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STABILITY FOR THE INFINITY-LAPLACE EQUATION WITH VARIABLE EXPONENT

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Abstract. We study the stability for the viscosity solutions of the differential equation

$$\sum u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \ln(|\nabla u|)\langle \nabla u, \nabla \ln p\rangle = 0$$

under perturbations of the function $p(x)$. The differential operator is the so-called $\infty(x)$-Laplacian.

1. Introduction

The object of our study is the curious differential equation

$$\sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \ln(|\nabla u|)\langle \nabla u, \nabla \ln p\rangle = 0 \tag{1}$$

in a bounded domain Ω in $\mathbb{R}^n$. Here $p(x)$ is a positive function, the so-called variable exponent, and it is of class $C^1(\overline{\Omega})$. The equation comes from the mini-max problem of determining

$$\min_u \max_x \left\{ |\nabla u(x)|^{p(x)} \right\}.$$

The case of a constant $p(x) = p$ reduces to the celebrated ∞-Laplace equation

$$\Delta_\infty u \equiv \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 \tag{2}$$

found by G. Aronsson. In [6] "the curious equation" was derived as the limit of the Euler-Lagrange equations for the variational integrals

$$\left\{ \int_\Omega |\nabla u(x)|^{kp(x)}\, \mathrm{d}x \right\}^{\frac{1}{k}}$$

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as $k \to \infty$. Such integrals were first considered by Zhikov, cf. [10]. See also [8] and [9] for similar equations. For sufficiently smooth solutions the meaning of the equation is that

$$|\nabla u(x)|^{p(x)} = C$$

along any fixed stream line (different stream lines may have different constants attached). In general, solutions have to be interpreted in the *viscosity sense*, and we assume that the reader is acquainted with the basic theory of viscosity solutions, see [2, 5, 3].

The viscosity solution with prescribed Lipschitz continuous boundary values is unique, cf. [6]. Taking into account that, in contrast, uniqueness does not always hold for the ∞-Poisson equation

$$\Delta_\infty u = \varepsilon(x),$$

as an example with a uniformly continuous sign-changing function $\varepsilon(x)$ in [7] shows, the uniqueness for the curious equation (1) is pretty remarkable. Therefore we have found it worth our while to study the stability under variations of $p(x)$.

Our first result is about a perturbation of the ∞-Laplace equation (2).

Theorem 1. *Let $p \in C^1(\overline{\Omega})$ be a positive function and suppose that $u \in C(\overline{\Omega})$ is the viscosity solution of*

$$\Delta_\infty u + |\nabla u|^2 \ln(|\nabla u|)\langle \nabla u, \nabla \ln(p)\rangle = 0$$

and that $v \in C(\overline{\Omega})$ is the viscosity solution of

$$\Delta_\infty v = 0,$$

both having the same Lipschitz continuous boundary values f. Then the estimate

$$\|v-u\|_{L^\infty(\Omega)} \leq C_1\|\nabla \ln p\|_{L^\infty(\Omega)} + C_2\|\nabla \ln p\|^{\frac{1}{5}}_{L^\infty(\Omega)} |\ln(c\|\nabla \ln p\|_{L^\infty(\Omega)})| \quad (3)$$

is valid with constants depending only on $\|f\|_{W^{1,\infty}(\Omega)}$ and $\operatorname{diam}(\Omega)$.

An interpretation is that when $p(x)$ deviates only a little from a constant value, then u is close to v. But, as we have pointed out, the perturbation term $|\nabla u|^2 \ln(|\nabla u|)\langle \nabla u, \nabla \ln(p)\rangle$ cannot be replaced by an arbitrary small perturbation $\varepsilon(x)$, despite the possibility of selecting $p(x)$ in any manner. The exponent $\frac{1}{5}$ seems to be an artifact of the arrangements in our proof in Section 4.

We also address the problem with two variable positive exponents $p_1, p_2 \in C^1(\overline{\Omega})$, but now the result is considerably weaker. Suppose that $u_\nu \in C(\overline{\Omega})$ is a viscosity solution of

$$\Delta_\infty u_\nu + |\nabla u_\nu|^2 \ln(|\nabla u_\nu|)\langle \nabla u_\nu, \nabla \ln(p_\nu)\rangle = 0 \tag{4}$$

in Ω, $\nu = 1, 2$. If u_1 and u_2 have the same Lipschitz continuous boundary values, then

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \le \frac{\text{Const.}}{|\ln\big(\|\nabla \ln p_2 - \nabla \ln p_1\|_{L^\infty(\Omega)}\big)|^\kappa}, \tag{5}$$

where $\kappa > 0$ depends on $\max(p_\nu), \min(p_\nu)$. The constant depends on the boundary values and on the norms $\|\nabla \ln p_\nu\|_\infty$. Needless to say, the obtained modulus of stability appears to be far from sharp. Therefore we have only sketched out the proof in Section 5. In the one-dimensional case a sharp bound of the form

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \le \text{Const.}\|\nabla \ln p_2 - \nabla \ln p_1\|_{L^\infty(\Omega)}$$

is easily reached via the “first integrals” $|u_\nu'(x)|^{p_\nu(x)} = C_\nu$ of the differential equation. This is much better than (5).

2. Preliminaries

We briefly recall some basic concepts. Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose that $f : \partial\Omega \to \mathbb{R}$ is a Lipschitz continuous function satisfying

$$|f(x) - f(y)| \le L|x - y|.$$

By extension, we may as well assume that the inequality holds in the whole space, if needed. The abbreviation

$$\Delta_{\infty(x)} u \equiv \Delta_\infty u + |\nabla u|^2 \ln(|\nabla u|)\langle \nabla u, \nabla \ln p\rangle$$

is convenient.[1]

To be on the safe side, we assume that $p \in C^1(\overline{\Omega})$, $p(x) > 0$. Then viscosity solutions to the equation (1) can be defined in the standard way.

Definition 2. *We say that a lower semicontinuous function $v\colon \Omega \to (-\infty, \infty]$ is a* viscosity supersolution *if, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that*

(1) $\varphi(x_0) = u(x_0)$, *and*
(2) $\varphi(x) < v(x)$, *when* $x \neq x_0$,

[1] The suggestive subscript $\infty(x)$ symbolizes the “variable exponent infinity.”

then we have

$$\Delta_{\infty(x_0)}\varphi(x_0) \le 0.$$

The *viscosity subsolutions* have a similar definition; they are upper semicontinuous, the test functions touch from above and the differential inequality is reversed. Finally, a *viscosity solution* is both a viscosity supersolution and viscosity subsolution.

There is an alternative way of expressing the definition in terms of "semijets." *Super*solutions require *sub*jets and subsolutions superjets. We say that the pair $(\xi, \mathbb{X})$, where ξ is a vector in $\mathbb{R}^n$ and $\mathbb{X}$ is a symmetric $n \times n$-matrix, belongs to the *subjet* $J^{2,-}v(x)$ if

$$v(y) \ge v(x) + \langle \xi, y - x\rangle + \tfrac{1}{2}\langle y - x, \mathbb{X}(x)(y - x)\rangle + o(|y - x|^2)$$

as $y \to x$. See [5, 2.2, page 17] where also the closure[2] $\overline{J^{2,-}}v(x)$ of the subjet is defined via a natural limit procedure $(x_j, \xi_j, \mathbb{X}_j) \to (x, \xi, \mathbb{X})$. Notice the similarity to a Taylor polynomial. The requirement for a supersolution is that when $x \in \Omega$

$$\langle \mathbb{X}\xi, \xi\rangle + |\xi|^2 \ln(|\xi|)\langle \xi, \nabla \ln p(x)\rangle \le 0$$

for all $(\xi, \mathbb{X})$ belonging to $\overline{J^{2,-}}v(x)$. For the basic theory of viscosity solutions we refer to [2], [3], and [5].

3. Auxiliary Equations

Following R. Jensen in [4] we introduce two auxiliary equations. For a constant exponent p the situation is

$$\max\{\varepsilon - |\nabla u^+|, \Delta_\infty u^+\} = 0 \qquad \textit{Upper equation} \qquad (6)$$

$$\Delta_\infty u = 0 \qquad \textit{Equation}$$

$$\min\{|\nabla u^-| - \varepsilon, \Delta_\infty u^-\} = 0 \qquad \textit{Lower equation} \qquad (7)$$

where $\varepsilon > 0$. Given $\varepsilon > 0$, three viscosity solutions u^-, u, u^+ are constructed with the same boundary values f so that

$$\begin{aligned} u^- \le u \le u^+ \\ \|u^+ - u^-\|_{L^\infty(\Omega)} \le \varepsilon \operatorname{diam}(\Omega) \\ \|\nabla u^\pm\|_{L^\infty(\Omega)} \le K + \varepsilon = K_\varepsilon, \end{aligned} \qquad (8)$$

[2]The subjet may be empty at a given point x. It is a decisive part of the theory that its closure is non-empty in a dense subset.

where K depends only on the Lipschitz constant L of f. The virtue of the auxiliary equations is that

$$\varepsilon - |\nabla u^+| \leq 0, \quad |\nabla u^-| - \varepsilon \geq 0$$

in the viscosity sense. We refer to [4] and [6] about the construction via variational integrals.

We need a *strict* supersolution. We will construct a function $g(u^+) \approx u^+$ such that $\Delta_\infty g(u^+) < 0$. To this end we use the following *approximation of the identity*

$$g(t) = \tfrac{1}{\alpha} \ln\bigl(1 + A(e^{\alpha t} - 1)\bigr), \quad A > 1, \alpha > 0 \tag{9}$$

taken from [6]. For $t > 0, A > 1, \alpha > 0$ we have

$$\begin{gathered} 0 < g(t) - t < \frac{A-1}{\alpha} \\ A^{-1}(A-1)e^{-\alpha t} < g'(t) - 1 < A - 1 \\ \frac{g''(t)}{g'(t)} = -\alpha\bigl(g'(t) - 1\bigr), \end{gathered}$$

which are easy to verify.

Lemma 3. *Let $v > 0$ and consider $w = g(v)$. If*

$$\varepsilon - |\nabla v| \leq 0 \quad \textit{and} \quad \Delta_\infty v \leq 0$$

in the viscosity sense, then the inequality

$$\Delta_\infty w \leq -\alpha(A-1)A^{-1}e^{-\alpha\|v\|_\infty}\varepsilon^4 \equiv -\mu \tag{10}$$

holds in the viscosity sense.

Proof. Formally, the equation for $w = g(v)$ is

$$\begin{aligned} \Delta_\infty w &= g'(v)^3 \Delta_\infty v + g''(v) g'(v)^2 |\nabla v|^4 \\ &\leq 0 + g''(v) g'(v)^2 |\nabla v|^4 \\ &= -\alpha(A-1)A^{-1}e^{-\alpha v} g'(v)^4 |\nabla v|^4 \\ &\leq -\alpha(A-1)A^{-1}e^{-\alpha v} 1^4 \varepsilon^4. \end{aligned}$$

To conclude the proof, one has to pass the calculation over to test functions. □

We will apply the lemma on $w = g(u^+)$ and we assume that $f > 0$ so that the encountered functions are non-negative. It holds that

$$\min_{\partial\Omega}(f) = \min_{\partial\Omega}(u) \leq u \leq u^+ \leq \max_{\partial\Omega}(f) + \varepsilon \operatorname{diam}(\Omega) \tag{11}$$

by the maximum principle and (8). Fix

$$\alpha = \frac{1}{\|u^+\|_\infty}.$$

With this optimal choice, estimate (10) in the lemma above becomes

$$\Delta_\infty g(u^+) \leq -\mu = -\frac{(A-1)\varepsilon^4}{Ae\|u^+\|_\infty}. \tag{12}$$

4. Proof of the Stability

Suppose that u_1 is a viscosity (sub)solution of

$$\Delta_\infty u_1 + |\nabla u_1|^2 \ln(|\nabla u_1|)\langle \nabla u_1, \nabla \ln p_1\rangle = 0$$

and that u_2 is a viscosity (super)solution of

$$\Delta_\infty u_2 = 0, \qquad (p_2 = \text{constant})$$

both with boundary values f. Our task is to estimate $|u_1 - u_2|$. Adding the same constant to f, u_1, and u_2, we may assume that $f \geq 0$ and $u_2 \geq 0$. Given $\varepsilon > 0$, write

$$v_2 = u_2^+, \quad w_2 = g(v_2) = g(u_2^+).$$

We obtain the estimate

$$\begin{aligned} u_1 - u_2 &= (u_1 - w_2) + (w_2 - v_2) + (v_2 - u_2) \\ &< (u_1 - w_2) + \frac{A-1}{\alpha} + \varepsilon \operatorname{diam}(\Omega). \end{aligned} \tag{13}$$

The last two terms could be made as small as we please, but the term $u_1 - w_2$ requires our attention, since there w_2 depends also on A and ε.

Lemma 4. *We have*

$$u_1 - w_2 \leq C_\varepsilon^3 \, \|u_2^+\|_\infty^2 \, \varepsilon^{-4} \ln\Big(\frac{C_\varepsilon}{\varepsilon}\Big) \, \|\nabla \ln p_1\|_\infty, \tag{14}$$

where $C_\varepsilon = C(1+\varepsilon)$.

Remark. The magnitude of $C_\varepsilon = C + O(\varepsilon)$ is decisive later.

Proof. Let $\sigma = \max(u_1 - w_2)$. If $\sigma \leq 0$ there is nothing to prove. Assume thus that $\sigma > 0$. In order to use the theorem on sums for viscosity solutions, we double the variables writing

$$M_j = \sup_{x\in\Omega,\ y\in\Omega} \Big(u_1(x) - w_2(y) - \frac{j}{2}|x-y|^2 \Big) \tag{15}$$

as usual. Then $M_j \geq \sigma$ (take $x = y$ to see this). The supremum is attained at some points x_j, y_j. Now $|x_j - y_j| \to 0$ as $j \to \infty$ and $x_j \to \hat{x}$, $y_j \to \hat{y} = \hat{x}$, at least for a subsequence. We claim that $\hat{x}$ is an interior point of Ω. Indeed, if $\hat{x} \in \partial\Omega$, then

$$\begin{aligned} u_1(\hat{x}) - w_2(\hat{x}) &= (u_1(\hat{x}) - v_2(\hat{x})) + (v_2(\hat{x}) - w_2(\hat{x})) \\ &= 0 + (v_2(\hat{x}) - g(v_2(\hat{x})) \leq 0 \end{aligned}$$

and hence

$$u_1(\hat{x}) - w_2(y) = (u_1(\hat{x}) - w_2(\hat{x})) + (w_2(\hat{x}) - w_2(y)) \leq w_2(\hat{x}) - w_2(y),$$

which, by continuity, is less than $\sigma/2 < \sigma \leq M_j$ provided that $|\hat{x} - y|$ is small. Hence, $\hat{x} \in \Omega$.

We conclude that also x_j and y_j are interior points for large indices j. We need the bounds

$$\varepsilon \leq j|x_j - y_j| \leq C_\varepsilon. \tag{16}$$

The upper bound follows from

$$u_1(x_j) - w_2(y_j) - \frac{j}{2}|x_j - y_j|^2 \geq u_1(x_j) - w_2(x_j),$$

$$\frac{j}{2}|x_j - y_j|^2 \leq w_2(x_j) - w_2(y_j) \leq \|g'(v_2)\nabla v_2\|_\infty |x_j - y_j| \leq AK_\varepsilon |x_j - y_j|,$$

where we used the fact that $g'(v_2) < A$. We had $K_\varepsilon = K + \varepsilon$ and we will later see that $A \leq 2$. Then $C_\varepsilon = 2K_\varepsilon$ will do. The lower bound is deduced from the fact that $\varepsilon - |\nabla w_2| \leq 0$ in the viscosity sense ($\nabla w_2 = g'(v_2)\nabla v_2$, $1 \leq g'(v_2)$, $\varepsilon \leq |\nabla v_2|$). To wit,

$$u_1(x_j) - w_2(y_j) - \frac{j}{2}|x_j - y_j|^2 \geq u_1(x_j) - w_2(y) - \frac{j}{2}|x_j - y|^2,$$

from which it follows that the function

$$\varphi(y) = w_2(y_j) + \frac{j}{2}|x_j - y_j|^2 - \frac{j}{2}|x_j - y|^2$$

touches $w_2(y)$ from below at the point y_j. Thus $\varepsilon \leq |\nabla\varphi(y_j)|$, and this is the desired inequality, indeed.

According to the theorem on sums there exist symmetric $n \times n$-matrices $\mathbb{X}_j$ and $\mathbb{Y}_j$ such that $\mathbb{X}_j \leq \mathbb{Y}_j$ and

$$(j(x_j - y_j), \mathbb{X}_j) \in \overline{J^{2,+}}u_1(x_j), \qquad (j(x_j - y_j), \mathbb{Y}_j) \in \overline{J^{2,-}}w_2(y_j),$$

where $\overline{J^{2,+}}u_1(x_j)$ and $\overline{J^{2,-}}w_2(y_j)$ are the closures of the super- and subjets. (Caution: *super*solutions are tested with *sub*jets.) For the jets and their closures we refer to [2], [3], [5]. The meaning of the notion is that we can rewrite the equations as

$$\begin{aligned}
&j^2\langle \mathbb{Y}_j(x_j-y_j), x_j-y_j\rangle \le -\mu,\\
&j^2\langle \mathbb{X}_j(x_j-y_j), x_j-y_j\rangle\\
&\quad + j^3|x_j-y_j|^2\ln(j|x_j-y_j|)\langle x_j-y_j, \nabla \ln p_1(x_j)\rangle \ge 0,\\
&j|x_j-y_j| \ge \varepsilon, \qquad j|x_j-y_j|\le C_\varepsilon.
\end{aligned}$$

It follows that

$$j^2\langle (\mathbb{Y}_j-\mathbb{X}_j)(x_j-y_j), x_j-y_j\rangle \le -\mu + C_\varepsilon^3 \ln\Bigl(\frac{C_\varepsilon}{\varepsilon}\Bigr)\|\nabla \ln p_1\|_\infty.$$

The left-hand member is a positive semidefinite quadratic form, since $\mathbb{Y}_j - \mathbb{X}_j \ge 0$; in other words, it is non-negative. Thus,

$$\mu \le C_\varepsilon^3 \ln\Bigl(\frac{C_\varepsilon}{\varepsilon}\Bigr)\|\nabla \ln p_1\|_\infty.$$

Recall the expression for μ in (12). The above estimate can be written as

$$\frac{(A-1)\varepsilon^4}{Ae\,\|u_2^+\|_\infty} \le C_\varepsilon^3 \ln\Bigl(\frac{C_\varepsilon}{\varepsilon}\Bigr)\|\nabla \ln p_1\|_\infty.$$

We fix $A>1$ so that $\frac{A-1}{\alpha}=\sigma$, where we had $\alpha^{-1}=\|u_2^+\|_\infty$. Then

$$\sigma \le Ae\varepsilon^{-4}C_\varepsilon^3 \ln\Bigl(\frac{C_\varepsilon}{\varepsilon}\Bigr)\|\nabla \ln p_1\|_\infty\,\|u_2^+\|_\infty^2.$$

Further, $A\le 2$ so that Ae can be absorbed into the constant C_ε. (Indeed, $\sigma = \max(u_1-w_2)\le \max(u_1) = u_1(\xi)$ where ξ is some boundary point. Now

$$A = 1+\alpha\sigma \le 1+\frac{u_1(\xi)}{\|u_2^+\|_\infty} \le 1+\frac{u_1(\xi)}{u_2^+(\xi)} = 2,$$

because the functions have the same boundary values.) This concludes the proof of the estimate (14). □

We return to (13). Using (14) we obtain

$$\begin{aligned}
u_1-u_2 &\le \sigma+\sigma+\varepsilon\mathrm{diam}(\Omega)\\
&\le 2\varepsilon^{-4}C_\varepsilon^3\ln\Bigl(\frac{C_\varepsilon}{\varepsilon}\Bigr)\|u_2^+\|_\infty^2\,\|\nabla \ln p_1\|_\infty + \varepsilon\mathrm{diam}(\Omega).
\end{aligned}$$

It remains to determine ε nearly optimally. To simplify, we use (11):

$$u_2^+ \le u_2+\varepsilon\mathrm{diam}(\Omega) \le \|f\|_\infty + \varepsilon\mathrm{diam}(\Omega).$$

Recall that $C_\varepsilon = C(1+\varepsilon)$. We agree to keep $\varepsilon \le 1$ and we rename C so that $C_\varepsilon \le C + \varepsilon$. We have to optimize

$$2\,(C+\varepsilon)^3 \ln\Big(\frac{C+\varepsilon}{\varepsilon}\Big)(\|f\|_\infty + \varepsilon)^2 \|\nabla \ln p_1\|_\infty\, \varepsilon^{-4} + \varepsilon \mathrm{diam}(\Omega),$$

which, renaming constants, is the same as an expression of the form

$$\Big(\frac{C+\varepsilon}{\varepsilon}\Big)^5 \ln\Big(\frac{C+\varepsilon}{\varepsilon}\Big)\|\nabla \ln p_1\|_\infty\, \varepsilon + \varepsilon a$$

multiplied by some constant. One may assume that $C > 1$, for example.

We consider two cases. The case of a large $\|\nabla \ln p_1\|_\infty$ is plain. Namely, if $a \le 32C^5\,\|\nabla \ln p_1\|_\infty$ we just take $\varepsilon = 1$ and obtain immediately a majorant of the form $C_1\|\nabla \ln p_1\|_\infty$. If not, we can determine ε from the equation

$$\Big(\frac{C+\varepsilon}{\varepsilon}\Big)^5 \|\nabla \ln p_1\|_\infty = a.$$

This yields a majorant like $C_2\|\nabla \ln p_1\|_\infty^{\frac{1}{5}} \ln(c\,\|\nabla \ln p_1\|_\infty)$. Combining the two cases we arrive at the desired estimate (3), yet so far only for $\max(u_1 - u_2)$. The corresponding estimate for $\max(u_2 - u_1)$ is still missing; the situation is not symmetric.

To complete the proof, observe that

$$u_2(x) - u_1(x) = (k - u_1(x)) - (k - u_2(x))$$

where the constant is large enough to make the new viscosity solution $k - u_2$ positive; $k = \max(f)$ will do. Now $\Delta_\infty(k - u_2(x)) = 0$ and the situation has been reduced to the previous case. Instead, we could have repeated the proof, this time using the lower equation (7). □

5. Two Varying Exponents

In the case of two exponents none of which is constant, an extra complication arises: the parameter α must be taken very large, say $\alpha \approx \varepsilon^{-1}$, and then the exponential factor in the counterpart to (12) is extremely small. This weakens the final result.

In principle, the proof is a repetition of the previous one. Only an outline is provided below. First, the auxiliary equations in Section 3 are modified so that Δ_∞ is replaced by $\Delta_{\infty(x)}$. As in [6] one then obtains the estimate

$$\|u^+ - u^-\|_\infty \le B\varepsilon^\kappa \tag{17}$$

where $\kappa > 0$ (either $\kappa = \min(p(x))$ or $\kappa = \max(p(x))$). Second, we need a strict supersolution to equation (1).

Lemma 5. *Consider $w = g(v)$ for $v > 0$. If*

$$\varepsilon - |\nabla v| \leq 0 \quad \textit{and} \quad \Delta_{\infty(x)} v \leq 0$$

in the viscosity sense, then

$$\Delta_{\infty(x)} w \leq -\varepsilon^3 (A-1) A^{-1} e^{-(1+\|\nabla \ln p\|_\infty)\|v\|_\infty \varepsilon^{-1}} \equiv -\mu \tag{18}$$

in the viscosity sense.

Proof. A routine calculation yields

$$\begin{aligned}\Delta_{\infty(x)} w &\leq g'(v)^3 (g'(v)-1)|\nabla v|^3 \left\{-\alpha|\nabla v| + |\nabla \ln p|\right\}\\ &\leq g'(v)^3 (g'(v)-1)|\nabla v|^3 \left\{-\alpha\varepsilon + \|\nabla \ln p\|_\infty\right\}.\end{aligned}$$

Given $\varepsilon > 0$, we fix $\alpha = \alpha(\varepsilon)$ so that

$$-\alpha\varepsilon + \|\nabla \ln p\|_\infty = -1.$$

The estimate (18) readily follows. □

Suppose now that u_ν is a viscosity solution of the equation (4), $\nu = 1, 2$. We assume that $u_1 = u_2 = f$ on $\partial\Omega$. By adding a constant, we reach the situation that $u_2^+ \geq u_2 > 0$. Write $v_2 = u_2^+$, $w_2 = g(v_2)$. Now

$$\begin{aligned}u_1 - u_2 &= (u_1 - w_2) + (w_2 - v_2) + (v_2 - u_2)\\ &\leq (u_1 - w_2) + \frac{A-1}{\alpha} + B\varepsilon^\kappa.\end{aligned}$$

Lemma 6. *We have*

$$u_1 - w_2 \leq 2\varepsilon^{-2} A C_\varepsilon^3 \ln\Big(\frac{C_\varepsilon}{\varepsilon}\Big) e^{(1+\|\nabla \ln p_2\|_\infty)\,\|v_2\|_\infty\,\varepsilon^{-1}} \|\nabla \ln p_2 - \nabla \ln p_1\|_\infty. \tag{19}$$

Proof. Denote $\sigma = \max(u_1 - u_2)$. We may assume that $\sigma > 0$. Double the variables as in (15). By the theorem on sums we again obtain symmetric matrices $\mathbb{X}_j$ and $\mathbb{Y}_j$ so that $\mathbb{X}_j \leq \mathbb{Y}_j$ and

$$\begin{aligned}&j^2 \langle \mathbb{Y}_j(x_j - y_j), x_j - y_j\rangle\\ &\quad + j^3 |x_j - y_j|^2 \ln(j|x_j - y_j|)\langle x_j - y_j, \nabla \ln p_2(y_j)\rangle \leq -\mu,\\ &j^2 \langle \mathbb{X}_j(x_j - y_j), x_j - y_j\rangle\\ &\quad + j^3 |x_j - y_j|^2 \ln(j|x_j - y_j|)\langle x_j - y_j, \nabla \ln p_1(x_j)\rangle \geq 0,\\ &\ j|x_j - y_j| \geq \varepsilon, \qquad j|x_j - y_j| \leq C_\varepsilon.\end{aligned}$$

Write $\ln p_1(x_j) = \ln p_1(x_j) - \ln p_2(x_j) + \ln p_2(x_j)$ and arrange the equations. It follows that

$$\begin{aligned}0 &\le j^2\langle(\mathbb{Y}_j - \mathbb{X}_j)(x_j - y_j),(x_j - y_j)\rangle \le -\mu \\ &+ j^3|x_j - y_j|^2 \ln(j|x_j - y_j|)\langle x_j - y_j, \nabla \ln p_2(x_j) - \nabla \ln p_2(y_j)\rangle \\ &+ j^3|x_j - y_j|^2 \ln(j|x_j - y_j|)\langle x_j - y_j, \nabla \ln p_1(x_j) - \nabla \ln p_2(x_j)\rangle \\ &\le -\mu + C_\varepsilon^3 \ln\Big(\frac{C_\varepsilon}{\varepsilon}\Big)|\nabla \ln p_2(x_j) - \nabla \ln p_2(y_j)| \\ &+ C_\varepsilon^3 \ln\Big(\frac{C_\varepsilon}{\varepsilon}\Big)\|\nabla \ln p_1 - \nabla \ln p_2\|_\infty.\end{aligned}$$

As $j \to \infty$, $x_j - y_j \to 0$, so that by continuity

$$\mu \le C_\varepsilon^3 \ln\Big(\frac{C_\varepsilon}{\varepsilon}\Big)\|\nabla \ln p_1 - \nabla \ln p_2\|_\infty. \tag{20}$$

To conclude the proof, we fix $A > 1$ so that $\frac{A-1}{\alpha} = \frac{\sigma}{2}$ and insert the expression for μ given in (18). Hence

$$\varepsilon^3 \frac{\sigma\alpha}{2} A^{-1} e^{-(1+\|\nabla \ln p_2\|_\infty)\|v_2\|_\infty \varepsilon^{-1}} \le C_\varepsilon^3 \ln\Big(\frac{C_\varepsilon}{\varepsilon}\Big)\|\nabla \ln p_1 - \nabla \ln p_2\|_\infty.$$

The estimate follows, since $\alpha \approx 1/\varepsilon$. □

In order to finish the proof of (4) we choose ε in the inequality

$$\begin{aligned}&u_1 - u_2 \\ &\quad\le 4A\varepsilon^{-2} C_\varepsilon^3 \ln\Big(\frac{C_\varepsilon}{\varepsilon}\Big) e^{(1+\|\nabla \ln p_2\|_\infty)\|v_2\|_\infty \varepsilon^{-1}} \|\nabla \ln p_1 - \nabla \ln p_2\|_\infty + B\varepsilon^\kappa\end{aligned}$$

so that

$$e^{(1+\|\nabla \ln p_2\|_\infty)\|v_2\|_\infty \varepsilon^{-1}} \|\nabla \ln p_1 - \nabla \ln p_2\|_\infty \approx \varepsilon^{\kappa+2}.$$

We omit the calculation.

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