Differential and Integral Equations Volume 25, Numbers 5-6 (2012), 579–588

UNIQUENESS SETS FOR MINIMIZATION FORMULAS

Yasuhiro Fujita
Department of Mathematics, University of Toyama
Toyama 930-8555, Japan

Hitoshi Ishii
Department of Mathematics, Waseda University
Shinjuku, Tokyo, 169-8050 Japan
and
Department of Mathematics, King Abdulaziz University
Jeddah, 2158 Saudi Arabia

(Submitted by: Yoshikazu Giga)

Abstract. In this paper, we consider minimization formulas which arise typically in optimal control and weak KAM theory for Hamilton-Jacobi equations. Given a minimization formula, we define a uniqueness set for the formula, which replaces the original region of minimization without changing its values. Our goal is to provide a necessary and sufficient condition that a given set be a uniqueness set. We also provide a characterization of the existence of a minimal uniqueness set with respect to set inclusion.

1. Introduction

In this paper we deal with minimization formulas of the form

$$\inf_{y\in A} (d(x,y) + f(y)) \quad \text{for } x \in \Omega, \tag{1.1}$$

where Ω is a set, d is a quasi-pseudo metric on Ω, A is a subset of Ω and f is a given real-valued function on A. Here, by the definition of quasi-pseudo metric, d is a real-valued function on $\Omega \times \Omega$ such that $d(x,x) = 0$ for $x \in \Omega$ and $d(x,y) \le d(x,z) + d(z,y)$ for $x, y, z \in \Omega$.

Minimizations of the form (1.1) arise typically in optimal control. We refer to [9] for a general overview of optimal control. For instance, let Ω be

Accepted for publication: September 2011.
AMS Subject Classifications: 54C40, 14E20; 46E25, 20C20.
The first author was supported in part by JSPS KAKENHI # 21540168, # 18204009.
The second author was supported in part by JSPS KAKENHI #18204009, #20340026, #21224001.

an open connected subset of $\mathbb{R}^n$ and consider the function d on $\Omega \times \Omega$ given by

$$d(x, y) = \inf \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds, \tag{1.2}$$

where L is a given continuous function on $\Omega \times \mathbb{R}^n$, called the Lagrangian, such that the functions $L(x, \cdot)$ are convex on $\mathbb{R}^n$ and

$$\lim_{|p| \to \infty} \frac{L(x, p)}{|p|} = \infty \quad \text{uniformly for } x \in \Omega;$$

the infimum is taken over all $t > 0$ and $\gamma : [0, t] \to \Omega$ such that $\gamma(t) = x$, $\gamma(0) = y$ and γ is absolutely continuous on $[0, t]$, and $\dot{\gamma}$ denotes the derivative $\mathrm{d}\gamma/\mathrm{d}s$. If we assume that $L(x, \xi) \geq 0$ for $(x, \xi) \in \Omega \times \mathbb{R}^n$, then (1.2) defines a nonnegative quasi-pseudo metric on Ω. Given a subset A of Ω and a bounded function f on A, the formula (1.1) now reads

$$\inf \Big(\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + f(\gamma(0)) \Big), \tag{1.3}$$

where the infimum is taken over all $t > 0$ and $\gamma : [0, t] \to \Omega$ such that γ is absolutely continuous on $[0, t]$, $\gamma(t) = x$ and $\gamma(0) \in A$. This is an optimization problem with running cost L, terminal cost f and initial set A. Moreover, it is known (see for instance [1, 2, 13]) that if we write $u(x)$ for (1.3), then u is a viscosity solution of

$$H(x, Du(x)) = 0 \quad \text{in } \operatorname{Int}(\Omega \setminus A),$$

where the Hamiltonian H is a continuous function on $\Omega \times \mathbb{R}^n$ given by $H(x, p) = \max_{\xi \in \mathbb{R}^n} (p \cdot \xi - L(x, \xi))$, $p \cdot \xi$ denotes the Euclidean inner product, and $\operatorname{Int} B$ denotes the interior of a subset B of a topological space. With the function u introduced above, the formula (1.1) can be rewritten as

$$u(x) = \inf\{d(x, y) + u(y) : y \in A\} \quad \text{for } x \in \Omega. \tag{1.4}$$

Indeed, as shown in Theorem 2.3, this is true if $\inf_{y \in A} (d(x_0, y) + f(y)) > -\infty$ for some point $x_0 \in \Omega$.

In recent developments of the theory of Hamilton-Jacobi equations, there has been much interest in weak KAM theory, where Hamilton-Jacobi equations of the form

$$H(x, Du(x)) = 0 \quad \text{in } \Omega \tag{1.5}$$

is studied in a general setting. We refer to [6] for an overview of weak KAM theory. One of the important ideas in weak KAM theory is the notion of the Aubry (or Aubry-Mather) sets for the Hamilton-Jacobi equation (1.5). When the Dirichlet boundary condition is our concern, then the Aubry set A

for (1.5) should be the subset of those points $y \in \Omega$ such that the functions $d(\cdot, y)$ are viscosity solutions of (1.5). With this Aubry set A and with the assumption that Ω has a sufficiently smooth boundary $\partial\Omega$, every viscosity solution u of (1.5) satisfying the Dirichlet condition $u = g$, with $g \in C(\partial\Omega)$, can be represented as $u(x) = \inf\{d(x, y) + u(y) : y \in A \cup \partial\Omega\}$ for $x \in \overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω and d is the unique extension by continuity of the original d to $\overline{\Omega} \times \overline{\Omega}$. (The regularity of the domain Ω comes in here for the original d to have a unique extension to $\overline{\Omega} \times \overline{\Omega}$ by continuity.) This formula has the same form as (1.4), and an interesting feature of the formula is in the point that the function u is identified only by its values on the set A, which raises the question of whether one can replace A by a smaller one in the formula (1.4). We refer to [12, 6, 7, 8, 11, 3, 4] for general formulas similar to the above.

In this paper we are concerned with the formula (1.4) for the function u, the value of which at x is given by formula (1.1) and seek the possibility of replacing the set A in (1.4) by other subsets of Ω.

In the next section we present our main results together with their proofs. In the third and last section we give a few simple examples to illustrate our main results in the theme of Hamilton-Jacobi equations.

Part of the results here has been announced in [10].

2. MAIN RESULTS

We are concerned with a quasi-pseudo metric space (Ω, d). Here, by saying that (Ω, d) is a quasi-pseudo metric space, we mean that Ω is a set and d is a real-valued function on $\Omega \times \Omega$ satisfying

$$d(x, x) = 0 \quad \text{for } x \in \Omega,$$
$$d(x, y) \le d(x, z) + d(z, y) \quad \text{for } x, y, z \in \Omega.$$

Let $A \subset \Omega$ be a nonempty set. We consider the real-valued functions $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ given by the formula

$$u(x) = \inf\{d(x, y) + f(y) : y \in A\},$$

where f is a real-valued function on A. We write $\mathrm{I}_A(f)(x)$ for the right side of the above formula.

We define $\mathcal{F}(A) = \{f : A \to \mathbb{R} : \mathrm{I}_A(f)(x) > -\infty \quad \text{for all } x \in \Omega\}$.

Proposition 2.1. *Let $f : A \to \mathbb{R}$. Then, $f \in \mathcal{F}(A)$ if and only if*

$$\mathrm{I}_A(f)(x_0) > -\infty \quad \textit{for some } x_0 \in \Omega. \tag{2.1}$$

Proof. Assume that (2.1) holds for some $x_0 \in \Omega$. Let $x \in \Omega$. Then we have

$$d(x_0, y) + f(y) \le d(x_0, x) + d(x, y) + f(y) \quad \text{for } y \in A,$$

and $\mathrm{I}_A(f)(x_0) \le d(x_0, x) + \mathrm{I}_A(f)(x)$. □

The monotonicity properties of I_A given in the following proposition are immediate consequences of its definition.

Proposition 2.2. (i) *Let $c \in \mathbb{R}$ and $f \in \mathcal{F}(A)$. Then $f + c \in \mathcal{F}(A)$ and $\mathrm{I}_A(f + c) = \mathrm{I}_A(f) + c$ on Ω.* (ii) *Let $f, g \in \mathcal{F}(A)$. If $f \le g$ on A, then $\mathrm{I}_A(f) \le \mathrm{I}_A(g)$ on Ω.* (iii) *Let $A \subset B \subset \Omega$. Then $\mathcal{F}(A) \supset \mathcal{F}(B)$ and, for any $f \in \mathcal{F}(B)$, $\mathrm{I}_A(f) \ge \mathrm{I}_B(f)$ on Ω.*

Theorem 2.3. *Let $f \in \mathcal{F}(A)$ and set $u = \mathrm{I}_A(f)$. Then $u \in \mathcal{F}(A)$ and $u = \mathrm{I}_A(u)$.*

In the above theorem, we use the loose notation that the function u on Ω, when needed, is identified with its restriction $u|_A$. If we regard I_A as a mapping on $\mathcal{F}(A)$, then Theorem 2.3 says that I_A is idempotent; i.e., $\mathrm{I}_A \circ \mathrm{I}_A = \mathrm{I}_A$.

Proof. By the definition of I_A, we have $u(x) \le d(x, z) + f(z)$ for $(x, z) \in \Omega \times A$. By the triangle inequality for d, we get

$$u(x) \le d(x, y) + d(y, z) + f(z) \quad \text{for } (x, y, z) \in \Omega \times \Omega \times A.$$

Hence, $u(x) \le d(x, y) + u(y)$ for $(x, y) \in \Omega \times \Omega$, and, moreover, $u \le \mathrm{I}_A(u)$ on Ω. In particular, we have $u \in \mathcal{F}(A)$.

Again, by the definition of I_A, we see that $u(x) \le f(x)$ for $x \in A$. Hence, by the monotonicity (Proposition 2.2 (ii)), we get $\mathrm{I}_A(u) \le \mathrm{I}_A(f) = u$ on Ω and conclude that $u = \mathrm{I}_A(u)$ on Ω. □

In view of Theorem 2.3, we introduce the set of functions $\mathcal{S}(A) = \{u : \Omega \to \mathbb{R} : u = \mathrm{I}_A(u)\}$. It is clear from Theorem 2.3 that $\mathcal{S}(A) = \{\mathrm{I}_A(f) : f \in \mathcal{F}(A)\}$. In the proof of Theorem 2.3, we have observed that, for any $u \in \mathcal{S}(A)$,

$$u(x) - u(y) \le d(x, y) \quad \text{for } x, y \in \Omega. \tag{2.2}$$

We note that property (2.2) is independent of the choice of A, and we introduce $\mathcal{S}^-$ as the set of functions $u : \Omega \to \mathbb{R}$ satisfying (2.2). Observe by the triangle inequality for d that $d(\cdot, x), -d(x, \cdot) \in \mathcal{S}^-$ for $x \in \Omega$.

Proposition 2.4. *Let $A \subset B \subset \Omega$ and $u \in \mathcal{S}(A)$. Then $u = \mathrm{I}_B(u)$, or equivalently, $u \in \mathcal{S}(B)$.*

We remark here that, if $y \in A \subset \Omega$, then $d(\cdot, y) \in \mathcal{S}(A)$. Indeed, noting that $d(x,y) = \mathrm{I}_{\{y\}}(d(\cdot, y))(x)$ for $x \in \Omega$, we find that $d(\cdot, y) \in \mathcal{S}(\{y\})$ for $y \in \Omega$. Hence, Proposition 2.4 ensures that the remark above is valid.

Proof. From (2.2), we see that $u \le \mathrm{I}_C(u)$ on Ω for any $C \subset \Omega$. In particular, we have $u \le \mathrm{I}_B(u)$ on Ω. By the monotonicity property (Proposition 2.2 (iii)), we have $\mathrm{I}_B(u) \le \mathrm{I}_A(u)$ on Ω. Accordingly, we have $u = \mathrm{I}_A(u) = \mathrm{I}_B(u)$ on Ω. □

In what follows we fix a nonempty set $A \subset \Omega$, and we deal with the question of finding the smallest $B \subset \Omega$, if exists, such that $u = \mathrm{I}_B(u)$ on Ω for any $u \in \mathcal{S}(A)$; i.e., $\mathcal{S}(A) \subset \mathcal{S}(B)$. We call $B \subset \Omega$ a *uniqueness set* for A (or for the minimization formula $u = \mathrm{I}_A(u)$) if $u = \mathrm{I}_B(u)$ on Ω for any $u \in \mathcal{S}(A)$.

We introduce the function λ on $\Omega \times \Omega$ by setting $\lambda(x,y) = d(x,y) + d(y,x)$, and observe that λ is a *pseudo metric*; i.e., λ has the properties: for any $x, y, z \in \Omega$

$$\lambda(x,x) = 0, \quad \lambda(x,y) = \lambda(y,x), \quad \lambda(x,y) \le \lambda(x,z) + \lambda(z,y).$$

We introduce the relation $\sim$ on the set Ω by $x \sim y$ if and only if $\lambda(x,y) = 0$. Since λ is a pseudo metric, it is clear that the relation $\sim$ is an equivalence relation. We denote by $[x]$ the equivalence class of $x \in \Omega$ under this relation. For any $B \subset \Omega$ we write $\widehat{B} = \{[x] : x \in B\}$. Let $B, C \subset \Omega$. We call C a *representation* of B (or more precisely, a representation of B under $\sim$) if for any $x \in B$ there exists a unique element $y \in C$ such that $y \sim x$. Similarly, given $\Theta \subset \widehat{\Omega}$, we call C a representation of Θ if for any $X \in \Theta$ the set $C \cap X$ is a singleton. If C is a representation of B, then we say as well that C represents B.

For any $x, y, p, q \in \Omega$ such that $x \sim p$ and $y \sim q$, we have

$$\lambda(x,y) \le \lambda(x,p) + \lambda(p,q) + \lambda(q,y) = \lambda(p,q),$$

and also, by symmetry, $\lambda(p,q) \le \lambda(x,y)$. Accordingly, we may define $\hat{\lambda} : \widehat{\Omega} \times \widehat{\Omega} \to \mathbb{R}$ by setting $\hat{\lambda}([x],[y]) = \lambda(x,y)$ for $x, y \in \Omega$. In this way, as is well known, we get a metric space $(\widehat{\Omega}, \hat{\lambda})$, which we refer to as the quotient metric space of (Ω, λ).

Theorem 2.5. *Let $u \in \mathcal{S}^-$, $B \in \widehat{\Omega}$ and $y, z \in B$. Then*

$$d(x,y) + u(y) = d(x,z) + u(z) \quad \text{for } x \in \Omega. \tag{2.3}$$

In particular, we have

$$\mathrm{I}_B(u)(x) = d(x,y) + u(y) \quad \text{for } x \in \Omega. \tag{2.4}$$

Proof. Let $y, z \in B$. Using the triangle inequality for d and the fact that $u \in \mathcal{S}^-$, we get

$$d(x,y)+u(y) \le d(x,z)+d(z,y)+u(z)+d(y,z)=d(x,z)+u(z) \quad \text{for } x \in \Omega.$$

By symmetry, we also have

$$d(x,z)+u(z) \le d(x,y)+u(y) \quad \text{for } x \in \Omega.$$

Consequently, we find that (2.3) is valid. Finally, (2.4) is an obvious consequence of (2.3). $\square$

Corollary 2.6. *Let $u \in \mathcal{S}^-$ and $B, C \subset \Omega$. Assume that C is a representation of B. Then*

$$\mathrm{I}_B(u)(x) = \inf_{X \in \widehat{B}} \mathrm{I}_X(u)(x) = \mathrm{I}_C(u)(x) \quad \textit{for } x \in \Omega.$$

Proof. Fix any $x \in \Omega$. Setting $D = \bigcup_{X \in \widehat{B}} X$, we see that $B \subset D$ and

$$\mathrm{I}_B(u)(x) \ge \mathrm{I}_D(u)(x) = \inf_{X \in \widehat{B}} \mathrm{I}_X(u)(x).$$

On the other hand, we have

$$\mathrm{I}_C(u)(x) = \inf_{y \in C}(d(x,y)+u(y)) = \inf_{y \in C} \mathrm{I}_{[y]}(u)(x) = \inf_{X \in \widehat{B}} \mathrm{I}_X(u)(x). \tag{2.5}$$

We may select a representation R of B so that $R \subset B$. Then we have $\mathrm{I}_R(u)(x) \ge \mathrm{I}_B(u)(x)$ and, by (2.5),

$$\mathrm{I}_R(u)(x) = \inf_{X \in \widehat{B}} \mathrm{I}_X(u)(x) = \mathrm{I}_C(u)(x).$$

Combining these observations, we conclude that

$$\mathrm{I}_B(u)(x) = \inf_{X \in \widehat{B}} \mathrm{I}_X(u)(x) = \mathrm{I}_C(u)(x). \qquad \square$$

Lemma 2.7. *Let $u \in \mathcal{S}^-$. Then*

$$\sup_{x \in \Omega} |\mathrm{I}_X(u)(x) - \mathrm{I}_Y(u)(x)| \le \hat{\lambda}(X,Y) \quad \textit{for } X,\, Y \in \widehat{\Omega}.$$

Proof. Let $X, Y \in \widehat{\Omega}$. Let $p \in X$ and $q \in Y$. As in the proof of Theorem 2.5, we observe that, for $x \in \Omega$,

$$\begin{aligned} \mathrm{I}_X(u)(x) &= d(x,p)+u(p) \le d(x,q)+d(q,p)+u(q)+d(p,q) \\ &= d(x,q)+u(q)+\lambda(p,q) = \mathrm{I}_Y(u)(x)+\hat{\lambda}(X,Y). \end{aligned}$$

By symmetry, we may conclude that

$$|\mathrm{I}_X(u)(x) - \mathrm{I}_Y(u)(x)| \le \hat{\lambda}(X,Y) \quad \text{for } x \in \Omega,$$

which is the end of the proof. $\square$

Theorem 2.8. *A set $D \subset \Omega$ is a uniqueness set for A if and only if $\widehat{D}$ is dense in $\widehat{A}$.*

Given a subset Θ of the metric space $(\widehat{\Omega},\, \hat{\lambda})$, we denote by $\mathrm{Cl}\,\Theta$ the closure of Θ in $(\widehat{\Omega},\, \hat{\lambda})$.

Proof. First of all, assume that $\widehat{D}$ is dense in $\widehat{A}$, so that $\widehat{A} \subset \mathrm{Cl}\,\widehat{D}$. Fix any $u \in \mathcal{S}(A)$ and $x \in \Omega$. Then we have

$$\mathrm{I}_A(u)(x) = \inf_{X\in\widehat{A}} \mathrm{I}_X(u)(x) \geq \inf_{X\in\,\mathrm{Cl}\,\widehat{D}} \mathrm{I}_X(u)(x).$$

By Lemma 2.7, the map $X \mapsto \mathrm{I}_X(u)(x)$ from $(\widehat{\Omega}, \hat{\lambda})$ to $\mathbb{R}$ is Lipschitz continuous, and consequently we have

$$\inf_{X\in\,\mathrm{Cl}\,\widehat{D}} \mathrm{I}_X(u)(x) = \inf_{X\in\widehat{D}} \mathrm{I}_X(u)(x) = \mathrm{I}_D(u)(x).$$

Accordingly, we get $u(x) \leq \mathrm{I}_D(u)(x) \leq \mathrm{I}_A(u)(x) = u(x)$, which shows that $\mathrm{I}_D(u)(x) = u(x)$. Since $x \in \Omega$ and $u \in \mathcal{S}(A)$ are arbitrary, we conclude that D is a uniqueness set for A.

Next, we assume that $\widehat{D}$ is not dense in $\widehat{A}$. There are a $Y \in \widehat{A}$ and an $\varepsilon > 0$ such that $\hat{\lambda}(Y, Z) \geq \varepsilon$ for all $Z \in \widehat{D}$. This implies that

$$d(y, z) + d(z, y) \geq \varepsilon \quad \text{for all } y \in Y \text{ and } z \in D.$$

Fix any $y \in Y \cap A$. Recall that $u := d(\cdot, y) \in \mathcal{S}(A)$. The above inequality now reads $u(y) = 0 < \varepsilon \leq \mathrm{I}_D(u)(y)$, from which we conclude that D is not a uniqueness set for A. The proof is complete. □

Let $\widehat{A}_0$ denote the set of all isolated points of $\widehat{A}$. That is, $X \in \widehat{A}_0$ if and only if there exists an $r > 0$ such that $B(X, r) \cap \widehat{A} = \{X\}$, where $B(X, r)$ denotes the closed ball $\{Y \in \hat{\Omega} : \hat{\lambda}(Y, X) \leq r\}$.

Lemma 2.9. *If $\Theta \subset \widehat{A}$ is dense in $\widehat{A}$ and X is an isolated point of Θ, then $X \in \widehat{A}_0$.*

Proof. Fix an $\varepsilon > 0$ so that $\Theta \cap B(X,\, \varepsilon) = \{X\}$. Since Θ is dense in $\widehat{A}$, it is easily seen that $\widehat{A} \cap B(X,\, \varepsilon/2) = \{X\}$. Thus, $X \in \widehat{A}_0$. □

Lemma 2.10. *Let $\Theta \subset \widehat{A}$ be a dense subset of $\widehat{A}$. Assume that Θ is minimal, with respect to set inclusion, among all dense subsets of $\widehat{A}$. Then every point of Θ is an isolated point of Θ.*

Proof. We argue by contradiction. Suppose that there were a limit point X of Θ. Then the closure of $\Theta \setminus \{X\}$ contains Θ and hence is identical to $\widehat{A}$. This contradicts the minimality of Θ. □

Let $\mathcal{D}$ denote the collection of all dense subsets of $\widehat{A}$.

Theorem 2.11. *The following three conditions are equivalent.*
(1) $\widehat{A}_0 \in \mathcal{D}$.
(2) *There exists a minimal element, with respect to set inclusion, of* $\mathcal{D}$.
(3) $\widehat{A}_0$ *is a minimum element, with respect to set inclusion, of* $\mathcal{D}$.

Proof. First of all, we assume that (1) is satisfied. Let $X \in \widehat{A}_0$. Since X is an isolated point of $\widehat{A}$, we have $X \notin \mathrm{Cl}\,(\widehat{A}_0 \setminus \{X\})$ and hence $\mathrm{Cl}\,(\widehat{A}_0 \setminus \{X\}) \neq \hat{A}$. This guarantees that the closure of any proper subset of Θ of $\widehat{A}_0$ is a proper subset of $\widehat{A}$. Accordingly, the set $\widehat{A}_0$ is minimal in $\mathcal{D}$. That is, (2) is valid.

Next, let $\Theta \in \mathcal{D}$ be a minimal element of $\mathcal{D}$. By Lemmas 2.10 and 2.9, we see that $\Theta \subset \widehat{A}_0$. Hence, $\widehat{A}_0$ is a dense subset of $\widehat{A}$ and, by the previous argument, $\widehat{A}_0$ is a minimal element of $\mathcal{D}$. The minimality of Θ now implies that $\Theta = \widehat{A}_0$. From these observations we conclude that (2) implies (3) and also that (3) implies (1). □

Corollary 2.12. *There exists a minimal uniqueness set for A if and only if $\widehat{A}_0$ is dense in $\widehat{A}$. Under the hypothesis that $\widehat{A}_0$ is dense in $\widehat{A}$, $D \subset A$ is a minimal uniqueness set for A if and only if D represents $\widehat{A}_0$.*

Proof. It is easy to see that, given a set $\Theta \subset \widehat{\Omega}$, $B \subset \Omega$ is minimal among those $C \subset \Omega$ which satisfy $\widehat{C} = \Theta$ if and only if B represents Θ. Using this and Theorem 2.11, we find that $D \subset \Omega$ is a minimal uniqueness set for A if and only if $\widehat{D}$ is a minimal element of $\mathcal{D}$ and D represents $\widehat{D}$.

Now, let $D \subset \Omega$ be a minimal uniqueness set for A. Then $\widehat{D}$ is a minimal element of $\mathcal{D}$ and D is a representation of $\widehat{D}$. Theorem 2.11 now guarantees that $\widehat{A}_0$ is dense in $\widehat{A}$ and D represents $\widehat{A}_0(= \widehat{D})$.

Next, assume that $\widehat{A}_0 \in \mathcal{D}$. Then $\widehat{A}_0$ is a minimum of $\mathcal{D}$ by Theorem 2.11. We may choose a representation $D \subset A$ of $\widehat{A}_0$. An immediate consequence is that $\widehat{D} = \widehat{A}_0$. Hence, D is a minimal uniqueness set for A. The proof is complete. □

3. EXAMPLES

In this section, we provide two examples, in the theme of Hamilton-Jacobi equations, to illustrate our results. In the following, we let $n = 1$ and $-\infty < a < b < \infty$.

Example 3.1. We consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = 0 \quad \text{in } (a,\, b), \tag{3.1}$$

where $H : [a,\, b] \times \mathbb{R} \to \mathbb{R}$ is the Hamiltonian given by $H(x, p) = |p|^2 - f(x)$ in $[a,\, b] \times \mathbb{R}$, f is a continuous nonnegative function on $[a,\, b]$ and Du denotes the derivative $\mathrm{d}u(x)/\,\mathrm{d}x$. We assume that $\min_{[a,\, b]} f = 0$, and set $A = \{x \in [a,\, b] : f(x) = 0\}$ and

$$d(x, y) = \Big| \int_y^x \sqrt{f(t)}\, dt \Big| \qquad \text{for } x, y \in [a,\, b],$$

According to [12] (see also [6, 7, 8]), A is the Aubry set associated with the state constraint problem for (3.1) and, if $u \in C([a,\, b])$ is a viscosity solution of the state constraint problem for (3.1), then

$$u(x) = \inf\{d(x, y) + u(y) : y \in A\} \quad \text{for } x \in [a,\, b]. \tag{3.2}$$

It is easily checked that d is a pseudo metric on $[a,\, b]$, which implies that $\lambda(x, y) := d(x, y) + d(y, x) = 2d(x, y)$ for $x, y \in [a,\, b]$. As in the previous section, we denote by $[x]$ the equivalence class $\{y \in [a,\, b] : d(y, x) = 0\}$ represented by $x \in [a,\, b]$ and set $\widehat{A} = \{[x] : x \in A\}$.

Consider the case where $A = \bigcup_{j=1}^N I_j$, with $N \in \mathbb{N}$ and the I_j's being mutually disjoint, closed intervals. We see easily that $\widehat{A} = \{I_j : j = 1, 2, ..., N\}$. By Corollary 2.12, we conclude that any set $\{x_j : j = 1, 2, ..., N\}$, with $x_j \in I_j$ for $j = 1, 2, ..., N$, is a minimal uniqueness set for A.

Consider next the case where $a = 0$, $b = 1$ and A is the Cantor set. Since the Cantor set A is totally disconnected, we have $[x] = \{x\}$ for all $x \in [a,\, b]$. Furthermore, since the Cantor set is perfect, there is no isolated point of $\widehat{A}$. Hence, by Corollary 2.12, there is no minimal uniqueness set for A.

Example 3.2. We consider the Hamilton-Jacobi equation (3.1), with the Hamiltonian H replaced by $H(x, p) = (|p|^2 - 1)_+ - f(x)$ for $(x, p) \in [a,\, b] \times \mathbb{R}$, where t_+ denotes the positive part of $t \in \mathbb{R}$; i.e., $t_+ = \max\{t, 0\}$ and f is a function on $[a,\, b]$ having the same properties as the one in the previous example. Also, A denotes the set defined as in the previous example. Here, the function d on $[a,\, b]^2$ is defined by

$$d(x, y) = \Big| \int_y^x \sqrt{1 + f(t)}\, dt \Big|.$$

It is easily seen that the function d is a metric on $[a,\, b]$ which induces the same topology as the Euclidean distance. Hence, we have $[x] = \{x\}$ for all $x \in [a,\, b]$. Moreover, if we denote by A_0 the set of all isolated points, in the

Euclidean distance, of A, then A has a minimal uniqueness set if and only if A equals the closure, in the Euclidean distance, of A_0. In particular, if we choose $A = [a, b]$, then there is no minimal set for A. If we choose A to be a set of a finite points of $[a, b]$, then A is the minimum uniqueness set for A.

In the above examples we have taken up the state constraint problem because of a simple presentation. In the case of the Dirichlet problem for (3.1), if we set $A = \{x \in (a, b) : f(x) = 0\} \cup \{a, b\}$, then we have formula (3.2) for any viscosity solution $u \in C([a, b])$ of (3.1). The Aubry set associated with the Dirichlet problem for (3.1) may be defined as the set $\{x \in (a, b) : f(x) = 0\}$. This observation on formula (3.2) for the solutions of the Dirichlet problem is irrelevant if one takes the Dirichlet condition in the classical sense or in the viscosity sense.

REFERENCES

1. M. Bardi and I. Capuzzo Dolcetta, "Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations," Birkhäuser, Boston, 1997.
2. G. Barles, "Solutions de viscosité des équations de Hamilton-Jacobi," Mathématiques & Applications (Berlin), 17, Springer-Verlag, Paris, 1994.
3. P. Bernard, *The asymptotic behaviour of solutions of the forced Burgers equation on the circle*, Nonlinearity **18** (2005), 101–124.
4. F. Camilli and A. Cesaroni, *A note on singular perturbation problems via Aubry-Mather theory*, Discrete Contin. Dyn. Syst. **17** (2007), 807–819.
5. A. Fathi, *Théoreme KAM et théorie de Mather pour les systèmes lagrangiens*, C.R. Acad. Sci. Paris Sér. I **324** (1997), 1034–1046.
6. A. Fathi, *Weak KAM theorem in Lagrangian dynamics*, Preliminary version No. 10.
7. A. Fathi and A. Siconolfi, *Existence of C^1 critical subsolutions of the Hamilton-Jacobi equation*, Invent. Math. **155** (2004), no. 2, 363–388.
8. A. Fathi and A. Siconolfi, *PDE aspects of Aubry-Mather theory for quasiconvex Hamiltonians*, Calc. Var. Partial Differential Equations **22** (2005), no. 2, 185–228.
9. W. H. Fleming and R. W. Rishel, "Deterministic and Stochastic Optimal Control," Applications of Mathematics, No. 1. Springer-Verlag, Berlin-New York, 1975.
10. Y. Fujita, *On minimal determining sets for Hamilton-Jacobi equations in Euclidean n space*, International conference for the 25th anniversary of viscosity solutions (Tokyo, 2006), pp. 89–95, GAKUTO Internat. Ser. Math. Sci. Appl. 30, Gakkōtosho Co., Ltd., Tokyo, 2008.
11. H. Ishii, *Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean n space*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), no. 2, 231–266.
12. H. Ishii and H. Mitake, *Representation formulas for solutions of Hamilton-Jacobi equations*, Indiana Univ. Math. J. **56** (2007), 2159–2184.
13. P.-L. Lions, "Generalized Solutions of Hamilton-Jacobi Equations," Research Notes in Mathematics, Vol. 69. Pitman (Advanced Publishing Program), Boston, London, 1982.