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EXISTENCE FOR CRITICAL HÉNON-TYPE EQUATIONS

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Abstract. This paper is concerned with the existence of a nontrivial solution for

$$-\Delta u = \lambda u + |x|^\alpha |u|^{2^*-2}u, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega, \tag{0.1}$$

where $\lambda > 0$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain. Let λ_k, $k = 1, 2, \ldots$, be eigenvalues of the operator $-\Delta$; we show for $\lambda_k < \lambda < \lambda_{k+1}$ that problem (0.1) possesses at least a solution and each λ_k is a bifurcation point.

1. Introduction

In this paper, we consider the existence of solutions of the following problem,

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha |u|^{2^*-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $\Omega \subset \mathbb{R}^n$ is smooth bounded domain, and $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent.

In [10], M. Hénon proposed the following problem,

$$-\Delta u = |x|^\alpha u^{p-1}, \quad u > 0, \quad x \in B_1(0); \quad u = 0, \quad x \in \partial B_1(0), \tag{1.2}$$

with $n \geq 3$, $\alpha > 0$, and $p > 1$, which stems from the study of rotating stellar structures and is called the Hénon equation. Such a problem has been extensively studied; see for instance [6, 15, 16, 17], etc. Interesting phenomena of problem (1.2) revealed recently include, among other things, that the exponent α affects the critical exponent for the existence of solutions. To be precise, it was shown in [15] that $2^*(\alpha) = \frac{2(n+\alpha)}{n-2}$ is the critical exponent for the existence of solutions for (1.2) if $\alpha \neq 0$. Moreover, the moving-plane

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method in [9] cannot be applied due to the fact that r^α is increasing. Actually, ground-state solutions of (1.2) concentrate at a boundary point of $B_1(0)$ for either $p \to 2^*$ or $\alpha \to \infty$; see [6] and [8].

If $\alpha = 0$, problem (1.1) has been widely considered. In particular, it is proved in [1] and [5] that problem (1.1) has at least a positive solution if $0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$, and then in [7], the authors showed that problem (1.1) possesses a solution for any $\lambda > 0$ and $n \geq 4$. The main ingredient is that the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\lambda\int_\Omega |u|^2\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx$$

associated with the problem satisfies $(PS)_c$ condition for $c \in (0, \frac{1}{n}S^{\frac{n}{2}})$, where S is the best Sobolev constant. So it is necessary to find a critical value of J in $(0, \frac{1}{n}S^{\frac{n}{2}})$. The same idea was also used for the problem

$$\begin{cases} -\Delta u = \lambda u + Q(x)|u|^{2^*-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

To control critical values of the functional related to problem (1.3), one always assumes, among other things, that $Q(x)$ has an interior maximum point in Ω. However, $|x|^\alpha$ attains its maximum on $\partial\Omega$, it causes difficulties in handling problem (1.1).

We showed in [11] that problem (1.1) has a positive solution for $\lambda < \lambda_1$. In this paper, we assume $\lambda > 0$. Without loss of generality, we assume that $(1, 0, \ldots, 0) \in \partial\Omega$ and Ω is contained in the unit ball of $\mathbb{R}^n$ centered at the origin. Let λ_k, $k = 1, 2, \ldots$, be eigenvalues of the operator $-\Delta$. Denote by φ_k the eigenfunction corresponding to λ_k. Our main result is as follows.

Theorem 1.1. *If $n \geq 7$ and α is small enough, problem* (1.1) *possesses at least one nontrivial solution for any $\lambda \neq \lambda_k$.*

The solution u of problem (1.1) obtained in Theorem 1.1 is sign-changing when $\lambda > \lambda_1$. Indeed, were it not the case, say $u \geq 0$, multiplying the equation in problem (1.1) by the eigenfunction φ_1 corresponding to λ_1 and integrating by parts, we obtain

$$(\lambda - \lambda_1)\int_\Omega u\varphi_1\,dx + \int_\Omega |x|^\alpha u^{2^*-1}\varphi_1\,dx = 0,$$

which is a contradiction.

If $\lambda = \lambda_k$, $k = 1, 2, \ldots$, using the theory of bifurcation for variational problems in [4] and [13], we have the following result.

Theorem 1.2. *Every eigenvalue λ_k of $-\Delta$ gives rise to a bifurcation point $(\lambda_k, 0)$ of problem* (1.1).

Theorem 1.1 is proved in Sections 3 and 4, and we show Theorem 1.2 in Section 5.

2. Preliminaries

We denote by $E = H_0^1(\Omega)$ the usual Sobolev space with the norm $\|\cdot\|$. To find solutions of (1.1) with $\lambda_k < \lambda < \lambda_{k+1}$, we decompose the space E as $E = E^+ \oplus E^-$, where E^+ is the subspace of E spanned by eigenfunctions corresponding to λ_j with $\lambda_j \geq \lambda_{k+1}$, and E^- is defined as the subspace of E spanned by eigenfunctions corresponding to λ_j with $\lambda_j \leq \lambda_{k+1}$. Then, we will look for critical points of the functional related to (1.1) by the following critical-point theorem, which is a modification of that in [2, Theorem 2.4].

Proposition 2.1. *Let H be a real Hilbert space and $f \in C^1(H; R)$ be a functional satisfying the following conditions:*

(1) *$f(u) = f(-u)$ for all $u \in H$ and $f(0) = 0$;*
(2) *There exists $\beta > 0$ such that f satisfies the $(PS)_c$ condition for $c \in (0, \beta)$;*
(3) *There exist two closed subspaces $V, W \subset H$, with $\operatorname{codim} V < \infty$ and constants $\rho > 0$ and $\delta > 0$ such that*

$$f(u) < \beta \quad \text{for all } u \in W$$

and

$$f(u) \geq \delta \quad \text{for all } u \in V \text{ with } \|u\| = \rho.$$

Then there exist at least m pairs of critical points, where

$$m = \dim(V \cap W) - \operatorname{codim}(V + W).$$

We recall that $f \in C^1(H, \mathbb{R})$ satisfies the Palais-Smale condition at level c ($(PS)_c$ condition for short) if every sequence $u_m \subset H$ such that $f(u_m) \to c$ and $f'(u_m) \to 0$ in H is relatively compact in H.

In the case $\lambda = \lambda_k$, we will consider the bifurcation phenomenon of problem (1.1) with λ being a parameter, which is based on a framework in [4] and [12]; see also Theorem 11.4 in [13]. We now state it as follows.

Proposition 2.2. *Suppose $\mathcal{H}$ is a real Hilbert space and $\Phi \in C^2(\mathcal{H}; R)$ with $D\Phi'(u) = Lu + H(u)$, where $L \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ is symmetric and $H(u) = o(\|u\|)$*

as $u \to 0$. *If* $\mu \in \sigma(L)$ *is an isolated eigenvalue of finite multiplicity, then* $(\mu, 0)$ *is a bifurcation point for*

$$\mathcal{F}(\lambda, u) \equiv D\Phi'(u) - \lambda u = Lu + H(u) - \lambda u.$$

Moreover there is an $r_0 > 0$ *such that*

(1) *For each* $r \in r(0, r_0)$ *there exist at least two distinct solutions* $(\lambda_i(r), u_i(r))$, $i = 1, 2$, *of* $\mathcal{F} = 0$ *having* $\|u_i\| = r$ *and* $|\lambda_i - \mu|$ *small;*
(2) *As* $r \to 0$, $(\lambda_i(r), u_i(r)) \to (\mu, 0)$.

3. Palais-Smale condition

In this and the next sections, we assume $\lambda_k < \lambda < \lambda_{k+1}$ for a fixed k. We will show that (1.1) possesses at least one solution. Define on E the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\lambda\int_\Omega |u|^2\,dx - \frac{1}{2^*}\int_\Omega |x|^\alpha |u|^{2^*}\,dx$$

related to (1.1). It is standard to show that I_λ is a C^1 functional.

Lemma 3.1. *Let* $\{u_m\} \subset E$ *be a* $(PS)_c$ *sequence of* I_λ*; then* $\{u_m\}$ *is bounded in* E.

Proof. Suppose for the sake of contradiction that $\|u_m\| \to \infty$ as $m \to \infty$. We derive from

$$I_\lambda(u_m) = c + o(1) \quad \text{and} \quad I'_\lambda(u_m) \to 0 \tag{3.1}$$

that

$$c + o(1) = I_\lambda(u_m) - \tfrac{1}{2}\langle I'_\lambda(u_m), u_m\rangle = (\tfrac{1}{2} - \tfrac{1}{2^*})\int_\Omega |x|^\alpha |u_m|^{2^*}\,dx. \tag{3.2}$$

Thus, for $\varphi \in E$,

$$\begin{aligned} |\int_\Omega |x|^\alpha |u_m|^{2^*-2} u_m \varphi\,dx| &\le \Big(\int_\Omega |x|^{\alpha\cdot\frac{2^*}{2^*-1}} |u_m|^{2^*}\,dx\Big)^{\frac{2^*-1}{2^*}} \Big(\int_\Omega |x|^\alpha \varphi^{2^*}\,dx\Big)^{\frac{1}{2^*}} \\ &\le C(c + o(1))\|\varphi\|. \end{aligned} \tag{3.3}$$

The fact that $I'_\lambda(u_m) \to 0$ leads to

$$\int_\Omega \frac{\nabla u_m}{\|u_m\|}\nabla\varphi\,dx - \lambda\int_\Omega \frac{u_m}{\|u_m\|}\varphi\,dx - \int_\Omega |x|^\alpha \frac{|u_m|^{2^*-2}u_m}{\|u_m\|}\varphi\,dx = o(1).$$

Let $v_m = \frac{u_m}{\|u_m\|}$ and assume $v_m \rightharpoonup v_0$; we get by letting $m \to \infty$ that

$$-\Delta v_0 = \lambda v_0 \quad \text{in} \quad \Omega.$$

We claim that $v_0 \not\equiv 0$. In fact, if $v_0 \equiv 0$, then $v_m \to 0$ in $L^2(\Omega)$. On the other hand, by (3.2) we have

$$\left(\tfrac{1}{2} - \tfrac{1}{2^*}\right) \int_\Omega |\nabla v_m|^2 \, dx - \lambda \int_\Omega |v_m|^2 \, dx = o(1),$$

i.e.,

$$\lim_{m\to\infty} \int_\Omega v_m^2 \, dx = \frac{1}{\lambda} \left(\tfrac{1}{2} - \tfrac{1}{2^*}\right) > 0.$$

This is a contradiction. So v_0 is a solution of the problem

$$\begin{cases} -\triangle v = \lambda v, & x \in \Omega \\ v = 0, & x \in \partial\Omega. \end{cases}$$

This is impossible since $\lambda \neq \lambda_k$. □

Lemma 3.2. *The functional I satisfies the $(PS)_c$ condition for $c \in (0, \frac{1}{n} S^{\frac{n}{2}})$, where*

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} u^{2^*} \, dx\right)^{\frac{2}{2^*}}}.$$

Proof. Let $\{u_m\} \subset E$ be such that $I_\lambda(u_m) \to c$ and $I'_\lambda(u_m) \to 0$ in E. By Lemma 3.1, $\{u_m\}$ is bounded in E. So we may assume that

$$\begin{aligned} u_m &\rightharpoonup u \ \text{ in } H_0^1(\Omega), \\ u_m &\to u \ \text{ in } L^2(\Omega), \\ u_m &\to u \ \text{ a.e. in } \Omega. \end{aligned}$$

Since $\{u_m\}$ is bounded in $L^{2^*}(\Omega)$, $\{u_m^{2^*-1}\}$ is bounded in $L^{\frac{2n}{n+2}}(\Omega)$ and

$$|x|^\alpha |u_m|^{2^*-2} u_m \rightharpoonup |x|^\alpha |u|^{2^*-2} u,$$

the limit function u is a solution of equation (1.1).

Setting $v_m = u_m - u$ and using Brézis–Lieb's lemma [3], we have

$$\int_\Omega |x|^\alpha |u_m|^{2^*} \, dx = \int_\Omega |x|^\alpha |u_m - u|^{2^*} \, dx + \int_\Omega |x|^\alpha |u|^{2^*} \, dx + o(1),$$

and then

$$I_\lambda(u_m) = I_\lambda(u) + I_0(u_m - u) + o(1), \tag{3.4}$$

where

$$I_0(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^*} \int_\Omega |x|^\alpha |u|^{2^*} \, dx.$$

Equation (3.4) together with

$$\langle u_m - u, I'_\lambda(u_m) \rangle = \int_\Omega |\nabla (u_m - u)|^2 \, dx - \int_\Omega |x|^\alpha |u_m - u|^{2^*} \, dx + o(1)$$

yield

$$I_0(u_m - u) = \frac{1}{n}\int_\Omega |\nabla(u_m - u)|^2\,dx + o(1) = \frac{1}{n}\|v_m\|^2.$$

Since u is a solution of problem (1.1), $I_\lambda(u) \geq 0$. Thus,

$$\frac{1}{n}\|v_m\|^2 = I_\lambda(u_m) - I_\lambda(u) + o(1) \leq I_\lambda(u_m) + o(1) = c + o(1) < \frac{1}{n}S^{\frac{n}{2}}$$

for $m \geq m_0$. By the Sobolev inequality and the fact that $\Omega \subset B_1(0)$,

$$\begin{aligned}\|v_m\|^2(1 - S^{-\frac{2^*}{2}}\|v_m\|^{2^*-2}) &\leq \int_\Omega |\nabla(u_m - u)|^2\,dx - \int_\Omega |u_m - u|^{2^*}\,dx \\ \leq \int_\Omega |\nabla(u_m - u)|^2\,dx &- \int_\Omega |x|^\alpha |u_m - u|^{2^*}\,dx = o(1), \qquad (3.5)\end{aligned}$$

which shows $u_m \to u$ in E. □

4. Proof of Theorem 1.1

To prove Theorem 1.1, we need to verify the conditions in Proposition 2.1. In the previous section, we showed that the $(PS)_c$ condition holds locally. In this section, we verify the rest of the conditions of Proposition 2.1.

Lemma 4.1. *There exist constants $\rho > 0$ and $\gamma > 0$ such that $I_\lambda(u) \geq \gamma$ for $u \in E^+$ with $\|u\|_E = \rho$.*

Proof. For $u \in V$, we have

$$\begin{aligned}I_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\lambda\int_\Omega |u|^2\,dx - \frac{1}{2^*}\int_\Omega |x|^\alpha |u|^{2^*}\,dx \\ &\geq \frac{1}{2}(1 - \frac{\lambda}{\lambda_{k+1}})\|u\|^2 - \frac{1}{2^*}\int_\Omega |x|^\alpha |u|^{2^*}\,dx \\ &\geq \frac{1}{2}(1 - \frac{\lambda}{\lambda_{k+1}})\|u\|^2 - C\|u\|^{2^*} \geq \gamma > 0\end{aligned}$$

if $\|u\|_E = \rho$ is small. □

Now, we will construct W in Proposition 2.1 and show that

$$\sup_W I_\lambda(u) < \frac{1}{n}S^{\frac{n}{2}}.$$

To this purpose, we choose $x_0 = (1 - \frac{1}{2}\sqrt[4]{\varepsilon}, 0, \ldots, 0)$. It is known from [14] that the function

$$U_\varepsilon(x) = \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x - x_0|^2]^{\frac{n-2}{2}}}$$

is a minimizer of the best Sobolev constant S. Let $\xi \in C_0^\infty(\Omega)$ be such that

$$\xi(x) = \begin{cases} 1, & x \in B_{\frac{1}{4}\sqrt[4]{\varepsilon}}(x_0), \\ 0, & \mathbb{R}^n \setminus B_{\frac{1}{2}\sqrt[4]{\varepsilon}}(x_0), \end{cases}$$

and $0 \le \xi(x) \le 1$, $|\nabla \xi(x)| \le C\frac{1}{\sqrt[4]{\varepsilon}}$ on $\mathbb{R}^n$. Set

$$u_\varepsilon(x) = \xi(x) U_\varepsilon(x), \quad x \in \Omega; \tag{4.1}$$

then $u_\varepsilon \in E$. We have the following estimates.

Lemma 4.2. *For $\varepsilon > 0$ and $n \ge 5$, there hold*

$$\|u_\varepsilon\|^2 = S^{\frac{n}{2}} + O(\varepsilon^{\frac{3n}{4}-\frac{3}{2}}), \tag{4.2}$$

$$\int_\Omega |x|^\alpha |u_\varepsilon|^{2^*}\, dx \ge (1 - \sqrt[4]{\varepsilon})^\alpha (S^{\frac{n}{2}} + O(\varepsilon^{\frac{3n}{4}})), \tag{4.3}$$

$$\int_\Omega |u_\varepsilon|^2\, dx \ge C\varepsilon^2 + O(\varepsilon^{\frac{3n}{4}-1}), \tag{4.4}$$

$$\int_\Omega |u_\varepsilon|\, dx \le C\varepsilon^{\frac{n-1}{2}}, \tag{4.5}$$

$$\int_\Omega |u_\varepsilon|^{2^*-1}\, dx \le C\varepsilon^{\frac{n-2}{2}}, \tag{4.6}$$

and

$$\frac{\|\nabla u_\varepsilon\|_2^2 - \lambda \|u_\varepsilon\|_2^2}{(\int_\Omega |x|^\alpha |u_\varepsilon|^{2^*}\, dx)^{\frac{2}{2^*}}} = \frac{1}{(1-\sqrt[4]{\varepsilon})^{\frac{2\alpha}{2^*}}}(S - C\varepsilon^2 + O(\varepsilon^{\frac{3}{4}n-\frac{3}{2}})). \tag{4.7}$$

Proof. It follows from [18] that

$$\int_{\mathbb{R}^n} |\nabla U_\varepsilon|^2\, dx = \int_{\mathbb{R}^n} |U_\varepsilon|^{2^*}\, dx = S^{\frac{n}{2}}.$$

Direct computation shows that

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx = \int_{\mathbb{R}^n} |\nabla U_\varepsilon|^2\, dx + O(\varepsilon^{\frac{3}{4}n-\frac{3}{2}});$$

i.e., (4.2) holds. Also, by direct computation, one can get

$$\int_\Omega |u_\varepsilon|^{2^*}\, dx = \int_{\mathbb{R}^n} |U_\varepsilon|^{2^*}\, dx + O(\varepsilon^{\frac{3}{4}n}),$$

which yields (4.3). We write

$$\int_\Omega |u_\varepsilon|^2 dx = \int_{B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{2}}}{[\varepsilon^2 + |x-x_0|^2]^{n-2}}\, dx$$
$$+ \int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} \frac{\xi^2[n(n-2)\varepsilon^n]^{\frac{n-2}{2}}}{[\varepsilon^2 + |x-x_0|^2]^{n-2}}\, dx,$$

and estimate

$$\Big|\int_{B_{\frac{1}{2}\sqrt[4]{\varepsilon}(x_0)} \setminus B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} \frac{\xi^2[n(n-2)\varepsilon^2]^{\frac{n-2}{2}}}{[\varepsilon^2 + |x-x_0|^2]^{n-2}}\, dx\Big|$$
$$\le C\varepsilon^{n-2} \int_{\frac{1}{4}\sqrt[4]{\varepsilon}}^{\frac{1}{2}\sqrt[4]{\varepsilon}} r^{-n+3}\, dr = C\varepsilon^{\frac{3}{4}n-1},$$

and

$$\int_{B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)}} \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{2}}}{[\varepsilon^2 + |x-x_0|^2]^{n-2}}\, dx$$
$$\ge \int_{B_\varepsilon(x_0)} \frac{C\varepsilon^{n-2}}{[2\varepsilon^2]^{n-2}}\, dx + \int_{B_{\frac{1}{4}\sqrt[4]{\varepsilon}(x_0)} \setminus B_\varepsilon(x_0)} \frac{C\varepsilon^{n-2}}{[2|x-x_0|^2]^{n-2}}\, dx$$
$$= C\varepsilon^2 + O(\varepsilon^{\frac{3n}{4}-1}).$$

Whence we obtain (4.4), i.e.,

$$\int_\Omega |u_\varepsilon|^2\, dx \ge C\varepsilon^2 + O(\varepsilon^{\frac{3n}{4}-1}).$$

Similarly, (4.5) holds. In addition, we have

$$\int_\Omega |u_\varepsilon|^{2^*-1} dx \le C\int_0^\varepsilon \Big(\frac{\varepsilon}{\varepsilon^2 + r^2}\Big)^{\frac{n+2}{2}} r^{n-1}\, dr + C\int_\varepsilon^{\frac{1}{2}\sqrt[4]{\varepsilon}} \Big(\frac{\varepsilon}{\varepsilon^2 + r^2}\Big)^{\frac{n+2}{2}} r^{n-1}\, dr$$
$$\le C\varepsilon^{\frac{n-2}{2}};$$

i.e., (4.6) holds. From (4.2), (4.3), and (4.4), we deduce

$$\frac{|\nabla u_\varepsilon|^2 - \lambda |u_\varepsilon|^2}{(\int_\Omega |x|^\alpha |u_\varepsilon|^{2^*}\, dx)^{\frac{2}{2^*}}} \le \frac{1}{(1-\sqrt[4]{\varepsilon})^{\frac{2\alpha}{2^*}}} \frac{S^{\frac{n}{2}} + O(\varepsilon^{\frac{3n}{4}-\frac{3}{2}}) - \lambda(C\varepsilon^2 + O(\varepsilon^{\frac{3}{4}n-1}))}{(S^{\frac{n}{2}} + O(\varepsilon^{\frac{3n}{4}}))^{\frac{2}{2^*}}}$$
$$= \frac{1}{(1-\sqrt[4]{\varepsilon})^{\frac{2\alpha}{2^*}}} (S - C\varepsilon^2 + O(\varepsilon^{\frac{3}{4}n-\frac{3}{2}}));$$

i.e., (4.7) holds. □

We set $\overline{W(\varepsilon)} = \{u \in H_0^1(\Omega) : u = u^- + tu_\varepsilon,\ u^- \in E^-,\ t \in \mathbb{R}\}$.

Lemma 4.3. *If $u \in \overline{W(\varepsilon)}$, then for any $\varepsilon > 0$*

$$\int_\Omega |x|^\alpha |u|^{2^*}\,dx \geq \int_\Omega |x|^\alpha |tu_\varepsilon|^{2^*}\,dx + \frac{1}{2}\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx - Ct^{2^*}\varepsilon^{\frac{n(n-2)}{n+2}}. \quad (4.8)$$

Proof. Using the identity

$$\int_\Omega |x|^\alpha |u|^{2^*} dx = 2^* \int_\Omega \int_0^u |x|^\alpha |s|^{2^*-2} s\,ds\,dx,$$

we derive

$$\begin{aligned} &\Big|\int_\Omega |x|^\alpha |u|^{2^*}\,dx - \int_\Omega |x|^\alpha |tu_\varepsilon|^{2^*}\,dx - \int_\Omega |x|^\alpha |u^-|^{2^*}\,dx\Big| \quad (4.9)\\ &= 2^* \int_0^1 d\tau \int_\Omega |x|^\alpha [|tu_\varepsilon + \tau u^-|^{2^*-2}(tu_\varepsilon + \tau u^-) - |\tau u^-|^{2^*-2}\tau u^-]u^-\,dx\\ &= 2^*(2^*-1)\int_0^1 d\tau \int_\Omega |x|^\alpha |\theta tu_\varepsilon + \tau u^-|^{2^*-2} tu_\varepsilon u^-\,dx, \end{aligned}$$

where $\theta = \theta(x)$ such that $0 < \theta < 1$. Using Hölder's inequality, Young's inequality, and the fact that norms in finite-dimensional space E^- are equivalent, we deduce from (4.5), (4.6), and (4.9) that

$$\begin{aligned} &\Big|\int_\Omega |x|^\alpha |u|^{2^*}\,dx - \int_\Omega |x|^\alpha |tu_\varepsilon|^{2^*}\,dx - \int_\Omega |x|^\alpha |u^-|^{2^*}\,dx\Big| \quad (4.10)\\ &\leq C\Big(\int_\Omega |x|^\alpha |u^-||tu_\varepsilon|^{2^*-1}\,dx + \int_\Omega |x|^\alpha |u^-|^{2^*-1}|tu_\varepsilon|\,dx\Big)\\ &\leq C\big(\|u^-\|_\infty \|tu_\varepsilon\|_{2^*-1}^{2^*-1} + \|u^-\|_\infty^{2^*-1}\|tu_\varepsilon\|_1\big)\\ &\leq C\Big[\|tu_\varepsilon\|_{2^*-1}^{2^*-1}\Big(\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx\Big)^{\frac{1}{2^*}} + \|tu_\varepsilon\|_1\Big(\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx\Big)^{\frac{2^*-1}{2^*}}\Big]\\ &\leq Ct^{2^*-1}\|u_\varepsilon\|_{2^*-1}^{2^*-1}\Big(\int_\Omega |x|^\alpha |u^-|^{2^*}dx\Big)^{\frac{1}{2^*}} + \frac{1}{4}\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx + Ct^{2^*}(\varepsilon^{\frac{n-1}{2}})^{\frac{2n}{n-2}}\\ &\leq \frac{1}{2}\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx + Ct^{2^*}(\varepsilon^{\frac{n-2}{2}})^{\frac{2n}{n+2}} + Ct^{2^*}\varepsilon^{\frac{n(n-1)}{n-2}}\\ &\leq \frac{1}{2}\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx + Ct^{2^*}(\varepsilon^{\frac{n-2}{2}})^{\frac{2n}{n+2}}. \end{aligned}$$

The proof is complete. □

Lemma 4.4. *For ε and α sufficiently small and $n \geq 7$,*

$$\sup_{\overline{W(\varepsilon)}} I_\lambda(u) < \frac{1}{n} S^{\frac{n}{2}}.$$

Proof. Fix $u \in \overline{W(\varepsilon)},\ u \neq 0$; we have

$$\max_t I_\lambda(tu) = \frac{1}{n}\Big(\frac{\|u\|^2 - \lambda\|u\|_2^2}{(\int_\Omega |x|^\alpha |u|^{2^*}\,dx)^{\frac{2}{2^*}}}\Big)^{n/2}.$$

In order to prove the lemma, we need to evaluate

$$\sup_{u\in\overline{W(\varepsilon)}}\{\|u\|^2 - \lambda\|u\|_2^2 : \int_\Omega |x|^\alpha |u|^{2^*}\,dx = 1\}.$$

By (4.8),

$$1 = \int_\Omega |x|^\alpha |u|^{2^*} dx \geq t^{2^*}\int_\Omega |x|^\alpha |u_\varepsilon|^{2^*} dx - Ct^{2^*}\varepsilon^{\frac{n(n-2)}{n+2}} + \frac{1}{2}\int_\Omega |x|^\alpha |u^-|^{2^*} dx,$$

which implies that t is bounded if ε is sufficiently small. We infer that

$$\begin{aligned}
&\|u\|^2 - \lambda\|u\|_2^2\\
\leq &\|\nabla u^-\|_2^2 - \lambda\|u^-\|_2^2 + t^2\|\nabla u_\varepsilon\|_2^2 - \lambda t^2\|u_\varepsilon\|_2^2\\
&\qquad + 2\int_\Omega \big(|tu_\varepsilon||\Delta u^-| + \lambda|tu_\varepsilon||u^-|\big)\,dx\\
&\leq \|\nabla u^-\|_2^2 - \lambda\|u^-\|_2^2\\
&\qquad + \|\nabla tu_\varepsilon\|_2^2 - \lambda\|tu_\varepsilon\|_2^2 + C(\|tu_\varepsilon\|_1\|\Delta u^-\|_\infty + \lambda\|tu_\varepsilon\|_1\|u^-\|_\infty)\\
&\leq (\lambda_k - \lambda)\|u^-\|_2^2 + \frac{\|\nabla tu_\varepsilon\|_2^2 - \lambda\|tu_\varepsilon\|_2^2}{(\int_\Omega |x|^\alpha |tu_\varepsilon|^{2^*}\,dx)^{\frac{2}{2^*}}}\Big(\int_\Omega |x|^\alpha |tu_\varepsilon|^{2^*}\,dx\Big)^{\frac{2}{2^*}}\\
&\qquad + Ct\|u^-\|_2\varepsilon^{\frac{n-1}{2}}.
\end{aligned} \tag{4.11}$$

We set $A = (\lambda_k - \lambda)\|u^-\|_2^2 + C\|u^-\|_2\varepsilon^{\frac{n-1}{2}}$ and observe that $A \leq C\varepsilon^{\frac{n-1}{2}}$. We now distinguish two cases:

(a) $\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx > 2Ct^{2^*}\varepsilon^{\frac{n(n-2)}{n+2}}$,

(b) $\int_\Omega |x|^\alpha |u^-|^{2^*}\,dx \leq 2Ct^{2^*}\varepsilon^{\frac{n(n-2)}{n+2}}$.

If the case (a) prevails, $\int_\Omega |x|^\alpha |tu_\varepsilon|^{2^*}\,dx < 1$. By (4.7) and (4.11),

$$\|u\|^2 - \lambda\|u\|_2^2 \leq \frac{1}{(1-\sqrt[4]{\varepsilon})^{\frac{2\alpha}{2^*}}}(S - C\varepsilon^2 + o(\varepsilon^{\frac{3}{4}n-\frac{3}{2}})) + A,$$

whence the conclusion follows if $n \geq 5$ and $\alpha > 0$ small.

If the case (b) occurs, by Lemma 4.3 and the boundedness of t,

$$\Big(\int_\Omega |x|^\alpha |tu_\varepsilon|^{2^*} \, dx \Big)^{\frac{2}{2^*}} \leq \big(1 + C\varepsilon^{\frac{n(n-1)}{n-2}}\big)^{\frac{2}{2^*}} \leq 1 + C\varepsilon^{\frac{(n-2)^2}{n+2}}.$$

It results that

$$\|u\|^2 - \lambda \|u\|_2^2 \leq \frac{1}{(1-\sqrt[4]{\varepsilon})^{\frac{2\alpha}{2^*}}} \Big(S - C\varepsilon^2 + o(\varepsilon^{\frac{3}{4}n - \frac{3}{2}}) \Big) \Big(1 + C\varepsilon^{\frac{(n-2)^2}{n+2}} \Big) + A.$$

If $n \geq 7$ and $\alpha > 0$ is small, we obtain $\|u\|^2 - \lambda\|u\|_2^2 < S$. The assertion follows. □

Proof of Theorem 1.1. Using Proposition 2.1 with $V = E^+$ and $W = W(\varepsilon)$, we have $V \oplus W = E$ and $V \cap W \neq \emptyset$. By Lemmas 4.1 and 4.4, the $(PS)_c$ condition holds for $c \in (0, \frac{1}{n} S^{\frac{n}{2}})$; the result readily follows. □

5. BIFURCATIONS AT $\lambda = \lambda_k$

In this section, we prove Theorem 1.2.

Proof of Theorem 1.2. Let

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2}\lambda \int_\Omega |u|^2 \, dx - \frac{1}{2^*} \int_\Omega |x|^\alpha |u|^{2^*} \, dx$$

with $u \in E$. It is standard to show that $I_\lambda(u) \in C^2$. Define the operator L by

$$(Lu, \varphi) = \int_\Omega \nabla u \nabla \varphi \, dx$$

and H by

$$H(u)\varphi = \int_\Omega |x|^\alpha |u|^{2^*-2} u\varphi \, dx$$

for $u, \varphi \in E$. Hence,

$$|H(u)\varphi| \leq \int_\Omega |x|^\alpha |u|^{2^*-1} |\varphi| \, dx \leq C\|u\|^{2^*-1} \|\varphi\|.$$

This yields $\|H(u)\| = o(\|u\|)$. So by Proposition 2.2, each eigenvalue of $-\Delta$ provides a bifurcation point of

$$-\Delta u = \lambda u + |x|^\alpha |u|^{2^*-2} u. \tag{5.1}$$

□

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