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GLOBAL EXISTENCE FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION IN LOWER ORDER SOBOLEV SPACES

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Abstract. We consider the Cauchy problem for the cubic nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = u^3, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}. \end{cases} \tag{0.1}$$

The aim of the present paper is to consider problem (0.1) in low-order Sobolev spaces, when the initial data $u_0 \in \mathbf{H}^{\alpha} \cap \mathbf{H}^{0,\alpha}$ with $\alpha > \frac{1}{2}$. In our previous paper [7] we proved the global existence of solutions to (0.1) if the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$. Also we find the large-time asymptotics of solutions.

1. Introduction

We consider the Cauchy problem for the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = u^3, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}. \end{cases} \tag{1.1}$$

Note that the time decay rate of the nonlinear term in equation (1.1) is critical with respect to the large-time asymptotic behavior of solutions. Different types of the cubic nonlinearities, including derivatives of the unknown function and the gauge invariant term $|u|^2u$, were considered previously (see papers [2], [4], [8], [10], [11] and literature cited therein).

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In paper [6], the nonlinear Schrödinger equation (1.1) with a more general cubic nonlinearity of the form

$$\lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3, \quad \lambda_j \in \mathbf{C}, j = 1, 2, 3$$

was studied. We assumed that there exists $\theta_0 > 0$ such that

$$\mathrm{Re}\Lambda(\omega) \geq C > 0 \text{ and } \omega \mathrm{Im}\Lambda(\omega) \geq C\omega^2 \text{ for all } |\omega| < \theta_0,$$

where the function

$$\Lambda(\omega) \equiv \lambda_1 e^{2i\omega} - i\lambda_2 \sqrt{3} e^{-2i\omega} + \lambda_3 e^{-4i\omega}.$$

Also we supposed in [6] that the initial data $u_1(x)$ are such that

$$\sup_{|\xi|\leq 1} |\arg \widehat{u_1}(\xi)| \leq C\varepsilon, \quad \inf_{|\xi|\leq 1} |\widehat{u_1}(\xi)| \geq C\varepsilon.$$

The method was based on the representation

$$\partial_\xi = i\Upsilon(1 - i\Upsilon\xi)^{-1}(-\widehat{\mathcal{I}} + 2t\partial_t) + (1 - i\Upsilon\xi)^{-1}\partial_\xi$$

with the operator $\widehat{\mathcal{I}} = -\xi\partial_\xi + 2t\partial_t$. Here the parameter $\Upsilon > 0$ has a time growth of order $\sqrt{t}$, so the terms $(1 - i\Upsilon\xi)^{-1}$ appearing in this identity help us to obtain better time decay properties of the solution. Unfortunately the application of this identity implies some derivative loss with respect to the operator $\widehat{\mathcal{I}}$. Hence to prove the global existence of solutions we need to assume that the initial data belong to some analytic function space. In our previous paper [7] we proved the global existence result for the case of the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$. The aim of the present paper is to consider problem (1.1) in the low-order Sobolev spaces, when the initial data $u_0 \in \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha}$ with $\alpha > \frac{1}{2}$. Here we use a method based on the factorization of [9] for the free Schrödinger evolution group $\mathcal{U}(t) = e^{\frac{it}{2}\partial_x^2} = \mathcal{F}^{-1} e^{-\frac{it}{2}\xi^2} \mathcal{F}$ related to the linear Schrödinger equation $i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$:

$$\mathcal{U}(t) = M(t)\mathcal{D}(t)\mathcal{V}(t)\mathcal{F} \tag{1.2}$$

with $M(t) = e^{\frac{i}{2t}x^2}$, the dilation operator $(\mathcal{D}(t)\phi)(x) = \frac{1}{\sqrt{it}}\phi(\frac{x}{t})$, and

$$\mathcal{V}(t)\phi = \mathcal{F}M(t)\mathcal{F}^{-1}\phi = \sqrt{\frac{it}{2\pi}} \int_{\mathbf{R}} e^{-\frac{it}{2}(\xi-\eta)^2} \phi(\eta) d\eta.$$

The Fourier transform is defined by

$$\mathcal{F}\phi = \widehat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x) dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{ix\xi}\phi(\xi)d\xi.$$

Formula (1.2) is very useful for studying the large-time asymptotic behavior of solutions to the Schrödinger equations. We also have

$$\mathcal{F}\mathcal{U}(-t) = i\mathcal{V}(-t)\overline{E}(t)\mathcal{D}(\tfrac{1}{t}), \tag{1.3}$$

where $E(t) = e^{\frac{it}{2}\xi^2}$. Here we used the commutation identity

$$\mathcal{D}(\tfrac{1}{t})M(t) = E(t)\mathcal{D}(\tfrac{1}{t}).$$

Denote the usual Lebesgue space $\mathbf{L}^p = \{\phi \in \mathbf{S}' : \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm

$$\|\phi\|_{\mathbf{L}^p} = \Big(\int_{\mathbf{R}} |\phi(x)|^p dx\Big)^{1/p}$$

if $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = ess.\sup_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. The weighted Lebesgue space is $\mathbf{L}^{p,k} = \{\phi \in \mathbf{S}' : \|\phi\|_{\mathbf{L}^{p,k}} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^{p,k}} = \|\langle\cdot\rangle^k\phi(\cdot)\|_{\mathbf{L}^p}$, $k \ge 0$. The weighted Sobolev space is

$$\mathbf{H}^{m,k} = \{\phi \in \mathbf{S}' : \|\phi\|_{\mathbf{H}^{m,k}} \equiv \|\langle i\partial\rangle^m\phi\|_{\mathbf{L}^{2,k}} < \infty\},$$

where $m, k \in \mathbf{R}$, $\langle x\rangle = \sqrt{1+x^2}$. The usual Sobolev space is $\mathbf{H}^m = \mathbf{H}^{m,0}$, so the index 0 we usually omit if it does not cause confusion. Different positive constants we denote by the same letter C.

In the present paper we prove the following result.

Theorem 1.1. *Let the initial data $u_0 \in \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha}$ with $\alpha > \frac{1}{2}$ and a norm $\|u_0\|_{\mathbf{H}^\alpha} + \|u_0\|_{\mathbf{H}^{0,\alpha}} \le \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Also suppose that*

$$\sup_{|\xi|\le 1} |\arg\widehat{u_0}(\xi)| < \frac{\pi}{8}, \quad \inf_{|\xi|\le 1} |\widehat{u_0}(\xi)| \ge \delta = \varepsilon^{\frac{5}{4}}. \tag{1.4}$$

Then there exists a unique solution $u \in \mathbf{C}([1,\infty); \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha})$ of the Cauchy problem (1.1). *Moreover,*

$$u(t,x) = \frac{(it)^{-\frac{1}{2}}|\widehat{u_0}(\frac{x}{t})|\exp(\frac{ix^2}{2t})}{(1+\frac{|\widehat{u_0}(\frac{x}{t})|^2}{\sqrt{3}}\log\frac{t}{1+\frac{x^2}{t}})^{\frac{1}{2}}} + O(t^{-\frac{1}{2}}(\log\frac{t}{1+\frac{x^2}{t}})^{-\frac{1}{2}-\gamma}) \tag{1.5}$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\gamma > 0$.

To make our strategy understandable to readers, we state our main points of the proof. We multiply both sides of equation (1.1) by the operator

$\mathcal{FU}(-t)$, and make the transformation similar to the Shatah normal forms (i.e., integrating by parts with respect to time t), to obtain

$$\partial_t(\varphi + \widetilde{R_1}) = -\frac{1}{\sqrt{3}t(1+it\xi^2)}\varphi^3 + \widetilde{R_2},$$

where C is a constant, $\varphi = \mathcal{FU}(-t)u$ and $\widetilde{R_j}$ are remainder terms coming from the cubic nonlinearity. Therefore, asymptotic behavior of φ is determined by the ordinary differential equation

$$\partial_t\varphi \simeq -\frac{1}{\sqrt{3}t(1+it\xi^2)}\varphi^3 = -\frac{1}{t\sqrt{3}B}\varphi^3 + \frac{1}{t\sqrt{3}B}(\frac{1}{B} - \frac{1}{1+it\xi^2})\varphi^3$$

where $B = \langle \xi\sqrt{t}\rangle$, since

$$\frac{1}{B} - \frac{1}{1+it\xi^2} = \frac{1+it\xi^2 - B}{B(1+it\xi^2)} = O((|\xi|\sqrt{t})^\gamma B^{-1})$$

is considered as the remainder term. Therefore the amplitude of φ is dominated by

$$|\varphi(t,\xi)| \le \frac{|\widehat{u_0}(\xi)|}{\left(1 + \frac{2|\widehat{u_0}(\xi)|^2}{\sqrt{3}}\log\frac{t}{1+t\xi^2}\right)^{\frac{1}{2}}}.$$

From this we deduce that

$$u = \mathcal{U}(t)\mathcal{F}\varphi \simeq MD\frac{|\widehat{u_0}(\xi)|}{\left(1 + \frac{2|\widehat{u_0}(\xi)|^2}{\sqrt{3}}\log\frac{t}{1+t\xi^2}\right)^{\frac{1}{2}}}.$$

We organize our paper as follows. In Section 2 we prove some preliminary estimates for the fractional derivatives. Section 3 is devoted to the $\mathbf{L}^2$-estimate of the fractional derivative $|\partial|^\alpha$ of the solution $\mathcal{FU}(-t)u$. Then in Section 4 we give estimates for the $\mathbf{L}^\infty$-norm of $\mathcal{FU}(-t)u$. Finally, Section 5 is devoted to the proof of Theorem 1.1.

2. Preliminary estimates

Define the operator

$$\mathcal{K}\phi = \sqrt{t}\int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t})\phi(\xi-\eta)d\eta$$

for $t \ge 1$, where $K(x,y)$ is a given kernel. Denote $B = \langle \xi\sqrt{t}\rangle$.

Lemma 2.1. *Let the kernel K satisfy the estimates $|K(x,y)| \le \langle x\rangle K_1(\langle x\rangle y)$ for $x, y \in \mathbf{R}$, and $|K(x,y)| \le K_2(y)$, for $|y| \ge \frac{|x|}{2}$, where K_1, $K_2 \in \mathbf{L}^1$. Then for any $\phi \in \mathbf{L}^2$*

$$\|\mathcal{K}\phi\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^2}.$$

Moreover, if $K_1 \in \mathbf{L}^{1,\beta}$, with $\beta \ge 0$, then

$$\|B^{-\beta}\mathcal{K}\phi\|_{\mathbf{L}^\infty} \le C\|B^{-\beta}\phi\|_{\mathbf{L}^\infty}$$

provided that the right-hand side is finite.

Remark 1. A typical example of the kernel to which we can apply Lemma 2.1 is $K(x,y) = \frac{\langle x\rangle \min(1,\langle x\rangle|y|)}{|\langle x\rangle y|^{1+\beta}}$, with $\beta \in (0,1)$. Indeed we can take $K_1(y) = \frac{\min(1,|y|)}{|y|^{1+\beta}}$ and $K_2(y) = C|y|^{\delta-1}\langle y\rangle^{-\beta}$ with $0 < \delta < \min(\beta, 1-\beta)$, since the estimates

$$\frac{\langle x\rangle \min(1,\langle x\rangle|y|)}{|\langle x\rangle y|^{1+\beta}} \le \frac{C\langle x\rangle^{\delta}}{|y|^{1-\delta}\langle y\rangle^{\beta+\delta}} \le \frac{C}{|y|^{1-\delta}\langle y\rangle^{\beta}}$$

are true for all $|y| \ge \frac{|x|}{2}$. Another typical example is $K(x,y) = \frac{\langle x\rangle}{\sqrt{\pi}}e^{-\langle x\rangle^2y^2}$.

Proof. By the condition of the lemma we have $|K(\xi\sqrt{t},\eta\sqrt{t})| \le K_2(\eta\sqrt{t})$ for $|\eta| \ge \frac{|\xi|}{2}$. Then application of the Young inequality for convolutions yields

$$\begin{aligned} &\|\sqrt{t}\int_{|y|\ge\frac{|\xi|}{2}} |K(\xi\sqrt{t},\eta\sqrt{t})||\phi(\xi-\eta)|d\eta\|_{\mathbf{L}^2} \\ \le\ & C\sqrt{t}\|\int_{\mathbf{R}} K_2(\eta\sqrt{t})|\phi(\xi-\eta)|d\eta\|_{\mathbf{L}^2} \\ \le\ & C\sqrt{t}\|\phi\|_{\mathbf{L}^2}\int_{\mathbf{R}} K_2(\eta\sqrt{t})d\eta \le C\|\phi\|_{\mathbf{L}^2}. \end{aligned}$$

By the condition of the lemma we have $|K(\xi\sqrt{t},\eta\sqrt{t})| \le BK_1(B\sqrt{t}\eta)$; then changing $z = B\sqrt{t}\eta$ and applying the Schwartz inequality we get

$$\begin{aligned} &\left\|\sqrt{t}\int_{|\eta|<\frac{|\xi|}{2}} |K(\xi\sqrt{t},\eta\sqrt{t})||\phi(\xi-\eta)|d\eta\right\|_{\mathbf{L}^2}^2 \\ &\le \left\|B\sqrt{t}\int_{|\eta|<\frac{|\xi|}{2}} K_1(B\sqrt{t}\eta)|\phi(\xi-\eta)|d\eta\right\|_{\mathbf{L}^2}^2 = \left\|\int_{|z|<B\sqrt{t}\frac{|\xi|}{2}} K_1(z)|\phi(\xi-\frac{z}{B\sqrt{t}})|dz\right\|_{\mathbf{L}^2}^2 \\ &= \int_{\mathbf{R}} K_1(\zeta)d\zeta\int_{\mathbf{R}} K_1(z)dz\int_{|\xi|\sqrt{t}B>2\max(|z|,|\zeta|)} |\phi(\xi-\frac{z}{B\sqrt{t}})\phi(\xi-\frac{\zeta}{B\sqrt{t}})|d\xi \\ &\le C\int_{\mathbf{R}} K_1(z)dz\int_{\frac{|z|}{B\sqrt{t}}<\frac{|\xi|}{2}} |\phi(\xi-\frac{z}{B\sqrt{t}})|^2d\xi. \end{aligned}$$

Now changing $\xi - \frac{z}{B\sqrt{t}} = \zeta$ we find

$$\begin{aligned} &\left\| \sqrt{t} \int_{|\eta|<\frac{|\xi|}{2}} |K(\xi\sqrt{t}, \eta\sqrt{t})||\phi(\xi-\eta)|d\eta \right\|^2_{\mathbf{L}^2} \\ \leq \;& C \int_{\mathbf{R}} K_1(z)dz \int_{\mathbf{R}} |\phi(\zeta)|^2 .(1+\frac{z\xi\sqrt{t}}{B^3})^{-1}|_{\frac{|z|}{B\sqrt{t}}<\frac{|\xi|}{2}} d\zeta \\ \leq \;& C \int_{\mathbf{R}} K_1(z)dz \int_{\mathbf{R}} |\phi(\zeta)|^2 d\zeta \leq C\|\phi\|^2_{\mathbf{L}^2}. \end{aligned}$$

Hence the first estimate of the lemma is true. Next we use the inequality

$$B^{-\beta}\langle(\xi-\eta)\sqrt{t}\rangle^\beta \leq C\langle\sqrt{t}\eta\rangle^\beta \leq C\langle B\sqrt{t}\eta\rangle^\beta,$$

then by the condition of the lemma we have

$$B^{-\beta}|K(\xi\sqrt{t}, \eta\sqrt{t})|\langle(\xi-\eta)\sqrt{t}\rangle^\beta \leq CB\langle B\sqrt{t}\eta\rangle^\beta K_1(B\sqrt{t}\eta).$$

Therefore, we obtain

$$\begin{aligned} &\|B^{-\beta}\mathcal{K}\phi\|_{\mathbf{L}^\infty} \leq \left\| \sqrt{t} \int_{\mathbf{R}} B^{-\beta}|K(\xi\sqrt{t}, \eta\sqrt{t})|\phi(\xi-\eta)d\eta \right\|_{\mathbf{L}^\infty} \\ &\leq C\|B^{-\beta}\phi\|_{\mathbf{L}^\infty} \left\| B\sqrt{t} \int_{\mathbf{R}} \langle B\sqrt{t}\eta\rangle^\beta K_1(B\sqrt{t}\eta)d\eta \right\|_{\mathbf{L}^\infty} \leq C\|B^{-\beta}\phi\|_{\mathbf{L}^\infty}. \end{aligned}$$

Thus the second estimate of the lemma follows. Lemma 2.1 is proved. □

We denote the fractional derivative for $\alpha \in (0,1)$ (see [12])

$$|\partial|^\alpha \phi = \mathcal{F}^{-1}(|\xi|^\alpha \widehat{\phi}(\xi)) = C \int_{\mathbf{R}} (\widetilde{\phi} - \phi) \frac{dy}{|y|^{1+\alpha}},$$

where we denote

$$\widetilde{\phi} = \phi(\cdot - y), C = -\frac{\alpha}{2\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})},$$

and Γ is the Euler gamma function. We can write the Leibnitz rule

$$|\partial|^\alpha(\phi\psi) = \phi|\partial|^\alpha\psi + [|\partial|^\alpha, \phi]\psi$$

with the commutator

$$[|\partial|^\alpha, \phi]\psi = C \int_{\mathbf{R}} (\widetilde{\phi} - \phi)\widetilde{\psi} \frac{dy}{|y|^{1+\alpha}}.$$

Also using the identity

$$\widetilde{\phi}\widetilde{\psi} - \phi\psi = \phi(\widetilde{\psi} - \psi) + \psi(\widetilde{\phi} - \phi) + (\widetilde{\phi} - \phi)(\widetilde{\psi} - \psi),$$

we can write the Leibnitz rule as follows:

$$|\partial|^\alpha(\phi\psi) = \phi|\partial|^\alpha\psi + \psi|\partial|^\alpha\phi + [|\partial|^\alpha, \phi, \psi],$$

where we denote the commutator

$$[|\partial|^{\alpha}, \phi, \psi] = C \int_{\mathbf{R}} (\widetilde{\phi} - \phi)(\widetilde{\psi} - \psi) \frac{dy}{|y|^{1+\alpha}}.$$

In the next lemma we find some estimates for the commutators $[|\partial|^{\alpha}, \phi]\psi$ and $[|\partial|^{\alpha}, \phi, \psi]$ for $\alpha \in (0,1)$.

Lemma 2.2. *The inequalities*

$$\|[|\partial|^{\alpha}, \phi]\psi\|_{\mathbf{L}^2} \le Ct^{\frac{\alpha}{2}} (\|B^{\beta}\phi\|_{\mathbf{L}^{\infty}} + t^{-\frac{1}{2}} \|B^{\beta}\partial\phi\|_{\mathbf{L}^{\infty}}) \|B^{-\beta}\psi\|_{\mathbf{L}^2}$$

are valid, where $0 \le \beta < \alpha < 1$,

$$\|[|\partial|^{\alpha}, \phi]\psi\|_{\mathbf{L}^2} \le Ct^{\frac{\alpha}{2}} (\|\phi\|_{\mathbf{L}^{\infty}} + t^{-\frac{1}{2}} \|B^{-1}\partial\phi\|_{\mathbf{L}^{\infty}}) \|B^{\alpha}\psi\|_{\mathbf{L}^2}$$

and

$$\|[|\partial|^{\alpha}, \phi, \psi]\|_{\mathbf{L}^2} \le Ct^{\frac{\alpha}{2}} \|\phi\|_{\mathbf{L}^{\infty}} \|\psi\|_{\mathbf{L}^2} + Ct^{\frac{1}{4}-\frac{\alpha}{2}} \||\partial|^{\alpha}\phi\|_{\mathbf{L}^2} \||\partial|^{\alpha}\psi\|_{\mathbf{L}^2},$$

provided that the right-hand sides are finite.

Proof. We have with $\widetilde{B} = \langle (\xi - y)\sqrt{t} \rangle$

$$\begin{aligned} \|[|\partial|^{\alpha}, \phi]\psi\|_{\mathbf{L}^2} &= C \int_{\mathbf{R}} (\widetilde{\phi} - \phi)\widetilde{\psi} \frac{dy}{|y|^{1+\alpha}} \\ \le\ & C \int_{|y|\sqrt{t} \le 1} \|\widetilde{B}^{\beta}(\widetilde{\phi} - \phi)\widetilde{B}^{-\beta}\widetilde{\psi}\|_{\mathbf{L}^2} \frac{dy}{|y|^{1+\alpha}} \\ & + C \int_{|y|\sqrt{t} > 1} (\|\widetilde{B}^{\beta}\widetilde{\phi}\widetilde{B}^{-\beta}\widetilde{\psi}\|_{\mathbf{L}^2} + \|\widetilde{B}^{\beta}\phi\widetilde{B}^{-\beta}\widetilde{\psi}\|_{\mathbf{L}^2}) \frac{dy}{|y|^{1+\alpha}} \end{aligned}$$

for the case $0 \le \beta < \alpha < 1$. Since

$$\widetilde{B}^{\beta}|\widetilde{\phi} - \phi| \le \widetilde{B}^{\beta} \int_0^y |\phi'(\cdot - h)| dh \le C|y| \|B^{\beta}\partial\phi\|_{\mathbf{L}^{\infty}}$$

for $|y|\sqrt{t} \le 1$ and $\widetilde{B}^{\beta} \le B^{\beta}(1 + |y\sqrt{t}|^{\beta})$ for $|y|\sqrt{t} \ge 1$, then we find the first estimate of the lemma

$$\begin{aligned} \|[|\partial|^{\alpha}, \phi]\psi\|_{\mathbf{L}^2} &\le C\|B^{\beta}\partial\phi\|_{\mathbf{L}^{\infty}} \|B^{-\beta}\psi\|_{\mathbf{L}^2} \int_{|y|\sqrt{t} \le 1} \frac{dy}{|y|^{\alpha}} \\ & + C\|B^{\beta}\phi\|_{\mathbf{L}^{\infty}} \|B^{-\beta}\psi\|_{\mathbf{L}^2} \int_{|y|\sqrt{t} > 1} (1 + |y\sqrt{t}|^{\beta}) \frac{dy}{|y|^{1+\alpha}} \\ \le\ & Ct^{\frac{\alpha}{2}} (t^{-\frac{1}{2}} \|B^{\beta}\partial\phi\|_{\mathbf{L}^{\infty}} + \|B^{\beta}\phi\|_{\mathbf{L}^{\infty}}) \|B^{-\beta}\psi\|_{\mathbf{L}^2}. \end{aligned}$$

To prove the second estimate we write the inequalities

$$\widetilde{B}^{-\alpha}|\widetilde{\phi} - \phi| \le C\|\phi\|_{\mathbf{L}^{\infty}}$$

for — y — $\sqrt{t} \geq 1$,

$$\widetilde{B}^{-\alpha}|\widetilde{\phi} - \phi| \leq CB^{-\alpha}\|\phi\|_{\mathbf{L}^\infty}$$

for $B^{-1} \leq$ — y — $\sqrt{t} \leq 1$, and also

$$\widetilde{B}^{-\alpha}|\widetilde{\phi} - \phi| \leq C\widetilde{B}^{-\alpha}\int_0^y |\phi'(\cdot - h)|dh \leq CB^{-\alpha}(|y|\sqrt{t}B)(t^{-\frac{1}{2}}\|B^{-1}\partial\phi\|_{\mathbf{L}^\infty})$$

for — y — $\sqrt{t} \leq B^{-1}$. Therefore, we find

$$\widetilde{B}^{-\alpha}|\widetilde{\phi} - \phi| \leq C\|\phi\|_{\mathbf{L}^\infty}\min(1, |y|\sqrt{t}) + CB^{-\alpha}(\|\phi\|_{\mathbf{L}^\infty} + t^{-\frac{1}{2}}\|B^{-1}\partial\phi\|_{\mathbf{L}^\infty})\min(1, |y|\sqrt{t}B)$$

for all $y \in \mathbf{R}$. Hence

$$\begin{aligned} &\|[|\partial|^\alpha, \phi]\psi\|_{\mathbf{L}^2} \leq C\|\int_{\mathbf{R}} \widetilde{B}^{-\alpha}(\widetilde{\phi} - \phi)\widetilde{B}^\alpha\widetilde{\psi}\frac{dy}{|y|^{1+\alpha}}\|_{\mathbf{L}^2} \\ \leq\ & C\|\phi\|_{\mathbf{L}^\infty}\|\int_{\mathbf{R}} \min(1, |y|\sqrt{t})|\widetilde{B}^\alpha\widetilde{\psi}|\frac{dy}{|y|^{1+\alpha}}\|_{\mathbf{L}^2} \\ &+Ct^{\frac{\alpha}{2}}(t^{-\frac{1}{2}}\|B^{-1}\partial\phi\|_{\mathbf{L}^\infty} + \|\phi\|_{\mathbf{L}^\infty}) \\ &\times\|(\sqrt{t}B)^{-\alpha}\int_{\mathbf{R}} \min(1, |y|\sqrt{t}B)|\widetilde{B}^\alpha\widetilde{\psi}|\frac{dy}{|y|^{1+\alpha}}\|_{\mathbf{L}^2}. \end{aligned} \tag{2.1}$$

For the first summand in the right-hand side of (2.1) we apply the Young inequality

$$\|\int_{\mathbf{R}} \frac{\min(1, |y|\sqrt{t})}{|y|^{1+\alpha}}|\widetilde{\Phi}|dy\|_{\mathbf{L}^2} \leq C\|\Phi\|_{\mathbf{L}^2}\int_{\mathbf{R}} \frac{\min(1, |y|\sqrt{t})}{|y|^{1+\alpha}}dy \leq Ct^{\frac{\alpha}{2}}\|\Phi\|_{\mathbf{L}^2}.$$

We now apply Lemma 2.1 with $K(\xi\sqrt{t}, y\sqrt{t}) = \frac{B\min(1,|y|\sqrt{t}B)}{|y\sqrt{t}B|^{1+\alpha}}$ (see Remark 1) to estimate the second summand in the right-hand side of (2.1):

$$\|\int_{\mathbf{R}} \frac{\min(1, |y|\sqrt{t}B)}{(\sqrt{t}B)^\alpha|y|^{1+\alpha}}|\widetilde{\Phi}|dy\|_{\mathbf{L}^2} \leq C\|\Phi\|_{\mathbf{L}^2}.$$

Therefore, by virtue of (2.1) the second estimate of the lemma follows.

Finally, to estimate the norm $\|[|\partial|^\alpha, \phi, \psi]\|_{\mathbf{L}^2}$ we write

$$\begin{aligned} &\|\int_{|y|\sqrt{t}>1} (\widetilde{\phi} - \phi)(\widetilde{\psi} - \psi)\frac{dy}{|y|^{1+\alpha}}\|_{\mathbf{L}^2} \\ \leq\ & C\|\phi\|_{\mathbf{L}^\infty}\|\psi\|_{\mathbf{L}^2}\int_{|y|\sqrt{t}>1} \frac{dy}{|y|^{1+\alpha}} \leq Ct^{\frac{\alpha}{2}}\|\phi\|_{\mathbf{L}^\infty}\|\psi\|_{\mathbf{L}^2}. \end{aligned}$$

By the Sobolev imbedding theorem we have $\|\phi\|_{\mathbf{L}^4} \le C\||\partial|^{\frac{1}{2}}\phi\|_{\mathbf{L}^2}^{\frac{1}{2}}\|\phi\|_{\mathbf{L}^2}^{\frac{1}{2}}$. Therefore applying the Plancherel theorem we find

$$\begin{aligned}\|\widetilde{\phi}-\phi\|_{\mathbf{L}^2} &= \|\int_{\mathbf{R}^n} e^{ix\xi}(e^{iy\xi}-1)\widehat{\phi}(\xi)d\xi\|_{\mathbf{L}^2} = \|(e^{iy\xi}-1)\widehat{\phi}(\xi)\|_{\mathbf{L}^2} \\ &\le C|y|^{\beta}\||\xi|^{\beta}\widehat{\phi}(\xi)\|_{\mathbf{L}^2} = C|y|^{\beta}\||\partial|^{\beta}\phi\|_{\mathbf{L}^2}\end{aligned}$$

for $0 \le \beta \le 1$. Hence $\|\widetilde{\phi}-\phi\|_{\mathbf{L}^4} \le C|y|^{\alpha-\frac{1}{4}}\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}$ and

$$\begin{aligned}&\|\int_{|y|\sqrt{t}\le 1}(\widetilde{\phi}-\phi)(\widetilde{\psi}-\psi)\frac{dy}{|y|^{1+\alpha}}\|_{\mathbf{L}^2} \\ &\le C\int_{|y|\sqrt{t}\le 1}\|\widetilde{\phi}-\phi\|_{\mathbf{L}^4}\|\widetilde{\psi}-\psi\|_{\mathbf{L}^4}\frac{dy}{|y|^{1+\alpha}} \\ &\le C\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}\||\partial|^{\alpha}\psi\|_{\mathbf{L}^2}\int_{|y|\sqrt{t}\le 1}|y|^{\alpha-\frac{3}{2}}dy \le Ct^{\frac{1}{4}-\frac{\alpha}{2}}\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}\||\partial|^{\alpha}\psi\|_{\mathbf{L}^2}.\end{aligned}$$

Thus we get the third estimate of the lemma. Lemma 2.2 is proved. □

Remark 2. Applying the first estimate of Lemma 2.2 we get

$$\|[|\partial|^{\alpha}, B^{-\beta}]\psi\|_{\mathbf{L}^2} \le Ct^{\frac{\alpha}{2}}\|B^{-\beta}\psi\|_{\mathbf{L}^2}$$

for $0 \le \beta < \alpha$. By the second estimate of Lemma 2.2 it follows that

$$\||\partial|^{\alpha}(E^2\psi)\|_{\mathbf{L}^2} \le C\||\partial|^{\alpha}\psi\|_{\mathbf{L}^2} + C\|(B\sqrt{t})^{\alpha}\psi\|_{\mathbf{L}^2}. \tag{2.2}$$

Finally, by the third estimate of Lemma 2.2 we get

$$\begin{aligned}\||\partial|^{\alpha}(\phi\psi)\|_{\mathbf{L}^2} &\le \|\phi\|_{\mathbf{L}^\infty}\||\partial|^{\alpha}\psi\|_{\mathbf{L}^2} + \|\psi\|_{\mathbf{L}^\infty}\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2} \\ &+Ct^{\frac{\alpha}{2}}\|\phi\|_{\mathbf{L}^\infty}\|\psi\|_{\mathbf{L}^2} + Ct^{\frac{1}{4}-\frac{\alpha}{2}}\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}\||\partial|^{\alpha}\psi\|_{\mathbf{L}^2}.\end{aligned}$$

Then taking $\psi = \phi^2$ we get

$$\||\partial|^{\alpha}(\phi^3)\|_{\mathbf{L}^2} \le Ct^{\frac{\alpha}{2}-\frac{1}{4}}(\|\phi\|_{\mathbf{L}^\infty} + t^{\frac{1}{4}}\|\phi\|_{\mathbf{L}^2} + t^{\frac{1}{4}-\frac{\alpha}{2}}\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2})^3. \tag{2.3}$$

Denote

$$K_0(\xi\sqrt{t}) = \sqrt{t}\int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t})d\eta = \int_{\mathbf{R}} K(\xi\sqrt{t}, \zeta)d\zeta.$$

In the next lemma we estimate the difference

$$\mathcal{K}\phi - K_0\phi = \sqrt{t}\int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t})(\phi(\xi-\eta)-\phi(\xi))dy.$$

A typical choice of a kernel is $K(x,y)= \frac{\langle x\rangle}{\sqrt{\pi}}e^{-\langle x\rangle^2 y^2}$ in applying Lemma 2.3 below in Section 3.

Lemma 2.3. *Let the kernel* $K \in \mathbf{C}^1(\mathbf{R}^2)$ *satisfy the estimates*

$$|K(x,y)| \le C\langle x\rangle \langle\langle x\rangle y\rangle^{-3} \quad \text{and} \quad |\partial_x K(x,y)| \le C\langle y\rangle^{-3}$$

for all $x, y \in \mathbf{R}$. *Then*

$$\||\partial|^\alpha(\mathcal{K}\phi - K_0\phi)\|_{\mathbf{L}^2} + \|(B\sqrt{t})^\alpha(\mathcal{K}\phi - K_0\phi)\|_{\mathbf{L}^2} \le C\||\partial|^\alpha \phi\|_{\mathbf{L}^2}$$

for all $t \ge 1$.

Proof. Denote

$$K_1(\xi\sqrt{t}, \eta\sqrt{t}) = \begin{cases} -\sqrt{t}\int_\eta^\infty K(\xi\sqrt{t}, \zeta\sqrt{t})d\zeta, \text{ for } \eta > 0, \\ \sqrt{t}\int_{-\infty}^\eta K(\xi\sqrt{t}, \zeta\sqrt{t})d\zeta, \text{ for } \eta < 0. \end{cases}$$

Hence,

$$\sqrt{t}K(\xi\sqrt{t}, \eta\sqrt{t}) = \partial_\eta K_1(\xi\sqrt{t}, \eta\sqrt{t})$$

for $\eta \ne 0$. We decompose the derivative as $\partial_\eta = |\partial_\eta|^{1-\alpha}|\partial_\eta|^\alpha \mathcal{H}$, where

$$\mathcal{H}\phi = \frac{1}{\pi}PV\int_{\mathbf{R}} \phi(\xi - y)\frac{dy}{y}$$

is the Hilbert transformation, then by the Plansherel theorem

$$\begin{aligned} \mathcal{K}\phi - K_0\phi &= \int_{\mathbf{R}} \partial_\eta K_1(\xi\sqrt{t}, \eta\sqrt{t})(\phi(\xi - \eta) - \phi(\xi))d\eta \\ &= -\int_{\mathbf{R}} |\partial_\eta|^{1-\alpha}(K_1(\xi\sqrt{t}, \eta\sqrt{t}))|\partial_\eta|^\alpha \mathcal{H}\phi(\xi - \eta)d\eta. \end{aligned} \tag{2.4}$$

Next we estimate the kernel $|\partial_\eta|^{1-\alpha}(K_1(\xi\sqrt{t}, \eta\sqrt{t}))$. By the condition of the lemma we have

$$\begin{aligned} &|K_1(\xi\sqrt{t}, (\eta - y)\sqrt{t}) - K_1(\xi\sqrt{t}, \eta\sqrt{t})| \\ \le\ & \sqrt{t}\int_{\eta - y}^\eta |K(\xi\sqrt{t}, \zeta\sqrt{t})|d\zeta \le CB\sqrt{t}\int_{\eta-y}^\eta d\zeta \le CB\sqrt{t}|y| \end{aligned}$$

for all $|y| \le \frac{\eta}{2}$, $0 < B\sqrt{t}\eta \le 1$. Also in view of the conditions of the lemma we get

$$|K_1(\xi\sqrt{t}, \eta\sqrt{t})| \le CB\sqrt{t}\int_\eta^\infty \langle B\sqrt{t}\zeta\rangle^{-3}d\zeta \le C\langle B\sqrt{t}\eta\rangle^{-2}$$

for all $\eta > 0$. (The case of $\eta < 0$ is considered in the same manner.) Then we find

$$||\partial_\eta|^{1-\alpha}(K_1(\xi\sqrt{t}, \eta\sqrt{t}))| \le C\int_{\mathbf{R}} |K_1(\xi\sqrt{t}, (\eta - y)\sqrt{t}) - K_1(\xi\sqrt{t}, \eta\sqrt{t})|\frac{dy}{|y|^{2-\alpha}}$$

$$\leq CB\sqrt{t}\int_{|y|\leq\frac{\eta}{2}}\frac{dy}{|y|^{1-\alpha}}+C\int_{|y|\geq\frac{\eta}{2}}\frac{dy}{|y|^{2-\alpha}}\leq CB\sqrt{t}\eta^{\alpha}+\frac{C}{\eta^{1-\alpha}}\leq C\frac{(B\sqrt{t})^{1-\alpha}}{|B\sqrt{t}\eta|^{1-\alpha}}$$

for all $0 < B\sqrt{t}\eta \leq 1$. Next by the condition of the lemma we have

$$|K_1(\xi\sqrt{t},(\eta-y)\sqrt{t})-K_1(\xi\sqrt{t},\eta\sqrt{t})|\leq\sqrt{t}\int_{\eta-y}^{\eta}|K(\xi\sqrt{t},\zeta\sqrt{t})|d\zeta$$

$$\leq \quad CB\sqrt{t}\int_{\eta-y}^{\eta}\langle B\sqrt{t}\zeta\rangle^{-3}d\zeta\leq CB\sqrt{t}|y|\langle B\sqrt{t}\eta\rangle^{-3}$$

for all $|y| \leq \frac{\eta}{2}$, $B\sqrt{t}\eta \geq 1$. Therefore we get

$$||\partial_\eta|^{1-\alpha}(K_1(\xi\sqrt{t},\eta\sqrt{t}))|\leq CB\sqrt{t}\langle B\sqrt{t}y\rangle^{-3}\int_{|y|\leq\frac{\eta}{2}}\frac{dy}{|y|^{1-\alpha}}$$

$$+C\int_{|y|\geq\frac{\eta}{2}}\langle B\sqrt{t}(y-\eta)\rangle^{-2}\frac{dy}{|y|^{2-\alpha}}+C\langle B\sqrt{t}\eta\rangle^{-2}\int_{|y|\geq\frac{\eta}{2}}\frac{dy}{|y|^{2-\alpha}}$$

$$\leq \quad \frac{CB\sqrt{t}\eta^{\alpha}}{\langle B\sqrt{t}\eta\rangle^{3}}+C\eta^{\alpha-2}\int_{\frac{\eta}{2}}^{2\eta}\frac{dy}{\langle B\sqrt{t}(y-\eta)\rangle^{2}}+\frac{C\eta^{\alpha-1}}{\langle B\sqrt{t}\eta\rangle^{2}}\leq\frac{C(B\sqrt{t})^{1-\alpha}}{(B\sqrt{t}\eta)^{2-\alpha}}$$

for $B\sqrt{t}\eta \geq 1$. Thus we obtain

$$||\partial_\eta|^{1-\alpha}(K_1(\xi\sqrt{t},\eta\sqrt{t}))|\leq C(B\sqrt{t})^{1-\alpha}\frac{\min(1,B\sqrt{t}|\eta|)}{|B\sqrt{t}\eta|^{2-\alpha}}$$

for all $\eta \in \mathbf{R}$. Applying the first estimate of Lemma 2.1 with a kernel

$$K(\xi\sqrt{t},\eta\sqrt{t})=\frac{B\min(1,B\sqrt{t}|\eta|)}{|B\sqrt{t}\eta|^{2-\alpha}}$$

(see Remark 1) we get from formula (2.4)

$$\begin{aligned}&\|(B\sqrt{t})^{\alpha}(\mathcal{K}\phi-K_0\phi)\|_{\mathbf{L}^2} \qquad (2.5)\\ &=\|(B\sqrt{t})^{\alpha}\int_{\mathbf{R}}|\partial_y|^{1-\alpha}(K_1(B\sqrt{t}y))|\partial_y|^{\alpha}\mathcal{H}\phi(\xi-y)dy\|_{\mathbf{L}^2}\\ &\leq C\|\sqrt{t}\int_{\mathbf{R}}\frac{B\min(1,B\sqrt{t}|\eta|)}{|B\sqrt{t}\eta|^{2-\alpha}}|\partial_y|^{\alpha}\mathcal{H}\phi(\xi-y)dy\|_{\mathbf{L}^2}\leq C\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}.\end{aligned}$$

Now we use the identity

$$|\partial|^{\alpha}((\mathcal{K}-K_0)\phi)=(\mathcal{K}-K_0)|\partial|^{\alpha}\phi+[|\partial|^{\alpha},(\mathcal{K}-K_0)]\phi,$$

where the commutator

$$[|\partial|^{\alpha},(\mathcal{K}-K_0)]\phi=|\partial|^{\alpha}(\mathcal{K}-K_0)\phi-(\mathcal{K}-K_0)|\partial|^{\alpha}\phi$$

$$= C\sqrt{t}\int_{\mathbf{R}}\int_{\mathbf{R}}(K((\xi-y)\sqrt{t},\eta\sqrt{t})-K(\xi\sqrt{t},\eta\sqrt{t}))$$
$$\times(\phi(\xi-y-\eta)-\phi(\xi-y))d\eta\frac{dy}{|y|^{1+\alpha}}.$$

By the condition of the lemma we have

$$|K((\xi-y)\sqrt{t},\eta\sqrt{t})-K(\xi\sqrt{t},\eta\sqrt{t})|$$
$$\leq \int_{\xi-y}^{\xi}|\partial_\xi K((\xi-\zeta)\sqrt{t},\eta\sqrt{t})|d\zeta \leq \frac{C|y\sqrt{t}|}{\langle\sqrt{t}\eta\rangle^3}$$

for $|y|\sqrt{t}\leq 1$. For $|y|\sqrt{t}\geq 1$

$$|K((\xi-y)\sqrt{t},\eta\sqrt{t})-K(\xi\sqrt{t},\eta\sqrt{t})|$$
$$\leq C\widetilde{B}\langle\widetilde{B}\eta\sqrt{t}\rangle^{-3}+CB\langle B\eta\sqrt{t}\rangle^{-3}\leq\frac{C}{|\eta\sqrt{t}|\langle\sqrt{t}\eta\rangle^2},$$

where $B=\langle\xi\sqrt{t}\rangle$, $\widetilde{B}=\langle(\xi-y)\sqrt{t}\rangle$. Hence we obtain

$$|K((\xi-y)\sqrt{t},\eta\sqrt{t})-K(\xi\sqrt{t},\eta\sqrt{t})|\leq\frac{C|y\sqrt{t}|}{\langle\sqrt{t}y\rangle|\eta\sqrt{t}|\langle\sqrt{t}\eta\rangle^2}$$

for all $y\in\mathbf{R}$. Therefore, we get

$$\|[|\partial|^\alpha,(\mathcal{K}-K_0)]\phi\|_{\mathbf{L}^2}\leq C\sqrt{t}\|\int_{\mathbf{R}}\int_{\mathbf{R}}\frac{|\phi(\xi-y-\eta)-\phi(\xi-y)||y\sqrt{t}|}{|\eta\sqrt{t}|\langle\eta\sqrt{t}\rangle^2\langle y\sqrt{t}\rangle|y|^{1+\alpha}}d\eta dy\|_{\mathbf{L}^2}$$
$$\leq C\sqrt{t}\int_{\mathbf{R}}\frac{\|\phi(\cdot-\eta)-\phi(\cdot)\|_{\mathbf{L}^2}}{|\eta\sqrt{t}|\langle\eta\sqrt{t}\rangle^2}d\eta\int_{\mathbf{R}}\frac{|y\sqrt{t}|dy}{\langle y\sqrt{t}\rangle|y|^{1+\alpha}}$$
$$\leq C\||\partial|^\alpha\phi\|_{\mathbf{L}^2}\int_{\mathbf{R}}\frac{t^{\frac{\alpha+1}{2}}|\eta|^\alpha d\eta}{|\eta\sqrt{t}|\langle\eta\sqrt{t}\rangle^2}\leq C\||\partial|^\alpha\phi\|_{\mathbf{L}^2}. \tag{2.6}$$

Thus by (2.5) and (2.6) the estimate of the lemma follows. Lemma 2.3 is proved. □

Remark 3. Integration by parts yields

$$\mathcal{K}\partial_\xi\phi=t\int_{\mathbf{R}}K_y(\xi\sqrt{t},\eta\sqrt{t})\phi(\xi-\eta)d\eta,$$

where we denote the kernel $K_y(x,y)=\partial_yK(x,y)$. Suppose that the kernel $\langle x\rangle^{-1}K_y(x,y)$ satisfies the conditions of Lemma 2.3. Since

$$\int_{\mathbf{R}}K_y(\xi\sqrt{t},y)dy\equiv 0,$$

by Lemma 2.3 we get the following estimate:

$$\|(B\sqrt{t})^{\alpha-1}\mathcal{K}\partial_\xi\phi\|_{\mathbf{L}^2} \le C\||\partial|^\alpha\phi\|_{\mathbf{L}^2}. \tag{2.7}$$

3. Estimates for the derivative $|\partial|^\alpha$

First we state the local existence result (see [1]).

Theorem 3.1. *Assume that the initial data* $u_0 \in \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha}$, $\alpha \in (\frac{1}{2}, 1)$. *Then for some time* $T > 1$ *there exists a unique solution* $u \in \mathbf{C}([0,T]; \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha})$ *of the Cauchy problem* (1.1).

In the same way as in paper [9] we change the variables

$$u(t,x) = (it)^{-\frac{1}{2}}Ev(t,\xi), \quad E = e^{\frac{it}{2}\xi^2}, \quad \text{and} \quad \xi = \frac{x}{t}$$

in equation (1.1) to get

$$\begin{cases} \mathcal{L}v = -it^{-1}E^2v^3, \ t \ge 1, \ \xi \in \mathbf{R}, \\ v(1,\xi) = v_1(\xi), \ \xi \in \mathbf{R}, \end{cases} \tag{3.1}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2t^2}\partial_\xi^2$. Denote

$$Q(t) = (1 + \delta^2\log(1+t))^{-\frac{1}{2}}, P(t) = t^\gamma + \varepsilon^2 t^{\frac{\alpha}{2}-\frac{1}{4}}Q^3(t),$$

where $\delta > 0$, $\gamma \in (0,1)$ are small as above.

Lemma 3.2. *Let* $v_1 \in \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha}$, $\alpha \in (\frac{1}{2}, 1)$, *and the norm* $\|v_1\|_{\mathbf{H}^\alpha\cap\mathbf{H}^{0,\alpha}} \le \varepsilon$, *where* $\varepsilon > 0$ *is sufficiently small. Also we suppose that*

$$\|v(t)\|_{\mathbf{L}^\infty} \le C\varepsilon, \ \|B^{-\gamma}v(t)\|_{\mathbf{L}^\infty} \le C\varepsilon Q(t) \tag{3.2}$$

for all $t \in [1,T]$. *Then the solutions* $v \in \mathbf{C}([1,T]; \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha})$ *of (3.1) satisfy the estimate*

$$\||\partial|^\alpha v(t)\|_{\mathbf{L}^2} < C\varepsilon P(t) \ \ \textit{for all } t \in [1,T]. \tag{3.3}$$

Proof. We prove estimate (3.3) by contradiction. By the continuity of v we can find a maximal time $\widetilde{T} \in (1,T]$ such that

$$\||\partial|^\alpha v(t)\|_{\mathbf{L}^2} \le C\varepsilon P(t) \tag{3.4}$$

for all $t \in [1,\widetilde{T}]$. We represent equation (3.1) in the form

$$\mathcal{L}v = -it^{-1}E^2(\mathcal{K}v)^3 + R_1, \tag{3.5}$$

where the regularizing operator

$$\mathcal{K}\phi = \sqrt{t}\int_{\mathbf{R}} K(\xi\sqrt{t}, \eta\sqrt{t})\phi(\xi-\eta)d\eta$$

with a kernel $K(x,y)=\frac{\langle x\rangle}{\sqrt{\pi}}e^{-\langle x\rangle^2y^2}$ and the remainder term

$$R_1 = -it^{-1}E^2(v^3 - (\mathcal{K}v)^3).$$

To eliminate the first summand in the right-hand side of (3.5) we use the identity

$$\mathcal{L}(E^2\phi) = \frac{i}{tA}E^2\phi + 2it^{-1}\xi E^2\phi_\xi + E^2\mathcal{L}\phi, \tag{3.6}$$

where $A =(1+3it\xi^2)^{-1}$. Then in view of (3.6) we get from (3.5)

$$\mathcal{L}(v + E^2\phi) = \frac{i}{tA}E^2(\phi - A(\mathcal{K}v)^3) + 2it^{-1}E^2\xi\phi_\xi + E^2\mathcal{L}\phi + R_1. \tag{3.7}$$

We choose $\phi = A(\mathcal{K}v)^3$ to eliminate the first summand in the right-hand side of (3.7). By the identity

$$\mathcal{L}(\psi\chi) = \chi\mathcal{L}\psi + \frac{1}{t^2}\psi_\xi\chi_\xi + \psi\mathcal{L}\chi,$$

in view of equation (3.1) we obtain

$$\begin{aligned}\mathcal{L}(A(\mathcal{K}v)^3) \quad &= \quad (\mathcal{K}v)^3\mathcal{L}A + \frac{1}{t^2}A_\xi\partial_\xi(\mathcal{K}v)^3 \\ &\quad +3A(\mathcal{K}v)^2\mathcal{L}(\mathcal{K}v) + 3At^{-2}(\mathcal{K}v)(\partial_\xi(\mathcal{K}v))^2. \end{aligned} \tag{3.8}$$

Therefore, by virtue of (3.8) we get from (3.7)

$$\mathcal{L}(v + AE^2(\mathcal{K}v)^3) = t^{-1}E^2a(\mathcal{K}v)^3 + \sum_{j=1}^{4} R_j, \tag{3.9}$$

where the coefficient $a = t\mathcal{L}A + 2i\xi A_\xi = -5iA + 14iA^2 - 12iA^3$, the remainder terms $R_2 = t^{-\frac{3}{2}}E^2b(\mathcal{K}v)^2\partial_\xi\mathcal{K}v$, $R_3 = 3At^{-2}E^2(\mathcal{K}v)(\partial_\xi\mathcal{K}v)^2$, $R_4 = 3AE^2(\mathcal{K}v)^2\mathcal{L}(\mathcal{K}v)$, and $b = 3t^{-\frac{1}{2}}A_\xi - 6i\sqrt{t}\xi A = 6i\sqrt{t}\xi A(1-3A)$. We now apply the derivative $|\partial|^\alpha$ to equation (3.9)

$$\mathcal{L}(|\partial|^\alpha v + |\partial|^\alpha(AE^2(\mathcal{K}v)^3)) = t^{-1}|\partial|^\alpha(E^2a(\mathcal{K}v)^3) + \sum_{j=1}^{4}|\partial|^\alpha R_j. \tag{3.10}$$

We now estimate the summands in equation (3.10). By estimate (2.2) we find

$$\||\partial|^\alpha R_1\|_{\mathbf{L}^2} \le Ct^{-1}\||\partial|^\alpha(v^3 - (\mathcal{K}v)^3)\|_{\mathbf{L}^2} + Ct^{-1}\|(B\sqrt{t})^\alpha(v^3 - (\mathcal{K}v)^3)\|_{\mathbf{L}^2}.$$

Denote $w = v - \mathcal{K}v$, then

$$v^3 - (\mathcal{K}v)^3 = (w^2 + 3vw + 3v^2)w.$$

Since $K_0 = \int_{\mathbf{R}} K(\xi\sqrt{t}, y)dy = 1$, by Lemma 2.3 we find

$$\||\partial|^{\alpha} w\|_{\mathbf{L}^2} + \|(B\sqrt{t})^{\alpha} w\|_{\mathbf{L}^2} \le C\varepsilon P(t),$$
$$\||\partial|^{\alpha} \mathcal{K}v\|_{\mathbf{L}^2} \le \||\partial|^{\alpha} w\|_{\mathbf{L}^2} + \||\partial|^{\alpha} v\|_{\mathbf{L}^2} \le C\varepsilon P(t).$$

Then we get

$$\|(B\sqrt{t})^{\alpha}(v^3 - (\mathcal{K}v)^3)\|_{\mathbf{L}^2} \le C(\|v\|^2_{\mathbf{L}^\infty} + \|w\|^2_{\mathbf{L}^\infty})\|(B\sqrt{t})^{\alpha} w\|_{\mathbf{L}^2} \le C\varepsilon^3 P(t).$$

Then by Lemma 2.2 (see Remark 2) we find $\||\partial|^{\alpha}(vw)\|_{\mathbf{L}^2} \le C\varepsilon^2 P(t)$, and then

$$\begin{aligned} &\||\partial|^{\alpha}(v^3 - (\mathcal{K}v)^3)\|_{\mathbf{L}^2} \\ \le\ & 3\||\partial|^{\alpha}(v^2 w)\|_{\mathbf{L}^2} + 3\||\partial|^{\alpha}(v w^2)\|_{\mathbf{L}^2} + \||\partial|^{\alpha}(w^3)\|_{\mathbf{L}^2} \le C\varepsilon^3 P(t). \end{aligned}$$

Therefore we find the estimate for the remainder term

$$\||\partial|^{\alpha} R_1\|_{\mathbf{L}^2} \le C\varepsilon^3 t^{-1} P(t). \tag{3.11}$$

Next let us estimate

$$|\partial|^{\alpha}(AE^2(\mathcal{K}v)^3) = |\partial|^{\alpha}(E^2 A v^3) - |\partial|^{\alpha}(E^2 A(v^3 - (\mathcal{K}v)^3)).$$

For the first summand we apply (2.2) and (2.3)

$$\||\partial|^{\alpha}(E^2 A v^3)\|_{\mathbf{L}^2} \le C\||\partial|^{\alpha}(B^{-\frac{2}{3}} v)^3\|_{\mathbf{L}^2} + Ct^{\frac{\alpha}{2}}\|B^{\alpha-2} v^3\|_{\mathbf{L}^2} \le C\varepsilon P(t).$$

The second term is estimated as above. Hence we get

$$\||\partial|^{\alpha}(AE^2(\mathcal{K}v)^3)\|_{\mathbf{L}^2} \le C\varepsilon P(t). \tag{3.12}$$

Similarly we write

$$t^{-1}|\partial|^{\alpha}(E^2 a(\mathcal{K}v)^3) = t^{-1}|\partial|^{\alpha}(E^2 a v^3) - t^{-1}|\partial|^{\alpha}(E^2 a(v^3 - (\mathcal{K}v)^3)).$$

Since $|a| \le CB^{-2}$, $|\partial_\xi a| \le C\sqrt{t}B^{-3}$, then as above we obtain

$$t^{-1}\||\partial|^{\alpha}(E^2 a(\mathcal{K}v)^3)\|_{\mathbf{L}^2} \le C\varepsilon t^{-1} P(t). \tag{3.13}$$

Next we estimate the remainder term

$$|\partial|^{\alpha} R_2 = t^{-\frac{3}{2}}|\partial|^{\alpha} E^2 b(\mathcal{K}v)^2 \partial_\xi \mathcal{K}v.$$

By estimate (2.7) we have

$$\|B^{\alpha-1}\mathcal{K}\partial_\xi v\|_{\mathbf{L}^2} \le Ct^{\frac{1-\alpha}{2}}\||\partial|^{\alpha} v\|_{\mathbf{L}^2}.$$

Also we can use the identity

$$\partial_\xi \mathcal{K}v = \mathcal{K}\partial_\xi v + t\xi B^{-2}\mathcal{K}_1 v = t\xi B^{-2}\mathcal{K}_1 v + B\sqrt{t}(\mathcal{K}_2 - K_{2,0})v, \tag{3.14}$$

where we denote the operators

$$\mathcal{K}_j\phi = \sqrt{t}\int_{\mathbf{R}} K_j(\xi\sqrt{t}, \eta\sqrt{t})\phi(\xi - \eta)d\eta,$$

with kernels

$$K_1(x,y) = \frac{\langle x\rangle}{\sqrt{\pi}}(1-2\langle x\rangle^2 y^2)e^{-\langle x\rangle^2 y^2}, K_2(x,y) = -\frac{2}{\sqrt{\pi}}\langle x\rangle^2 y e^{-\langle x\rangle^2 y^2},$$

which satisfy the conditions of Lemmas 2.1 and 2.3 and also

$$K_{2,0} = \int_{\mathbf{R}} K_2(x,y)dy = 0.$$

Note that $|b| \le CB^{-1}$ and $|\partial_\xi b| \le C\sqrt{t}B^{-2}$. Then application of Lemmas 2.2 and 2.3 yields

$$\begin{aligned} &\| |\partial|^\alpha R_2\|_{\mathbf{L}^2} \le Ct^{-1}\| |\partial|^\alpha ((\mathcal{K}v)^2(\mathcal{K}_2 - K_{2,0})v)\|_{\mathbf{L}^2} \\ &+Ct^{\frac{\alpha}{2}-1}\|B^\alpha(\mathcal{K}v)^2(\mathcal{K}_2 - K_{2,0})v\|_{\mathbf{L}^2} \\ &+Ct^{-1}\| |\partial|^\alpha (B^{-2}(\mathcal{K}v)^2\mathcal{K}_1 v)\|_{\mathbf{L}^2} + Ct^{\frac{\alpha}{2}-1}\|B^{\alpha-2}(\mathcal{K}v)^2\mathcal{K}_1 v\|_{\mathbf{L}^2} \\ \le\ & C\varepsilon t^{-1}P(t). \end{aligned} \tag{3.15}$$

In the same manner we estimate the remainder term $|\partial|^\alpha R_3$:

$$\begin{aligned} &\| |\partial|^\alpha R_3\|_{\mathbf{L}^2} \le Ct^{-1}\| |\partial|^\alpha (\mathcal{K}v)((\mathcal{K}_2 - K_{2,0})v)^2\|_{\mathbf{L}^2} \\ &+Ct^{\frac{\alpha}{2}-1}\|B^\alpha(\mathcal{K}v)((\mathcal{K}_2 - K_{2,0})v)^2\|_{\mathbf{L}^2} \\ &+Ct^{-1}\| |\partial|^\alpha (B^{-2}(\mathcal{K}v)(\mathcal{K}_1 v)^2)\|_{\mathbf{L}^2} + Ct^{\frac{\alpha}{2}-1}\|B^{\alpha-2}(\mathcal{K}v)(\mathcal{K}_1 v)^2\|_{\mathbf{L}^2} \\ \le\ & C\varepsilon t^{-1}P(t). \end{aligned} \tag{3.16}$$

Finally we consider $|\partial|^\alpha R_4 = 3i|\partial|^\alpha E^2 A(\mathcal{K}v)^2\mathcal{L}(\mathcal{K}v)$. Since by equation (3.1) $\mathcal{L}(\mathcal{K}v) = t^{-1}\mathcal{K}_3 v + \mathcal{K}\mathcal{L}v = t^{-1}\mathcal{K}_3 v - it^{-1}\mathcal{K}(E^2v^3) = t^{-1}\mathcal{K}_3 v + t^{-1}E^2\mathcal{K}_4(v^3)$ where

$$\mathcal{K}_j\phi = \sqrt{t}\int_{\mathbf{R}} K_j(\xi\sqrt{t}, \eta\sqrt{t})\phi(\xi-\eta)d\eta$$

for $j = 3, 4$, with kernels

$$K_3(\xi\sqrt{t}, \eta\sqrt{t}) = \frac{1}{\sqrt{\pi}}(\sqrt{t}\mathcal{L}(\sqrt{t}Be^{-tB^2\eta^2}) + t^{-1}\partial_\eta\partial_\xi(Be^{-tB^2\eta^2}))$$

and

$$K_4(x,y) = \frac{1}{\sqrt{\pi}}\langle x\rangle e^{-\langle x\rangle^2y^2 + iy^2 - 2ixy},$$

which satisfy the conditions of Lemmas 2.1 and 2.3. Then as above we get

$$\begin{aligned} &\| |\partial|^\alpha R_4\|_{\mathbf{L}^2} \le t^{-1}\| |\partial|^\alpha E^2 A(\mathcal{K}v)^2\mathcal{K}_3 v\|_{\mathbf{L}^2} + t^{-1}\| |\partial|^\alpha E^4 A(\mathcal{K}v)^2\mathcal{K}_4(v^3)\|_{\mathbf{L}^2} \\ &\le Ct^{-1}\| |\partial|^\alpha (B^{-2}(\mathcal{K}v)^2\mathcal{K}_3 v)\|_{\mathbf{L}^2} + Ct^{-1}\| |\partial|^\alpha (B^{-2}(\mathcal{K}v)^2\mathcal{K}_4 v)\|_{\mathbf{L}^2} \\ &+ Ct^{\frac{\alpha}{2}-1}\|B^{\alpha-2}(\mathcal{K}v)^2\mathcal{K}_3 v\|_{\mathbf{L}^2} + Ct^{\frac{\alpha}{2}-1}\|B^{\alpha-2}(\mathcal{K}v)^2\mathcal{K}_4 v\|_{\mathbf{L}^2} \end{aligned}$$

$$\leq C\varepsilon t^{-1}P(t). \tag{3.17}$$

In view of estimates (3.11), (3.13)-(3.17) we get from (3.10)

$$\frac{d}{dt}\||\partial|^{\alpha}v + |\partial|^{\alpha}(AE^2(\mathcal{K}v)^3)\|_{\mathbf{L}^2} < C\varepsilon t^{-1}P(t).$$

Hence the integration with respect to time yields

$$\begin{aligned}&\||\partial|^{\alpha}v + |\partial|^{\alpha}(AE^2(\mathcal{K}v)^3)\|_{\mathbf{L}^2}\\ \leq\ & C\varepsilon + C\varepsilon^3 t^{\frac{\alpha}{2}-\frac{1}{4}}Q^3 + C\varepsilon^3\int_1^t \tau^{\frac{\alpha}{2}-\frac{5}{4}}Q^3(\tau)d\tau < C\varepsilon P(t)\end{aligned}$$

for all $t \in [1, \widetilde{T}]$. Therefore by estimate (3.12) we find $\||\partial|^{\alpha}v\|_{\mathbf{L}^2} < C\varepsilon P(t)$ for all $t \in [1, \widetilde{T}]$. Thus we obtain a contradiction. Therefore estimate (3.3) is true for all $t \in [1, T]$. Lemma 3.2 is proved. □

4. Estimates for the uniform norm

First we obtain a representation for the nonlinear term $\mathcal{V}(-t)(E^2 a(\mathcal{K}v)^3)$, where $a = -5iA + 14iA^2 - 12iA^3$, $A = (1 + 3it\xi^2)^{-1}$ and estimate the remainder terms R_j, $j = 1, 2, 3, 4$, from equation (3.9), where, as in the proof of Lemma 3.2,

$$\begin{aligned}R_1 &= -it^{-1}E^2(v^3 - (\mathcal{K}v)^3), R_2 = t^{-\frac{3}{2}}E^2 b(\mathcal{K}v)^2\partial_{\xi}\mathcal{K}v,\\ R_3 &= 3At^{-2}E^2(\mathcal{K}v)(\partial_{\xi}\mathcal{K}v)^2, R_4 = 3AE^2(\mathcal{K}v)^2\mathcal{L}(\mathcal{K}v)\end{aligned}$$

and $b = 6i\sqrt{t}\xi A(1 - 3A)$. Define the norm

$$\|v\|_{\mathbf{X}_T} = \sup_{t\in[1,T]}(\|v(t)\|_{\mathbf{L}^{\infty}} + Q^{-1}(t)\|B^{-\gamma}v(t)\|_{\mathbf{L}^{\infty}} + P^{-1}(t)\||\partial|^{\alpha}v(t)\|_{\mathbf{L}^2}),$$

where $Q(t) = (1 + \delta^2\log(1 + t))^{-\frac{1}{2}}$, $P(t) = t^{\gamma} + \varepsilon^2 t^{\frac{\alpha}{2}-\frac{1}{4}}Q^3(t)$, where $\delta > 0$, $\gamma \in (0, 1)$ are small.

Lemma 4.1. *Suppose that $\|v\|_{\mathbf{X}_T} \leq C\varepsilon$. Then*

$$|\mathcal{V}(-t)R_j(t)| \leq C\varepsilon^3 t^{-\frac{3}{4}-\frac{\alpha}{2}}Q^{2\gamma}(t)P(t)$$

for all $t \in [1, T]$, $j = 1, 2, 3, 4$. Moreover the following representation is valid:

$$\mathcal{V}(-t)(E^2 a(\mathcal{K}v)^3) = -\frac{i}{\sqrt{3}B}\varphi^3 + O(\varepsilon^3 t^{\frac{1}{4}-\frac{\alpha}{2}}((|\xi|\sqrt{t})^{\gamma}B^{-1} + Q^{2\gamma}(t))P(t))$$

for all $t \in [1, T]$, where $\varphi = \mathcal{V}(-t)v$.

Proof. We first compute the commutator of $\mathcal{V}(-t)$ and $E^{\rho-1}$ (see [3]):

$$\mathcal{V}(-t)E^{\rho-1}\phi = \sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{\frac{it}{2}(\eta-\xi)^2}e^{\frac{it}{2}(\rho-1)\eta^2}\phi(\eta)d\eta$$
$$= E^{(1-\frac{1}{\rho})}\frac{1}{\rho}\sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{\frac{it}{2\rho}(\xi-\zeta)^2}\phi(\frac{\zeta}{\rho})d\zeta = E^{(1-\frac{1}{\rho})}\mathcal{V}(-\frac{t}{\rho})\sqrt{i}\mathcal{D}(\rho)\phi \qquad (4.1)$$

for $\rho > 0$. Hence by the Sobolev imbedding inequality

$$\|\phi\|_{\mathbf{L}^\infty} \le C\||\partial|^\alpha\phi\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|\phi\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}}$$

for $\alpha \in(\frac{1}{2},1)$ we get

$$|\mathcal{V}(-t)R_1| = t^{-1}|\mathcal{V}(-t)E^2(v^3-(\mathcal{K}v)^3)| = t^{-1}|\mathcal{V}(-\frac{t}{3})\mathcal{D}(3)(v^3-(\mathcal{K}v)^3)|$$
$$\le Ct^{-1}\||\partial|^\alpha(v^3-(\mathcal{K}v)^3)\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|v^3-(\mathcal{K}v)^3\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}}.$$

As in the proof of Lemma 3.2 we have $\||\partial|^\alpha(v^3-(\mathcal{K}v)^3)\|_{\mathbf{L}^2} \le C\varepsilon^3P(t)$ and by Lemma 2.3 we find

$$\begin{aligned}\|v^3-(\mathcal{K}v)^3\|_{\mathbf{L}^2} &\le C(\|B^{-\frac{\alpha}{2}}v\|_{\mathbf{L}^\infty}^2 + \|B^{-\frac{\alpha}{2}}\mathcal{K}v\|_{\mathbf{L}^\infty}^2)\|B^\alpha(v-\mathcal{K}v)\|_{\mathbf{L}^2}\\ &\le C\varepsilon^3t^{-\frac{\alpha}{2}}Q^2(t)P(t).\end{aligned}$$

Hence,

$$|\mathcal{V}(-t)R_1| \le C\varepsilon^3t^{-1}(P(t))^{\frac{1}{2\alpha}}(t^{-\frac{\alpha}{2}}Q^2(t)P(t))^{1-\frac{1}{2\alpha}} \le C\varepsilon^3t^{-\frac{3}{4}-\frac{\alpha}{2}}Q^{2\gamma}(t)P(t).$$

Similarly, as in estimate (3.15) and by (2.7) we find

$$\begin{aligned}|\mathcal{V}(-t)R_2| &= t^{-\frac{3}{2}}|\mathcal{V}(-\frac{t}{3})\mathcal{D}(3)b(\mathcal{K}v)^2\partial_\xi\mathcal{K}v|\\ &\le Ct^{-\frac{3}{2}}\||\partial|^\alpha(b(\mathcal{K}v)^2\partial_\xi\mathcal{K}v)\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|b(\mathcal{K}v)^2\partial_\xi\mathcal{K}v\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}}\\ &\le C\varepsilon^{\frac{3}{2\alpha}}t^{-\frac{3}{2}+\frac{1}{4\alpha}}P^{\frac{1}{2\alpha}}(t)\|B^{-\gamma}v(t)\|_{\mathbf{L}^\infty}^{2-\frac{1}{\alpha}}\|\partial_\xi\mathcal{K}v\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}}\\ &\le C\varepsilon^3t^{-\frac{3}{4}-\frac{\alpha}{2}}Q^{2\gamma}(t)P(t).\end{aligned}$$

As in estimate (3.16) we get

$$|\mathcal{V}(-t)R_3| = t^{-2}|\mathcal{V}(-\frac{t}{3})\mathcal{D}(3)A(\mathcal{K}v)(\partial_\xi\mathcal{K}v)^2|$$
$$\begin{aligned}&\le Ct^{-2}\||\partial|^\alpha(A(\mathcal{K}v)(\partial_\xi\mathcal{K}v)^2)\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|A(\mathcal{K}v)(\partial_\xi\mathcal{K}v)^2\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}}\\ &\le C\varepsilon^3t^{-\frac{3}{4}-\frac{\alpha}{2}}Q^{2\gamma}(t)P(t).\end{aligned}$$

Finally as in the proof of estimate (3.17) we have

$$\begin{aligned}|\mathcal{V}(-t)R_4| &\le Ct^{-1}|\mathcal{V}(-\frac{t}{3})\mathcal{D}(3)A(\mathcal{K}v)^2(\mathcal{K}_3v)| \\ &+ Ct^{-1}|\mathcal{V}(-\frac{t}{5})\mathcal{D}(5)A(\mathcal{K}v)^2(\mathcal{K}_4(v^3))| \\ &\le Ct^{-1}\||\partial|^\alpha(A(\mathcal{K}v)^2(\mathcal{K}_3v))\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|A(\mathcal{K}v)^2(\mathcal{K}_3v)\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}} \\ &+ Ct^{-1}\||\partial|^\alpha(A(\mathcal{K}v)^2(\mathcal{K}_4(v^3)))\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|A(\mathcal{K}v)^2(\mathcal{K}_4(v^3))\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}} \\ &\le C\varepsilon^3 t^{-\frac{3}{4}-\frac{\alpha}{2}}Q^{2\gamma}(t)P(t).\end{aligned}$$

Thus the first estimate of the lemma is true. By (4.1) we have

$$\mathcal{V}(-t)(E^2a(\mathcal{K}v)^3) = E^{\frac{2}{3}}\sqrt{i}\mathcal{V}(-\frac{t}{3})\mathcal{D}(3)a(\mathcal{K}v)^3 = E^{\frac{2}{3}}\sqrt{i}\mathcal{V}(-\frac{t}{3})a_1\mathcal{D}(3)(\mathcal{K}v)^3,$$

where $a_1 = \sqrt{3i}\mathcal{D}(3)a = -5iA_1 + 14iA_1^2 - 12iA_1^3$, $A_1 = (1+\frac{it}{3}\xi^2)^{-1}$. Hence, we can write the representation

$$\mathcal{V}(-t)(E^2a(\mathcal{K}v)^3) = -\frac{i}{\sqrt{3}B}\varphi^3 + R_5 + R_6 + R_7,$$

where the remainder terms

$$\begin{aligned}R_5 &= E^{\frac{2}{3}}\sqrt{i}\mathcal{D}(3)\varphi^3\mathcal{V}(-\frac{t}{3})a_1 + \frac{i}{\sqrt{3}B}\varphi^3, \\ R_6 &= E^{\frac{2}{3}}\sqrt{i}\mathcal{D}(3)((\mathcal{K}v)^3-\varphi^3)\mathcal{V}(-\frac{t}{3})a_1, \\ R_7 &= E^{\frac{2}{3}}\sqrt{i}\mathcal{V}(-\frac{t}{3})a_1\mathcal{D}(3)(\mathcal{K}v)^3 - E^{\frac{2}{3}}\sqrt{i}\mathcal{D}(3)(\mathcal{K}v)^3\mathcal{V}(-\frac{t}{3})a_1.\end{aligned}$$

Denote

$$\Lambda = \sqrt{\frac{t}{6\pi i}}\int_{\mathbf{R}} e^{\frac{it}{6}\eta^2}a_1 d\eta.$$

To compute explicitly Λ we change $\sqrt{\frac{t}{3}}\eta = z$, then we obtain

$$\Lambda = \frac{1}{\sqrt{2\pi i}}\int_{\mathbf{R}} e^{\frac{i}{2}z^2}a_2(z)dz,$$

where $a_2 = -5iA_2 + 14iA_2^2 - 12iA_2^3$, $A_2 = (1+iz^2)^{-1}$. Then we can extract the full derivative in $e^{\frac{i}{2}z^2}a_2(z)$, by the identity

$$e^{\frac{i}{2}z^2}a_2(z) = ie^{\frac{i}{2}z^2}(-\frac{5}{1+iz^2} + \frac{14}{(iz^2+1)^2} - \frac{12}{(1+iz^2)^3})$$

$$= \quad -i\frac{d}{dz}(ze^{\frac{i}{2}z^2}(-\frac{1}{1+iz^2}+\frac{3}{(1+iz^2)^2})) - ie^{\frac{i}{2}z^2}.$$

Hence we find

$$\Lambda = -\frac{i}{\sqrt{2\pi i}}\int_{\mathbf{R}} e^{\frac{i}{2}z^2}dz = -i.$$

Next we estimate the difference $\mathcal{V}(-\frac{t}{3})a_1 - \Lambda B^{-1}$. Using the identity $e^{\frac{it}{6}\eta^2} = G\partial_\eta(\eta e^{\frac{it}{6}\eta^2})$ with $G = (1+\frac{it}{3}\eta^2)^{-1}$, we integrate by parts

$$\mathcal{V}(-\frac{t}{3})a_1 - \Lambda B^{-1} = B^{-1}\sqrt{\frac{t}{6\pi i}}\int_{\mathbf{R}} e^{\frac{it}{6}\eta^2}(B\widetilde{a_1} - a_1)d\eta$$

$$= \quad -B^{-1}\sqrt{\frac{t}{6\pi i}}\int_{\mathbf{R}} e^{\frac{it}{6}\eta^2}\eta\partial_\eta(G(B\widetilde{a_1} - a_1))d\eta,$$

where $\widetilde{a_1} = a_1(\eta - \xi)$. We have the estimate for $|\xi|\sqrt{t} \leq 1$

$$|\eta\partial_\eta(G(B\widetilde{a_1} - a_1))| \leq C|\xi|\sqrt{t}\langle\eta\sqrt{t}\rangle^{-2}$$

and for $|\xi|\sqrt{t} \geq 1$

$$\begin{aligned} |\eta\partial_\eta(G(B\widetilde{a_1} - a_1))| &\leq C\langle\eta\sqrt{t}\rangle^{-1}\langle\xi\sqrt{t}\rangle\langle(\eta-\xi)\sqrt{t}\rangle^{-3} + C\langle\eta\sqrt{t}\rangle^{-4} \\ &\leq C\langle(\eta-\xi)\sqrt{t}\rangle^{-2} + C\langle\eta\sqrt{t}\rangle^{-2}. \end{aligned}$$

Hence

$$|\eta\partial_\eta(G(B\widetilde{a_1} - a_1))| \leq \frac{|\xi|\sqrt{t}}{\langle\xi\sqrt{t}\rangle}(\langle\eta\sqrt{t}\rangle^{-2} + \langle(\eta-\xi)\sqrt{t}\rangle^{-2})$$

for all $\xi \in \mathbf{R}$. Therefore, we get

$$\begin{aligned} |\mathcal{V}(-\frac{t}{3})a_1 + iB^{-1}| &\leq CB^{-1}\frac{|\xi|\sqrt{t}}{\langle\xi\sqrt{t}\rangle}\int_{\mathbf{R}}(\langle\eta\sqrt{t}\rangle^{-2} + \langle(\eta-\xi)\sqrt{t}\rangle^{-2})\sqrt{t}d\eta \\ &\leq C|\xi|\sqrt{t}B^{-2}. \end{aligned}$$

By the Sobolev inequality we have

$$\|\phi(\xi) - \phi(\rho\xi)\|_{\mathbf{L}^\infty} \leq C\||\partial|^\alpha\phi\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|\phi(\xi) - \phi(\rho\xi)\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}} \leq C|\xi|^{\alpha-\frac{1}{2}}\||\partial|^\alpha\phi\|_{\mathbf{L}^2},$$

therefore,

$$\begin{aligned} |R_5| &\leq |\mathcal{D}(3)\varphi^3(\mathcal{V}(-\frac{t}{3})a_1 + iB^{-1})| + |B^{-1}(E^{\frac{2}{3}}\sqrt{i}\mathcal{D}(3)\varphi^3 - \frac{1}{\sqrt{3}}\varphi^3)| \\ &\leq C\varepsilon^3|\xi|\sqrt{t}B^{3\gamma-2}t^{\frac{1}{4}-\frac{\alpha}{2}}P(t) + C\varepsilon^3|\xi|^{\alpha-\frac{1}{2}}B^{-1}P(t) \\ &\leq C\varepsilon^3(|\xi|\sqrt{t})^{\alpha-\frac{1}{2}}B^{-1}t^{\frac{1}{4}-\frac{\alpha}{2}}P(t). \end{aligned}$$

By the Plancherel theorem we have

$$\begin{aligned}\|(\mathcal{V}(t)-1)\phi\|_{\mathbf{L}^2} &= \|\mathcal{F}(M(t)-1)\mathcal{F}^{-1}\phi\|_{\mathbf{L}^2} \\ &\le Ct^{-\frac{\alpha}{2}}\||x|^{\alpha}\mathcal{F}^{-1}\phi\|_{\mathbf{L}^2} \le Ct^{-\frac{\alpha}{2}}\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}.\end{aligned}$$

Hence we find

$$\begin{aligned}\|(\mathcal{V}(t)-1)\phi\|_{\mathbf{L}^\infty} &\le C\||\partial|^{\alpha}(\mathcal{V}(t)-1)\phi\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|(\mathcal{V}(t)-1)\phi\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}} \\ &\le Ct^{\frac{1}{4}-\frac{\alpha}{2}}\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}.\end{aligned}$$

Since $\|v\|_{\mathbf{X}_T} \le C\varepsilon$ we find

$$\begin{aligned}\|R_6\|_{\mathbf{L}^\infty} &\le C(\|B^{-\gamma}\mathcal{K}v\|_{\mathbf{L}^\infty}^2 + \|B^{-\gamma}\varphi\|_{\mathbf{L}^\infty}^2)\|\mathcal{K}v-\varphi\|_{\mathbf{L}^\infty} \\ &\le C\varepsilon^2 Q^2(t)\||\partial|^{\alpha}(\mathcal{K}v-\varphi)\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|\mathcal{K}v-\varphi\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}} \\ &\le C\varepsilon^3 t^{\frac{1}{4}-\frac{\alpha}{2}}Q^{2\gamma}(t)P(t)\end{aligned}$$

for $t \in [1,T]$. Finally to estimate the remainder R_7 we need to consider a commutator $\mathcal{V}(-\frac{t}{3})\phi^3 a_1 - \phi^3\mathcal{V}(-\frac{t}{3})a_1$. Again integrating by parts via the identity $e^{\frac{it}{6}\eta^2} = G\partial_\eta(\eta e^{\frac{it}{6}\eta^2})$ with $G = (1+\frac{it}{3}\eta^2)^{-1}$, we find

$$\begin{aligned}&\mathcal{V}(-\frac{t}{3})\phi^3 a_1 - \phi^3\mathcal{V}(-\frac{t}{3})a_1 \\ = &-3\sqrt{\frac{t}{6\pi i}}\int_{\mathbf{R}} e^{\frac{it}{6}\eta^2}\phi^2(\xi-\eta)\eta G a_1(\sqrt{t}(\xi-\eta))\partial_\eta\phi(\xi-\eta)d\eta \\ &-\sqrt{\frac{t}{6\pi i}}\int_{\mathbf{R}} e^{\frac{it}{6}\eta^2}(\phi^3(\xi-\eta)-\phi^3(\xi))\eta\partial_\eta(G a_1(\sqrt{t}(\xi-\eta)))d\eta.\end{aligned}$$

Then by the Cauchy-Schwartz inequality we obtain the estimate

$$\begin{aligned}&\|\mathcal{V}(-\frac{t}{3})\phi^3 a_1 - \phi^3\mathcal{V}(-\frac{t}{3})a_1\|_{\mathbf{L}^\infty} \\ \le\ & C\|B^{-\gamma}\phi\|_{\mathbf{L}^\infty}^2\|B^{\alpha-1}\partial_\xi\phi\|_{\mathbf{L}^2}(\int_{\mathbf{R}}\langle\eta\sqrt{t}\rangle^{-2}d\eta)^{\frac{1}{2}} \\ &+C\sqrt{t}\|B^{-\gamma}\phi\|_{\mathbf{L}^\infty}^2\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}\int_{\mathbf{R}}\langle\eta\sqrt{t}\rangle^{2\gamma-1}\langle\sqrt{t}(\xi-\eta)\rangle^{2\gamma-2}|\eta|^{\alpha-\frac{1}{2}}d\eta \\ \le\ & Ct^{-\frac{1}{4}}\|B^{-\gamma}\phi\|_{\mathbf{L}^\infty}^2\|B^{\alpha-1}\partial_\xi\phi\|_{\mathbf{L}^2} + Ct^{\frac{1}{4}-\frac{\alpha}{2}}\|B^{-\gamma}\phi\|_{\mathbf{L}^\infty}^2\||\partial|^{\alpha}\phi\|_{\mathbf{L}^2}.\end{aligned}$$

Hence, taking here $\phi = \mathcal{D}(3)(\mathcal{K}v)^3$, by (3.14) we get

$$\begin{aligned}&\|R_7\|_{\mathbf{L}^\infty} \le \|\mathcal{V}(-\frac{t}{3})a_1\mathcal{D}(3)(\mathcal{K}v)^3 - \mathcal{D}(3)(\mathcal{K}v)^3\mathcal{V}(-\frac{t}{3})a_1\|_{\mathbf{L}^\infty} \\ \le\ & Ct^{-\frac{1}{4}}\|B^{-\gamma}\mathcal{K}v\|_{\mathbf{L}^\infty}^2\|B^{\alpha-1}\partial_\xi\mathcal{K}v\|_{\mathbf{L}^2} + Ct^{\frac{1}{4}-\frac{\alpha}{2}}\|B^{-\gamma}\mathcal{K}v\|_{\mathbf{L}^\infty}^2\||\partial|^{\alpha}\mathcal{K}v\|_{\mathbf{L}^2}\end{aligned}$$

$$\leq \quad C\varepsilon^3 t^{\frac{1}{4}-\frac{\alpha}{2}} Q^{2\gamma}(t)P(t)$$

for $t \in [1,T]$. Therefore the second result of the lemma follows. Lemma 4.1 is proved. □

We consider now the Cauchy problem for the ordinary differential equation depending on the parameter $\xi \in \mathbf{R}$

$$\begin{cases} y_t = -\frac{1}{t\sqrt{3}B} y^3 + h, \ t \geq 1, \\ y(1,\xi) = y_1(\xi), \end{cases} . \tag{4.2}$$

where

$$|h| \leq C\varepsilon^3 t^{\frac{1}{4}-\frac{\alpha}{2}}((|\xi|\sqrt{t})^\gamma B^{-1} + Q^{2\gamma}(t))P(t)$$

for all $t > 1, \xi \in \mathbf{R}$, where we denote as above

$$Q(t) = (1+\delta^2 \log(1+t))^{-\frac{1}{2}}, P(t) = t^\gamma + \varepsilon^2 t^{\frac{\alpha}{2}-\frac{1}{4}} Q^3(t),$$

$\delta > 0$ and $\gamma \in (0,1)$ are small.

Lemma 4.2. *Suppose that there exists a solution $y \in \mathbf{C}([1,T] \times \mathbf{R})$ of (4.2) with the initial data y_1 satisfying the following conditions:*

$$\delta \leq |y_1(\xi)| \leq \varepsilon, \ \ |\arg y_1(\xi)| < \frac{\pi}{8}$$

for $|\xi| \leq 1$, where $\varepsilon > 0$ is small, and $\delta = \varepsilon^{\frac{5}{4}}$. Then

$$|y(t)| \leq C\varepsilon(1+\delta^2 \log(tB^{-2}))^{-\frac{1}{2}}$$

and

$$|\arg y(t)| \leq C(1+\delta^2 \log(tB^{-2}))^{-\gamma}$$

for all $t \in [1,T], \xi \in \mathbf{R}$.

Proof. For the case of — ξ — > 1 we have $B^{-2} \leq (1+t)^{-1}$, so that by (4.2)

$$y_t = O(\varepsilon^3 t^{-1-\gamma}) + O(\varepsilon^5 t^{-1} Q^3(t)).$$

Then integration with respect to time yields

$$|y(t)| \leq \varepsilon + \varepsilon^2 \leq 2\varepsilon(1+\delta^2 \log(tB^{-2}))^{-\frac{1}{2}}.$$

So next we consider the case of — ξ — ≤ 1. We change the dependent variable $y = re^{i\omega}$, where $r > 0$ and ω is a real-valued function. Then from (4.2) we get

$$r_t + ir\omega_t = -\frac{1}{t\sqrt{3}B} r^3 e^{2i\omega} + he^{-i\omega}.$$

Hence taking the real and imaginary part we get

$$r_t = -\frac{1}{t\sqrt{3}B} r^3 \cos 2\omega + \mathrm{Re} he^{-i\omega} \tag{4.3}$$

and

$$\omega_t = -\frac{1}{t\sqrt{3}B} r^2 \sin 2\omega + \mathrm{Im} h r^{-1} e^{-i\omega}, \tag{4.4}$$

with the initial conditions $r(1,\xi) =$ — $y_1(\xi)$— , $\omega(1,\xi) = \arg y_1(\xi)$. Let us prove the estimates

$$\frac{1}{2} + \frac{\delta^2}{2}\log(tB^{-2}) < \frac{|y_1(\xi)|^2}{r^2(t)} < 2 + 4\varepsilon^2 \log(tB^{-2}) \tag{4.5}$$

and

$$|\omega(t,\xi)| < \frac{\pi}{8} \tag{4.6}$$

for all $t \in [1,T]$, — ξ — ≤ 1. To the contrary we suppose that there exists a maximal time $\widetilde{T} \in (1,T]$, such that

$$\frac{1}{2} + \frac{\delta^2}{2}\log(tB^{-2}) \leq \frac{|y_1(\xi)|^2}{r^2(t)} \leq 2 + 4\varepsilon^2 \log(tB^{-2}) \tag{4.7}$$

and

$$|\omega(t,\xi)| \leq \frac{\pi}{8}$$

for all $t \in [1,\widetilde{T}]$, — $\xi| \leq 1$. Dividing equation (4.3) by r^3 we get

$$\frac{1}{t\sqrt{3}B} - 2r^{-3}|h| \leq \partial_t r^{-2} \leq \frac{2}{t\sqrt{3}B} + 2r^{-3}|h|. \tag{4.8}$$

Then integration of the above inequality in time yields

$$|y_1(\xi)|^{-2} + \frac{1}{\sqrt{3}}\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1} d\tau - 2\int_1^t r^{-3}|h| d\tau \leq r^{-2}(t)$$
$$\leq |y_1(\xi)|^{-2} + \frac{2}{\sqrt{3}}\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1} d\tau + 2\int_1^t r^{-3}|h| d\tau. \tag{4.9}$$

For the case of $t\xi^2 \leq 1$ we get

$$\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1} d\tau \leq \log t = \log(tB^{-2}) + \log B^2 \leq \log 2 + \log(tB^{-2})$$

and

$$\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1} d\tau \geq \frac{1}{2}\log t = \frac{1}{2}\log(tB^{-2}) + \frac{1}{2}\log B^2 \geq \frac{1}{2}\log(tB^{-2}).$$

For the case of $t\xi^2 \geq 1$ we find

$$\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1} d\tau \leq \int_{\xi^2}^{\infty} \frac{d\eta}{\eta\sqrt{1+\eta}} \leq 1 + \log\frac{1}{\xi^2} \leq C + \log(tB^{-2})$$

and

$$\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1}d\tau \geq \int_{\xi^2}^1 \frac{d\eta}{\eta\sqrt{1+\eta}} \geq \frac{1}{2}\log\frac{1}{\xi^2} \geq C + \frac{1}{2}\log(tB^{-2}).$$

Thus we get the two-sided estimate

$$\frac{1}{2}\log(tB^{-2}) - C \leq \int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1}d\tau \leq C + \log(tB^{-2}).$$

Then by (4.7) and by the assumption on h we obtain

$$\int_1^t r^{-3}|h|d\tau \leq C\varepsilon^3|y_1(\xi)|^{-3}\int_1^t (2+4\varepsilon^2\log(\tau B^{-2}))^{\frac{3}{2}}$$
$$\times(\tau^{-\frac{3}{4}-\frac{\alpha}{2}}Q^{2\gamma}(\tau)P(\tau) + \tau^{-1}Q^3(\tau)(\sqrt{\tau}|\xi|)^{\gamma}B^{-1})d\tau.$$

We have the estimates

$$\begin{aligned}
&\varepsilon^3\delta^{-1}\int_1^t (2+4\varepsilon^2\log(\tau B^{-2}))^{\frac{3}{2}}Q^3(\tau)(\sqrt{\tau}|\xi|)^{\gamma}B^{-1}\frac{d\tau}{\tau}\\
\leq\ & C\varepsilon^5\delta^{-4}\int_1^t \frac{(\sqrt{\tau}|\xi|)^{\gamma}d\tau}{\tau\langle\sqrt{\tau}|\xi|\rangle^{1-3\gamma}} = C\int_{|\xi|}^{\sqrt{t}|\xi|}\frac{d\eta}{\eta^{1-\gamma}\langle\eta\rangle^{1-3\gamma}} \leq C
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon^3\delta^{-1}\int_1^t (2+C\varepsilon^2\log(\tau B^{-2}))^{\frac{3}{2}}\tau^{-\frac{3}{4}-\frac{\alpha}{2}}Q^{2\gamma}(\tau)P(\tau)d\tau\\
\leq\ & \varepsilon^3\delta^{-1}\int_1^t (2+C\varepsilon^2\log(\tau B^{-2}))^{\frac{3}{2}}\tau^{\gamma-\frac{3}{4}-\frac{\alpha}{2}}d\tau\\
&+\varepsilon^8\delta^{-4}\int_1^t \tau^{-1}(1+\delta^2\log(1+\tau))^{-\gamma}d\tau\\
\leq\ & C(1+\delta^2\log(tB^{-2}))^{1-\gamma}
\end{aligned}$$

for all $t \in [1,\widetilde{T}]$, — $\xi| \leq 1$. Hence

$$\int_1^t r^{-3}|h|d\tau \leq C(1+\delta^2\log(tB^{-2}))^{1-\gamma}.$$

Therefore by (4.9) we get

$$\frac{|y_1(\xi)|^2}{r^2(t)} < 2 + 4\varepsilon^2\log(tB^{-2})$$

and

$$\frac{|y_1(\xi)|^2}{r^2(t)} \geq 1 + \delta^2\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-2}d\tau - C\varepsilon^2 - C\varepsilon^3\log(tB^{-2})$$

$$> \frac{1}{2} + \frac{\delta^2}{2}\log(tB^{-2})$$

for all $t \in [1, \widetilde{T}]$, — $\xi| \leq 1$. Hence the estimate in (4.5) is true for all $t \in [1, \widetilde{T}]$. Multiplying both sides of equation (4.4) by ω in view of (4.7) and the conditions on h we see that

$$\partial_t \omega^2 = -\frac{2}{t\sqrt{3}B} r^2 \omega \sin 2\omega + 2\omega r^{-1} \mathrm{Im} h e^{-i\omega} \leq \omega^2 \partial_t \log r^2 + Cr^{-1}|h| \quad (4.10)$$

since $2\omega \sin 2\omega \geq \omega^2$ for — ω— $\leq \frac{\pi}{8}$. We integrate (4.10) with respect to time

$$|\omega(t)|^2 \leq r^2(|\arg y_1|^2 + C\int_1^t r^{-3}|h|d\tau) \leq Cr^2(1 + \delta^2 \log(tB^{-2}))^{1-\gamma}$$

for all $t \in [1, \widetilde{T}]$, — $\xi| \leq 1$. This yields estimate (4.6) for all $t \in [1, \widetilde{T}]$. We get a contradiction, therefore the estimates of the lemma are true for all $t \in [1, T]$. Lemma 4.2 is proved. □

In the next lemma we obtain the estimates of the function v in the norm $\mathbf{L}^\infty$. Denote

$$Q(t) = (1 + \delta^2 \log(1+t))^{-\frac{1}{2}}, P(t) = t^\gamma + \varepsilon^2 t^{\frac{\alpha}{2} - \frac{1}{4}} Q^3(t),$$

where $\delta > 0$, $\gamma \in (0, 1)$ are small.

Lemma 4.3. *Let $v_1 \in \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha}$, $\alpha \in (\frac{1}{2}, 1)$, and the norm $\|v_1\|_{\mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha}} \leq \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Also we suppose that*

$$\||\partial|^\alpha v(t)\|_{\mathbf{L}^2} \leq C\varepsilon P(t). \quad (4.11)$$

for all $t \in [1, T]$. Then the solutions $v \in \mathbf{C}([1, T]; \mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha})$ of (3.1) satisfy the estimates

$$\|v(t)\|_{\mathbf{L}^\infty} < C\varepsilon, \ \|B^{-\gamma} v(t)\|_{\mathbf{L}^\infty} < C\varepsilon Q(t) \quad (4.12)$$

for all $t \in [1, T]$.

Proof. We prove estimates (4.12) by contradiction. By the continuity of v we can find a maximal time $\widetilde{T} \in (1, T]$ such that

$$\|v(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon, \ \|B^{-\gamma} v(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon Q(t) \quad (4.13)$$

for all $t \in [1, \widetilde{T}]$. Note that

$$\|B^{-\gamma}\varphi\|_{\mathbf{L}^\infty} \leq \|B^{-\gamma} v\|_{\mathbf{L}^\infty} + \|(\mathcal{V}(-t) - 1)v\|_{\mathbf{L}^\infty} \leq C\varepsilon Q(t) + C\varepsilon t^{\frac{1}{4} - \frac{\alpha}{2}} P(t).$$

Hence we have

$$\|\varphi(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon, \ \|B^{-\gamma}\varphi(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon Q(t) \quad (4.14)$$

for all $t \in [1, \widetilde{T}]$. We apply the operator $\mathcal{V}(-t)$ to equation (3.9) to get

$$i\partial_t(\varphi + \mathcal{V}(-t)AE^2(\mathcal{K}v)^3) = t^{-1}\mathcal{V}(-t)(E^2a(\mathcal{K}v)^3) + \sum_{j=1}^{4}\mathcal{V}(-t)R_j, \quad (4.15)$$

for all $t \in [1, \widetilde{T}]$, where $\varphi = \mathcal{V}(-t)v$. Using Lemma 4.1 we find

$$\begin{aligned}&\partial_t(\varphi + \mathcal{V}(-t)AE^2(\mathcal{K}v)^3)\\ &= -\frac{1}{t\sqrt{3}B}\varphi^3 + O(\varepsilon^3 t^{-\frac{3}{4}-\frac{\alpha}{2}}((|\xi|\sqrt{t})^{\gamma}B^{-1} + Q^{2\gamma}(t))P(t)). \quad (4.16)\end{aligned}$$

By (4.14) we obtain

$$|\mathcal{V}(-t)AE^2(\mathcal{K}v)^3| \le C\varepsilon^3 t^{\frac{1}{4}-\frac{\alpha}{2}}P(t). \quad (4.17)$$

We now make a change $y = \varphi + \mathcal{V}(-t)AE^2(\mathcal{K}v)^3$, then (4.16) yields

$$y_t = -\frac{1}{t\sqrt{3}B}y^3 + O(\varepsilon^3 t^{-\frac{3}{4}-\frac{\alpha}{2}}((|\xi|\sqrt{t})^{\gamma}B^{-1} + Q^{2\gamma}(t))P(t)) \quad (4.18)$$

for all $t \in [1, \widetilde{T}]$. By the result of Lemma 4.2 we get

$$|y(t)| \le 2\varepsilon(1 + \delta^2\log(tB^{-2}))^{-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}|\varphi(t)| &\le 2\varepsilon(1 + \delta^2\log(tB^{-2}))^{-\frac{1}{2}} + O(\varepsilon^3 Q^3(t)) \quad (4.19)\\ &< C\varepsilon(1 + \delta^2\log(tB^{-2}))^{-\frac{1}{2}}\end{aligned}$$

for all $t \in [1, \widetilde{T}]$. In particular $\|\varphi(t)\|_{\mathbf{L}^\infty} < C\varepsilon$. Next, if $t\xi^2 \le t^{\frac{1}{2}}$, then

$$(1 + \delta^2\log(tB^{-2}))^{-\frac{1}{2}} \le \sqrt{2}Q(t)$$

and if $t\xi^2 > t^{\frac{1}{2}}$, then $B^{-\gamma} \le t^{-\frac{\gamma}{2}} \le \sqrt{2}Q(t)$. Thus by (4.19) we find

$$\|B^{-\gamma}\varphi(t)\|_{\mathbf{L}^\infty} < C\varepsilon\sup_{\xi\in\mathbf{R}} B^{-\gamma}(1 + \delta^2\log(tB^{-2}))^{-\frac{1}{2}} < C\varepsilon Q(t).$$

Therefore by the estimates

$$\|B^{-\gamma}v\|_{\mathbf{L}^\infty} \le \|B^{-\gamma}\varphi\|_{\mathbf{L}^\infty} + \|(\mathcal{V}(-t) - 1)v\|_{\mathbf{L}^\infty} \le C\varepsilon Q(t) + C\varepsilon t^{\frac{1}{4}-\frac{\alpha}{2}}P(t),$$

we obtain

$$\|v(t)\|_{\mathbf{L}^\infty} < C\varepsilon,\ \|B^{-\gamma}v(t)\|_{\mathbf{L}^\infty} < C\varepsilon Q(t)$$

for all $t \in [1, \widetilde{T}]$. We obtain a contradiction. Therefore estimates (4.12) are valid for all $t \in [1, T]$. Lemma 4.3 is proved. □

5. Proof of Theorem 1.1

By Lemma 3.2 it is shown that the a priori estimate of the norm

$$\sup_{t\in[1,T]} (\|v(t)\|_{\mathbf{L}^\infty} + Q^{-1}(t)\|B^{-\gamma}v(t)\|_{\mathbf{L}^\infty})$$

implies the a priori estimate of the norm $\sup_{t\in[1,T]} P^{-1}(t)\||\partial|^\alpha v(t)\|_{\mathbf{L}^2}$ and by Lemma 4.3 it is shown that the a priori estimate of the norm

$$\sup_{t\in[1,T]} P^{-1}(t)\||\partial|^\alpha v(t)\|_{\mathbf{L}^2}$$

yields the a priori estimate of the norm

$$\sup_{t\in[1,T]} (\|v(t)\|_{\mathbf{L}^\infty} + Q^{-1}(t)\|B^{-\gamma}v(t)\|_{\mathbf{L}^\infty}).$$

Therefore, the global existence of a solution $u \in \mathbf{C}([1,\infty);\mathbf{H}^\alpha \cap \mathbf{H}^{0,\alpha})$ of the Cauchy problem (1.1) satisfying the a priori estimate

$$\sup_{t\geq 1}(\|v(t)\|_{\mathbf{L}^\infty} + Q^{-1}(t)\|B^{-\gamma}v(t)\|_{\mathbf{L}^\infty} + P^{-1}(t)\||\partial|^\alpha v(t)\|_{\mathbf{L}^2}) \leq C\varepsilon$$

follows by a standard continuation argument from Lemma 3.2, Lemma 4.3 and the local existence result Theorem 3.1. We need only to prove the asymptotic formula (1.5). By Lemmas 4.2, 4.3 we have with $\varphi(t)= \mathcal{F}\mathcal{U}(-t)u(t)= \mathcal{V}(-t)v(t)$

$$\begin{aligned} u(t) &= \mathcal{U}(t)\mathcal{F}^{-1}\varphi(t) = M(t)\mathcal{D}(t)\mathcal{V}(t)\varphi(t) \\ &= M(t)\mathcal{D}(t)\varphi(t) + M\mathcal{D}(t)(\mathcal{V}(t)-1)\varphi(t) \\ &= M(t)\mathcal{D}(t)r(t,\xi)e^{i\omega(t,\xi)} + O(t^{-\frac{1}{2}}(\log t)^{-\frac{3}{2}}) \end{aligned}$$

for $t \to \infty$, since by the Sobolev imbedding theorem

$$\begin{aligned} \|M\mathcal{D}(t)(\mathcal{V}(t)-1)\varphi\|_{\mathbf{L}^\infty} &\leq Ct^{-\frac{1}{2}}\||\partial|^\alpha(\mathcal{V}(t)-1)\varphi\|_{\mathbf{L}^2}^{\frac{1}{2\alpha}}\|(\mathcal{V}(t)-1)\varphi(t)\|_{\mathbf{L}^2}^{1-\frac{1}{2\alpha}} \\ &\leq Ct^{-\frac{1}{4}-\frac{\alpha}{2}}\||\partial|^\alpha\varphi\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{4}-\frac{\alpha}{2}}(\log t)^{-\frac{3}{2}}\||\partial|^\alpha v\|_{\mathbf{L}^2}. \end{aligned}$$

As in the proof of Lemma 4.2 we obtain

$$\begin{aligned} r^{-2}(t,\xi) = |\widehat{u_0}(\xi)|^{-2} + \frac{1}{\sqrt{3}}\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1}\cos(2\omega(\tau,\xi))d\tau \\ + O((1+\delta^2\log(tB^{-2}))^{1-\gamma}) \end{aligned}$$

and $|\omega(t,\xi)| \leq C(1+\delta^2\log(tB^{-2}))^{-\gamma}$. Hence,

$$r(t,\xi) = \frac{|\widehat{u_0}(\xi)|}{\sqrt{1 + \frac{|\widehat{u_0}(\xi)|^2}{\sqrt{3}}\int_1^t \tau^{-1}\langle\sqrt{\tau}\xi\rangle^{-1}d\tau + O((1+\delta^2\log(tB^{-2}))^{1-\gamma})}}$$

$$= |\widehat{u_0}(\xi)|(1 + \frac{|\widehat{u_0}(\xi)|^2}{\sqrt{3}} \log(tB^{-2}))^{-\frac{1}{2}} + O((\log(tB^{-2}))^{-\gamma-\frac{1}{2}}).$$

Therefore, we get

$$u(t) = M(t)\mathcal{D}(t)r(t,\xi)e^{i\omega(t,\xi)} + O(t^{-\frac{1}{2}}(\log t)^{-\frac{3}{2}})$$

$$= M(t)\mathcal{D}(t)|\widehat{u_0}(\xi)|(1 + \frac{|\widehat{u_0}(\xi)|^2}{\sqrt{3}} \log \frac{t}{1+t\xi^2})^{-\frac{1}{2}} + O(t^{-\frac{1}{2}}(\log \frac{t}{1+t\xi^2})^{-\gamma-\frac{1}{2}}),$$

which implies the asymptotics (1.5). Theorem 1.1 is proved.

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