obtain

$$|u_j(t)| \leq t(A,R) \quad \forall\, t \in [\tfrac{R}{3}, R], \quad j = 2, \cdots, m. \tag{2.7}$$

Now we need the following well-known lemma, whose proof is left to the reader.

Lemma 1 *Let $g \in C^2[\alpha, \beta]$. We have*

$$||g'||_\infty \leq \frac{2}{\beta - \alpha}||g||_\infty + \frac{\beta - \alpha}{2}||g''||_\infty.$$

Since $u_1'' + f_2(u_2) = 0$, Lemma 1, (2.6), (2.7) and (H_2) imply

$$|u_1'(t)| \leq \frac{3}{R} t(A,R) + \frac{R}{3} f_2(t(A,R)) \quad \forall\, t \in [\frac{R}{3}, R].$$

In the same way we obtain

$$|u_j'(t)| \leq \frac{3}{R} t(A,R) + \frac{R}{3} f_{j+1}(t(A,R)) \quad \forall\, t \in [\frac{R}{3}, R],$$

for $j = 2, \cdots, m-1$ (if $m \geq 3$) and

$$|u_m'(t)| \leq \frac{3}{R} t(A,R) + \frac{R}{3} f_1(t(A,R)) \quad \forall\, t \in [\frac{R}{3}, R].$$

Since u_j' is nonincreasing for $j = 1, \cdots, m$ we get

$$||u_j'||_\infty \leq \frac{3}{R} t(A,R) + \frac{R}{3} f_{j+1}(t(A,R)), \quad j = 1, \cdots, m-1,$$

and

$$||u_m'||_\infty \leq \frac{3}{R} t(A,R) + \frac{R}{3} f_1(t(A,R)).$$

We have

$$||u_j||_\infty = u_j(0) = -\int_0^R u_j'(t)\, dt \leq R||u_j'||_\infty, \quad j = 1, \cdots, m,$$

and (2.2) is proved.

Now we need the following fixed-point theorem, which is a modified version of a result due to Krasnosel'skii [12] and Benjamin [1].

Proposition 1 [7, page 56] *Let C be a cone in a Banach space X and $\Phi : C \to C$ a compact map such that $\Phi(0) = 0$. Assume that there exist numbers $0 < r < M$ and $x_0 > 0$ such that*

(i) $u \neq \theta\Phi(u)$ for $\theta \in [0,1]$ and $u \in C$ such that $||u|| = r$,

(ii) there exists a compact map $F : \overline{B_M} \times [0,\infty) \to C$ (where $B_\rho = \{u \in C : ||u|| < \rho\}$) such that $F(u,0) = \Phi(u)$ for $||u|| = M$, $F(u,x) \neq u$ for $||u|| = M$ and $0 \leq x < \infty$ and $F(u,x) = u$ has no solution $u \in \overline{B_M}$ for $x \geq x_0$.

Then if $U = \{u \in C : r < ||u|| < M\}$, one has

$$i_C(\Phi, B_M) = 0, \quad i_C(\Phi, B_r) = 1, \quad i_C(\Phi, U) = -1,$$

where $i_C(\Phi, W)$ denotes the fixed-point index of Φ on W. In particular Φ has a fixed point in U.

Let X denote the Banach space $(C[0,R])^m$ endowed with the norm $||u|| = \max_{1\leq j\leq m}(||u_j||_\infty)$. Define the cone

$$C = \{u = (u_1, \cdots, u_m) \in X : u_j \geq 0, \, j = 1, \cdots, m\}.$$

For $(u,x) \in C \times [0,\infty)$ we define

$$F(u,x)(t) = (F_1(u,x)(t), \cdots, F_m(u,x)(t)), \quad 0 \leq t \leq R,$$

where

$$F_j(u,x)(t) = \int_0^R G(t,s) f_{j+1}(u_{j+1}(s) + x) \, ds, \quad j = 1, \cdots, m-1,$$
$$F_m(u,x)(t) = \int_0^R G(t,s) f_1(u_1(s) + x) \, ds,$$

and $\Phi(u) = F(u,0)$. We want to apply Proposition 1 to F. By (H_1), F maps $C \times [0,\infty)$ into C. Since G is continuous, it is well known that F is compact. (H_1) and (H_4) imply that $f_j(0) = 0$ for $j = 1, \cdots, m$, hence $\Phi(0) = 0$.

Let $\varepsilon > 0$ be such that $\varepsilon R^2/2 < 1$. (H_4) with $c = R^2/2$ implies that there exists $r > 0$ such that

$$(f_1 \circ cf_2 \circ \cdots \circ cf_m)(t) \leq \varepsilon t, \quad 0 \leq t \leq r. \tag{2.8}$$

Suppose that there exist $\theta \in [0,1]$ and $u \in C$ with $||u|| = r$ such that $u = \theta\Phi(u)$. Then

$$u_j'' + \theta f_{j+1}(u_{j+1}) = 0 \quad \text{in } [0,R), \quad j = 1, \cdots, m-1,$$
$$u_m'' + \theta f_1(u_1) = 0 \quad \text{in } [0,R),$$
$$u_j(R) = u_j'(0) = 0, \quad j = 1, \cdots, m.$$

By the maximum principle, for each $j \in \{1, \cdots, m\}$, $u_j > 0$ on $[0, R)$ or $u_j \equiv 0$ on $[0, R]$. Now, if there exists $i \in \{1, \cdots, m\}$ such that $u_i \equiv 0$, we easily show that $u_j \equiv 0$ on $[0, R]$ for $j = 1, \cdots, m$. Thus $u_j > 0$ on $[0, R)$ for $j = 1, \cdots, m$. Using (H_2) and (2.8) we can write

$$\begin{aligned} ||u_m||_\infty = u_m(0) = \theta \int_0^R G(0,s) f_1(u_1(s))\, ds &\le \frac{R^2}{2} f_1(u_1(0)) \\ &\le \frac{R^2}{2}(f_1 \circ \frac{R^2}{2} f_2 \circ \cdots \circ \frac{R^2}{2} f_m)(u_m(0)) \le \frac{R^2 \varepsilon}{2} u_m(0), \end{aligned}$$

which implies that $u_m \equiv 0$ and we have a contradiction. Thus condition (i) of Proposition 1 is satisfied.

Let $x_0 > t(A, R)$ where $t(A, R)$ is defined in the proof of (2.2). We will show that

$$F(u, x) \ne u \quad \text{for all } u \in C \text{ and } x \ge x_0. \tag{2.9}$$

Indeed suppose that there exist $u \in C$ and $x \ge x_0$ such that $F(u, x) = u$. If $u \equiv 0$ then $f_j(x) = 0$ for $j = 1, \cdots, m$, a contradiction to (2.3)-(2.5). Thus $u \not\equiv 0$ and $u_j > 0$ in $[0, R)$ for $j = 1, \cdots, m$ by the maximum principle. Now as in the proof of (2.2) we have

$$u_1(t) \ge \frac{AR^2}{9}(u_1(t) + x) \quad \forall\, t \in [\frac{R}{3}, \frac{2R}{3}],$$

and we reach a contradiction. Thus (2.9) holds and the third condition of (ii) is verified.

Now we note that the constant in (2.2) can be chosen independently of the parameter $x \in [0, x_0]$ for each fixed $x_0 \in (0, \infty)$ if we consider positive solutions of (2.1) for the family of nonlinearities $f_{j,x}(t) = f_j(t + x)$, $t \ge 0$. Thus we can find $M > r$ such that

$$F(u, x) \ne u \quad \text{for all } x \in [0, x_0] \text{ and } u \in C \text{ with } ||u|| = M. \tag{2.10}$$

Therefore (2.9) and (2.10) prove the second condition of (ii).

Then we may apply Proposition 1 to conclude that Φ has a nontrivial fixed point $u \in C$. Using the same arguments as before we can show that any nontrivial fixed point of Φ in C yields a positive solution of (2.1) in $(C^2[0, R])^m$. The proof of the theorem is complete.

We conclude this section with the following theorem.

Theorem 3. *Let f_j, $j = 1, \cdots, m$ satisfy (H_1) and*

(H_6) *There exist a_j, b_j, p_j, p'_j, q_j such that $0 < a_j \le b_j$, $0 < p'_j \le p_j \le q_j$ for $j = 1, \cdots, m$ with $p'_1 \cdots p'_m > 1$ and*

$$a_j t^{p_j} \le f_j(t) \le b_j g_j(t) \quad \forall\, t \ge 0,\ j = 1, \cdots, m,$$

where $g_j(t) = t^{p'_j}$ *for* $0 \leq t \leq 1$ *and* $g_j(t) = t^{q_j}$ *for* $t \geq 1$.
Then problem (2.1) *has at least one positive solution* $u = (u_1, \cdots, u_m) \in (C^2[0,R])^m$.

Proof. Let $u = (u_1, \cdots, u_m) \in (C^2[0,R])^m$ be a positive solution of (2.1). Then $u'_j < 0$ in $(0,R)$ for $j = 1, \cdots, m$. Now the proof is close enough to the proof of Theorem 1 that we leave it to the reader.

3. Proof of Theorem 2

Since we are interested in positive radial solutions, problem (1.2) reduces to the one-dimensional boundary-value problem

$$\begin{aligned} &\Delta u_j + f_{j+1}(u_{j+1}) = 0 \quad \text{in } (a,b), \quad j = 1, \cdots, m-1, \\ &\Delta u_m + f_1(u_1) = 0 \quad \text{in } (a,b), \\ &u_j(a) = u_j(b) = 0, \quad j = 1, \cdots, m, \end{aligned} \tag{3.1}$$

where Δ denotes the polar form of the Laplacian; i.e.,

$$\Delta = t^{1-n}\frac{d}{dt}(t^{n-1}\frac{d}{dt}).$$

We will prove that there exists $C > 0$ such that

$$||u_j||_\infty \leq C, \quad j = 1, \cdots, m, \tag{3.2}$$

for all positive solutions $u = (u_1, \cdots, u_m) \in (C^2[a,b])^m$ of (3.1).

The Green's function of the operator $-\Delta$ on (a,b) with Dirichlet boundary conditions is given by

$$G(t,s) = \frac{s}{(n-2)t^{n-2}(b^{n-2} - a^{n-2})} \times \begin{cases} (s^{n-2} - a^{n-2})(b^{n-2} - t^{n-2}) & a \leq s \leq t \leq b\,, \\ (t^{n-2} - a^{n-2})(b^{n-2} - s^{n-2}) & a \leq t \leq s \leq b, \end{cases}$$

if $n \geq 3$ and

$$G(t,s) = \frac{s}{\ln b - \ln a} \begin{cases} (\ln b - \ln t)(\ln s - \ln a) & a \leq s \leq t \leq b, \\ (\ln t - \ln a)(\ln b - \ln s) & a \leq t \leq s \leq b, \end{cases}$$

if $n = 2$. We denote by λ_1 the first eigenvalue of $-\Delta$ on (a,b) with Dirichlet boundary conditions and φ_1 is the corresponding (positive) eigenfunction. Define $\rho(t) = (t-a)(b-t)$, $a \leq t \leq b$. Then we have

$$0 \leq G(t,s) \leq c_1\rho(s), \quad a \leq s, t \leq b, \tag{3.3}$$

and

$$c_2\rho(t) \leq \varphi_1(t) \leq c_3\rho(t), \quad a \leq t \leq b, \tag{3.4}$$

for some positive constants c_j, $j = 1, \cdots, 3$.

Let $u = (u_1, \cdots, u_m) \in (C^2[a, b])^m$ be a positive solution of (3.1). We will denote by K a positive constant depending on a, b, n, λ_1, a_j, b_j, p_j and c_1 and possibly changing value from line to line. We have three cases to consider.

Case 1: $m = 2$. Then we may assume that $p_2 \leq 1$ and (necessarily) $p_1 > 1$. We have

$$\begin{aligned} \lambda_1 \int_a^b t^{n-1}\varphi_1 u_1\,dt &= -\int_a^b t^{n-1} u_1 \Delta\varphi_1\,dt = -\int_a^b t^{n-1}\varphi_1 \Delta u_1\,dt \\ &= \int_a^b t^{n-1}\varphi_1 f_2(u_2)\,dt \geq K \int_a^b t^{n-1}\varphi_1 u_2^{p_2}\,dt \\ &= K \int_a^b t^{n-1}\varphi_1(t)(\int_a^b G(t,s) f_1(u_1(s))\,ds)^{p_2}\,dt \\ &\geq K \int_a^b t^{n-1}\varphi_1(t)(\int_a^b G(t,s)^{p_2} f_1(u_1(s))^{p_2}\,ds)\,dt \\ &\geq K \int_a^b t^{n-1}\varphi_1(t)(\int_a^b G(t,s) f_1(u_1(s))^{p_2}\,ds)\,dt. \end{aligned} \tag{3.5}$$

Let

$$z(t) = \int_a^b G(t,s) f_1(u_1(s))^{p_2}\,ds, \quad a \leq t \leq b.$$

Since

$$\Delta z = -f_1(u_1)^{p_2} \quad \text{and } z(a) = z(b) = 0,$$

we can write

$$\begin{aligned} \int_a^b t^{n-1}\varphi_1 z\,dt &= -\frac{1}{\lambda_1}\int_a^b t^{n-1} z\Delta\varphi_1\,dt = -\frac{1}{\lambda_1}\int_a^b t^{n-1}\varphi_1 \Delta z\,dt \\ &= \frac{1}{\lambda_1}\int_a^b t^{n-1}\varphi_1 f_1(u_1)^{p_2}\,dt. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we deduce

$$\int_a^b t^{n-1}\varphi_1 u_1\,dt = \frac{1}{\lambda_1}\int_a^b t^{n-1}\varphi_1 f_2(u_2)\,dt \geq K \int_a^b t^{n-1}\varphi_1 u_1^{p_1 p_2}\,dt. \tag{3.7}$$

Since $p_1 p_2 > 1$, for any $T > 0$ there exists a positive constant K_1 depending on T and $p_1 p_2$ such that

$$s^{p_1 p_2} \geq Ts - K_1 \quad \forall\, s \geq 0. \tag{3.8}$$

Using (3.7) and (3.8) with T such that $TK > 1$ we obtain

$$\int_a^b t^{n-1}\varphi_1 u_1\,dt = \frac{1}{\lambda_1}\int_a^b t^{n-1}\varphi_1 f_2(u_2)\,dt \geq TK\int_a^b t^{n-1}\varphi_1 u_1\,dt - K_2 \quad (3.9)$$

for another positive constant K_2. From (3.9) we deduce that there exists a constant $C > 0$ such that

$$\int_a^b \varphi_1 f_2(u_2)\,dt \leq C.$$

Then using (3.3) and (3.4) we get

$$u_1(t) = \int_a^b G(t,s) f_2(u_2(s))\,ds \leq \frac{c_1}{c_2}\int_a^b \varphi_1(s) f_2(u_2(s))\,ds \leq \frac{c_1 C}{c_2}$$

for $t \in [a,b]$. Since

$$u_2(t) = \int_a^b G(t,s) f_1(u_1(s))\,ds,$$

we deduce that u_2 is also bounded. Then (3.2) is proved.

Case 2: $m = 3$. We can always assume that

(i) p_2, $p_3 \leq 1$ and $p_1 > 1$, or

(ii) $p_2 \leq 1$ and p_1, $p_3 > 1$.

Then as in case 1 we get

$$\lambda_1\int_a^b t^{n-1}\varphi_1 u_1\,dt = \int_a^b t^{n-1}\varphi_1 f_2(u_2)\,dt \geq K\int_a^b t^{n-1}\varphi_1 f_3(u_3)^{p_2}\,dt$$
$$\geq K\int_a^b t^{n-1}\varphi_1 u_3^{p_2 p_3}\,dt.$$

(i) Since $p_2 p_3 \leq 1$ using the same arguments we get

$$\int_a^b t^{n-1}\varphi_1 u_3^{p_2 p_3}\,dt \geq K\int_a^b t^{n-1}\varphi_1 u_1^{p_1 p_2 p_3}\,dt.$$

From the fact that $p_1 p_2 p_3 > 1$ we conclude as in case 1 that there exists a constant $C > 0$ such that

$$\int_a^b \varphi_1 f_2(u_2)\,dt \leq C,$$

from which we deduce that u_1 is bounded and finally that u_j is bounded for $j = 1, \cdots, 3$.

(ii) If $p_2p_3 \le 1$ we argue as in (i). Now assume that $p_2p_3 > 1$. Then for any $T > 0$ there exists a constant $K_1 > 0$ such that

$$\begin{aligned}
&\int_a^b t^{n-1}\varphi_1 u_3^{p_2p_3}\,dt \ge T\int_a^b t^{n-1}\varphi_1 u_3\,dt - K_1\\
&= -\frac{T}{\lambda_1}\int_a^b t^{n-1}u_3\Delta\varphi_1\,dt - K_1 = -\frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1\Delta u_3\,dt - K_1\\
&= \frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1 f_1(u_1)\,dt - K_1 \ge \frac{Ta_1}{\lambda_1}\int_a^b t^{n-1}\varphi_1 u_1^{p_1}\,dt - K_1.
\end{aligned}$$

Since $p_1 > 1$ for any $T > 0$ there exists a constant $K_2 > 0$ such that

$$\int_a^b t^{n-1}\varphi_1 u_1^{p_1}\,dt \ge T\int_a^b t^{n-1}\varphi_1 u_1\,dt - K_2,$$

and we conclude as in case 1. Then (3.2) is proved.

Case 3: $m \ge 4$. We further divide the proof into two subcases.

(i) $k \in \{2, \cdots, m-1\}$. As in case 2 we get

$$\begin{aligned}
&\lambda_1\int_a^b t^{n-1}\varphi_1 u_1\,dt = \int_a^b t^{n-1}\varphi_1 f_2(u_2)\,dt\\
&\ge K\int_a^b t^{n-1}\varphi_1 f_{k+1}(u_{k+1})^{p_2\cdots p_k}\,dt \ge K\int_a^b t^{n-1}\varphi_1 u_{k+1}^{p_2\cdots p_{k+1}}\,dt.
\end{aligned}$$

If $p_2\cdots p_j \le 1$ for $j \in \{k+1, \cdots, m\}$, as before we obtain

$$\int_a^b t^{n-1}\varphi_1 u_{k+1}^{p_2\cdots p_{k+1}}\,dt \ge K\int_a^b t^{n-1}\varphi_1 u_m^{p_2\cdots p_m}\,dt \ge K\int_a^b t^{n-1}\varphi_1 u_1^{p_1\cdots p_m}\,dt.$$

Since $p_1\cdots p_m > 1$ we conclude as in case 1 that there exists a constant $C > 0$ such that

$$\int_a^b \varphi_1 f_2(u_2)\,dt \le C,$$

from which we deduce again that (3.2) holds. Otherwise let $j \in \{k+1, \cdots, m\}$ denote the first integer such that $p_2\cdots p_j > 1$. Then in the same way we have

$$\int_a^b t^{n-1}\varphi_1 u_{k+1}^{p_2\cdots p_{k+1}}\,dt \ge K\int_a^b t^{n-1}\varphi_1 u_j^{p_2\cdots p_j}\,dt.$$

Suppose first that $j = m$. Then for any $T > 0$ there exists a constant $K_1 > 0$ such that

$$\int_a^b t^{n-1}\varphi_1 u_m^{p_2\cdots p_m}\,dt \ge T\int_a^b t^{n-1}\varphi_1 u_m\,dt - K_1$$

$$= -\frac{T}{\lambda_1}\int_a^b t^{n-1}u_m\Delta\varphi_1\,dt - K_1 = -\frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1\Delta u_m\,dt - K_1$$
$$= \frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1 f_1(u_1)\,dt - K_1 \geq \frac{Ta_1}{\lambda_1}\int_a^b t^{n-1}\varphi_1 u_1^{p_1}\,dt - K_1.$$

Since $p_1 > 1$ using the same arguments we conclude that (3.2) holds. Now if $j \leq m-1$, then for any $T > 0$ there exists a constant $K_2 > 0$ such that

$$\int_a^b t^{n-1}\varphi_1 u_j^{p_2\cdots p_j}\,dt \geq T\int_a^b t^{n-1}\varphi_1 u_j\,dt - K_2$$
$$= -\frac{T}{\lambda_1}\int_a^b t^{n-1}u_j\Delta\varphi_1\,dt - K_2 = -\frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1\Delta u_j\,dt - K_2$$
$$= \frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1 f_{j+1}(u_{j+1})\,dt - K_2 \geq \frac{Ta_{j+1}}{\lambda_1}\int_a^b t^{n-1}\varphi_1 u_{j+1}^{p_{j+1}}\,dt - K_2.$$

Since $p_{j+1} > 1$ for any $T > 0$ there exists a constant $K_3 > 0$ such that

$$\int_a^b t^{n-1}\varphi_1 u_{j+1}^{p_{j+1}}\,dt \geq T\int_a^b t^{n-1}\varphi_1 u_{j+1}\,dt - K_3.$$

Using an induction argument we obtain that for any $T > 0$ there exists a constant $K_4 > 0$ such that

$$\int_a^b t^{n-1}\varphi_1 u_j^{p_2\cdots p_j}\,dt \geq T\int_a^b t^{n-1}\varphi_1 u_m\,dt - K_4$$
$$= -\frac{T}{\lambda_1}\int_a^b t^{n-1}u_m\Delta\varphi_1\,dt - K_4 = -\frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1\Delta u_m\,dt - K_4$$
$$= \frac{T}{\lambda_1}\int_a^b t^{n-1}\varphi_1 f_1(u_1)\,dt - K_4 \geq \frac{Ta_1}{\lambda_1}\int_a^b t^{n-1}\varphi_1 u_1^{p_1}\,dt - K_4,$$

and (3.2) follows as before.

(ii) $k = m$. As in Case 2 we get

$$\lambda_1\int_a^b t^{n-1}\varphi_1 u_1\,dt = \int_a^b t^{n-1}\varphi_1 f_2(u_2)\,dt$$
$$\geq K\int_a^b t^{n-1}\varphi_1 f_1(u_1)^{p_2\cdots p_m}\,dt \geq K\int_a^b t^{n-1}\varphi_1 u_1^{p_1\cdots p_m}\,dt.$$

Since $p_1\cdots p_m > 1$ we conclude as in (i) that there exists a constant $C > 0$ such that

$$\int_a^b \varphi_1 f_2(u_2)\,dt \leq C,$$

and again (3.2) holds.

Now we want to apply Proposition 1. Let X denote the Banach space $(C[a,b])^m$ endowed with the norm $||u|| = \max_{1\leq j\leq m}(||u_j||_\infty)$. Define the cone

$$C = \{u = (u_1, \cdots, u_m) \in X : u_j \geq 0,\ j = 1, \cdots, m\}.$$

For $(u,x) \in C \times [0,\infty)$ we define

$$F(u,x)(t) = (F_1(u,x)(t), \cdots, F_m(u,x)(t)), \quad a \leq t \leq b,$$

where

$$F_j(u,x)(t) = \int_a^b G(t,s) f_{j+1}(u_{j+1}(s) + x)\,ds, \quad j = 1, \cdots, m-1,$$
$$F_m(u,x)(t) = \int_a^b G(t,s) f_1(u_1(s) + x)\,ds,$$

and $\Phi(u) = F(u,0)$. We have to show that F satisfies the conditions of Proposition 1. Since the arguments are similar to those used in the proof of Theorem 1 we will be sketchy. F maps $C \times [0,\infty)$ into C and F is compact. (H_5) implies that $f_j(0) = 0$ for $j = 1, \cdots, m$, hence $\Phi(0) = 0$.

Define

$$c = b_1 b_2^{p_1'} \cdots b_m^{p_{m-1}'} \big(c_1 \frac{(b-a)^3}{4}\big)^{1+p_1'+\cdots+p_1'\cdots p_{m-1}'}.$$

Let $\varepsilon \in (0,1)$. There exists $r \in (0,1)$ such that

$$ct^{p_1'\cdots p_m'} \leq \varepsilon t, \quad 0 \leq t \leq r. \tag{3.10}$$

Suppose that there exist $\theta \in [0,1]$ and $u \in C$ with $||u|| = r$ such that $u = \theta\Phi(u)$. As in the proof of Theorem 1 $u_j > 0$ on (a,b) for $j = 1, \cdots, m$. Using (H_5) and (3.3) we can write

$$||u_m||_\infty = \sup_{a\leq t\leq b} \theta \int_a^b G(t,s) f_1(u_1(s))\,ds \leq c_1 \frac{(b-a)^3}{4} b_1 ||u_1||_\infty^{p_1'}, \tag{3.11}$$

and for $j \in \{1, \cdots, m-1\}$

$$||u_j||_\infty \leq c_1 \frac{(b-a)^3}{4} b_{j+1} ||u_{j+1}||_\infty^{p_{j+1}'}. \tag{3.12}$$

Now from (3.10)-(3.12)we obtain

$$||u_m||_\infty \leq \varepsilon ||u_m||_\infty,$$

which implies that $u_m \equiv 0$ and we have a contradiction. Thus condition (i) of Proposition 1 is satisfied.

We will show that we can find $x_0 > 0$ such that

$$F(u,x) \neq u \quad \text{for all } u \in C \text{ and } x \geq x_0. \tag{3.13}$$

Indeed suppose that there exist $u \in C$ and $x \geq x_0$ such that $F(u,x) = u$ for some $x_0 > 0$. If $u \equiv 0$ then $f_j(x) = 0$ for $j = 1, \cdots, m$, a contradiction to (H_5). Thus $u \not\equiv 0$ and $u_j > 0$ in (a,b) for $j = 1, \cdots, m$ by the maximum principle. Assume first that $m = 2$. For any $S > 0$ there exists $x_0(S) > 0$ such that

$$(u+x)^{p_1p_2} \geq S(u+x) \geq Su \quad \text{for } u \geq 0 \text{ and } x \geq x_0(S). \tag{3.14}$$

Let $x \geq x_0(S)$. Then as in the proof of (3.2) we have

$$\int_a^b t^{n-1}\varphi_1 u_1 \, dt \geq K \int_a^b t^{n-1}\varphi_1(t)(u_1(t)+x)^{p_1p_2} \, dt \tag{3.15}$$

for some constant $K > 0$. Take S in (3.14) such that $SK > 1$ and let $x_0 = x_0(S)$. Then (3.14) and (3.15) imply that

$$\int_a^b t^{n-1}\varphi_1 u_1 \, dt \geq SK \int_a^b t^{n-1}\varphi_1 u_1 \, dt,$$

and we reach a contradiction. Now let $m = 3$. For any $S > 0$ there exists $x_0(S) > 0$ such that

$$(u+x)^{p_1} \geq S(u+x) \geq Su \quad \text{for } u \geq 0 \text{ and } x \geq x_0(S), \tag{3.16}$$

$$(u+x)^{p_1p_2p_3} \geq S(u+x) \geq Su \quad \text{for } u \geq 0 \text{ and } x \geq x_0(S), \tag{3.17}$$

and if $p_2p_3 > 1$

$$(u+x)^{p_2p_3} \geq S(u+x) \geq Su \quad \text{for } u \geq 0 \text{ and } x \geq x_0(S). \tag{3.18}$$

Let $x \geq x_0(S)$. Then as in the proof of (3.2) we have

$$\int_a^b t^{n-1}\varphi_1 u_1 \, dt \geq K \int_a^b t^{n-1}\varphi_1(t)(u_1(t)+x)^{p_1p_2p_3} \, dt$$

for some constant $K > 0$ if p_2, $p_3 \leq 1$ or $p_2p_3 \leq 1$. If $p_2p_3 > 1$, using (3.18) and arguing as in the proof of (3.2) we have

$$\int_a^b t^{n-1}\varphi_1 u_1 \, dt \geq K \int_a^b t^{n-1}\varphi_1(t)(u_1(t)+x)^{p_1} \, dt$$

for some constant $K > 0$. In the first situation we take S in (3.17) such that $SK > 1$ and $x_0 = x_0(S)$. Otherwise we take S in (3.16) such that $SK > 1$ and $x_0 = x_0(S)$. In both cases we obtain

$$\int_a^b t^{n-1}\varphi_1 u_1 \, dt \geq SK \int_a^b t^{n-1}\varphi_1 u_1 \, dt,$$

and we have a contradiction. The case $m \geq 4$ can be handled using similar arguments. Thus (3.13) holds and the third condition of (ii) in Proposition 1 is verified.

Now we note that the constant in (3.2) can be chosen independently of the parameter $x \in [0, x_0]$ for each fixed $x_0 \in (0, \infty)$ if we consider positive solutions of (2.1) for the family of nonlinearities $f_{j,x}(t) = f_j(t + x)$, $t \geq 0$. Thus we can find $M > r$ such that

$$F(u, x) \neq u \quad \text{for all } x \in [0, x_0] \text{ and } u \in C \text{ with } ||u|| = M. \tag{3.19}$$

Therefore, (3.13) and (3.19) prove the second condition of (ii) in Proposition 1.

Then we conclude as in the proof of Theorem 1.

4. Examples

In this section we give some examples to illustrate our theorems.

Example 1. Let p_j, p'_j, q_j be such that $0 < p'_j \leq p_j \leq q_j$ for $j = 1, \cdots, m$ with $p'_1 \cdots p'_m > 1$. Define

$$f_j(t) = \begin{cases} t^{p'_j} & 0 \leq t \leq 1, \\ t^{q_j} & t \geq 1. \end{cases}$$

Then Theorems 1 and 3 apply. If moreover p_j, $j = 1, \cdots, m$ satisfy the conditions in (H_5), then Theorem 2 holds.

Example 2. Let p_j, p'_j, q_j be as in Example 1. Define

$$f_j(t) = \begin{cases} (1 + \sin^2 t) t^{p'_j} & 0 \leq t \leq 1, \\ (1 + \cos^2 t) t^{q_j} & t \geq 1. \end{cases}$$

Then Theorem 3 applies. If moreover p_j, $j = 1, \cdots, m$ satisfy the conditions in (H_5), then Theorem 2 holds.

Example 3. Let

$$f_j(t) = a_{j1} t^{p_{j1}} + \cdots + a_{jk_j} t^{p_{jk_j}}$$

for $t \geq 0$ and $j = 1, \cdots, m$ where $k_j \geq 1$, a_{ji}, $p_{ji} > 0$, $1 \leq i \leq k_j$, $p_{j1} < \cdots < p_{jk_j}$ if $k_j \geq 2$ and $p_{11} \cdots p_{m1} > 1$. Then Theorems 1 and 3 apply. If moreover $p_j = p_{j1}$, $j = 1, \cdots, m$ satisfy the conditions in (H_5), then Theorem 2 holds.

Example 4. Let f_j be as in example 3 but with a_{ji} depending continuously on t. Assume that there exist constants b_{ji}, $c_{ji} > 0$ such that

$$b_{ji} \leq a_{ji}(t) \leq c_{ji} \quad \forall t \geq 0.$$

If (H_2) holds then Theorems 1 and 3 apply. If (H_2) does not hold then Theorem 3 still applies. If $p_j = p_{j1}$, $j = 1, \cdots, m$ satisfy the conditions in (H_5), then Theorem 2 holds.

References

[1] T.B. Benjamin, *A unified theory of conjugate flows,* Philos. Trans. Roy. Soc., 269 A (1971), 587–643.

[2] Ph. Clément, D.G. de Figueiredo, and E. Mitidieri, *Positive solutions of semilinear elliptic systems,* Comm. Part. Diff. Eq., 17 (1992), 923–940.

[3] R. Cui, J. Shi, and Y. Wang, *Uniqueness of the positive solution for a class of semilinear elliptic systems,* Nonlinear Analysis, 67 (2007), 1710–1714.

[4] R. Dalmasso, *Existence and uniqueness of positive solutions of semilinear elliptic systems,* Nonlinear Analysis, 39 (2000), 559–568.

[5] R. Dalmasso, *Positive solutions of nonlinear elliptic systems,* Annales Polonici Mathematici, 58 (1993), 201–212.

[6] R. Dalmasso, *Existence of positive radial solutions for a semilinear elliptic system without variational structure,* Nonlinear Analysis, 67 (2007), 1255–1259.

[7] D.G. de Figueiredo, P.L. Lions and R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations,* J. Math. Pures Appl., 61 (1982), 41–63.

[8] M. Fila and P. Quittner, *The blow-up rate for a semilinear parabolic system,* J. Math. Anal. Appl., 238 (1999), 468–476.

[9] D. G. de Figueiredo, B. Sirakov, *Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems,* Mat. Ann., 333 (2005) 231–260.

[10] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle,* Comm. Math. Phys., 68 (1979), 209–243.

[11] J. Hulshof and R.C.A.M. van der Vorst, *Differential systems with strongly indefinite variational structure,* J. Funct. Anal., 114 (1993), 32–58.

[12] M.A. Krasnosel'skii, *Fixed points of cone-compressing and cone-extending operators,* Soviet Math. Dokl., 1 (1960), 1285–1288.

[13] Y. Li, Q. Liu and C. Xie, *Semilinear reaction-diffusion systems of several components,* J. Differential Equations, 187 (2003), 510–519.

[14] M. Maniwa, *Uniqueness and existence of positive solutions for some semilinear elliptic systems,* Nonlinear Analysis, 59 (2004), 993–999.

[15] D. O' Regan and H. Wang, *On the number of positive solutions of elliptic systems,* Math. Nach., 280 (2007), 1417–1430.

[16] L.A. Peletier and R.C.A.M. van der Vorst, *Existence and nonexistence of positive solutions of nonlinear elliptic systems and the biharmonic equation,* Differential Integral equations, 5 (1992), 747–767.

[17] W.C. Troy, *Symmetry properties in systems of semilinear elliptic equations,* J. Differential Equations, 42 (1981), 400–413.

[18] H. Wang, *On the number of positive solutions of nonlinear systems,* J. Math. Anal. Appl., 281 (2003), 287–306.

[19] H. Zou, *A priori estimates for a semilinear elliptic system without variational structure and their applications,* Math. Ann., 323 (2002), 713–735.