

## MULTIPLE SOLUTIONS TO THE PLANAR PLATEAU PROBLEM

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**Abstract.** We give existence and non-uniqueness results for simple planar curves with prescribed geodesic curvature.

### 1. INTRODUCTION

We are interested in the planar Plateau problem: Given two points  $p_1$  and  $p_2$  in the plane and a smooth function  $k : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}$ , find an immersed curve  $\gamma \in C^2([0, 1], \mathbb{R}^2)$ , such that  $\gamma(0) = p_1$ ,  $\gamma(1) = p_2$ , and for every  $t \in [0, 1]$  the (signed) geodesic curvature  $k_\gamma(t)$  of  $\gamma$  at  $t$ ,

$$k_\gamma(t) := |\dot{\gamma}(t)|^{-3} \langle \ddot{\gamma}(t), J\dot{\gamma}(t) \rangle,$$

is given by  $k(\gamma(t), t)$ , where  $J$  denotes the rotation by  $\pi/2$ . We choose the orientation, such that the circle of radius  $r$  with counterclockwise parameterization has positive curvature  $r^{-1}$ .

Without loss of generality after a rotation and a translation we may assume that  $p_1 = (a, 0)$  and  $p_2 = (-a, 0)$  for some  $a > 0$ . Then the planar Plateau problem is equivalent to the following ordinary differential equation:

$$\ddot{\gamma} = |\dot{\gamma}|k(\gamma(t), t)J(\dot{\gamma}), \quad \gamma(0) = (a, 0), \quad \gamma(1) = (-a, 0). \quad (1.1)$$

If the function  $k \equiv k_0$  is constant, by elementary geometry, the planar Plateau problem is only solvable for  $|k_0| \leq a^{-1}$ ; the solutions in this case are given by subarcs connecting  $(a, 0)$  and  $(-a, 0)$  of  $n$ -fold iterates of a circle of radius  $|k_0|$  with clockwise or counterclockwise parameterization depending on the sign of  $k_0$ . If the analysis is restricted to simple solutions, then there are 2 solutions if  $|k_0| < a^{-1}$ , the *small* and the *large* solution corresponding to the subarcs subtending an angle strictly smaller or strictly larger than  $\pi$ . If  $k_0 = \pm a^{-1}$ , then the unique simple solution is given by the half circle lying

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above or below the  $x$ -axis depending on the sign of  $k_0$ . We will be mainly interested in the case when  $k$  is a positive function.

If the prescribed curvature function is independent of the variable  $t$ , then the planar Plateau problem is “geometric,” in the sense that the set of solutions is invariant under reparameterizations. If in this case the function  $k$  satisfies  $\|k\|_\infty < a^{-1}$ , then from [1] there exists a stable solution  $\gamma_s$  to (P). We refer to  $\gamma_s$  as a *small* solution. In the higher-dimensional case and in the context of  $H$ -surfaces analogous results are given in [10, 11]. For closed curves with prescribed curvature we refer to [6, 4, 5, 16, 17].

Concerning the existence of a second, *large* solution for non-constant functions  $k$  there are only perturbative results; i.e., the function  $k$  is assumed to be close to a constant  $k_0$  (see [1]). Concerning the existence of a large  $H$ -surface we refer to [3, 19, 18], if  $H$  is constant, and to [2, 7, 20, 15, 14, 21] for non-constant functions  $H$ .

We give existence criteria for a large solution that are non-perturbative.

**Theorem 1.1.** *Let  $a > 0$  and  $k \in C(\mathbb{R}^2 \times [0, 1], \mathbb{R})$  be given, such that*

$$0 < \inf_{\mathbb{R}^2 \times [0, 1]} k \leq \sup_{\mathbb{R}^2 \times [0, 1]} k < a^{-1};$$

*then there is a simple curve that solves (1.1). If, moreover,*

$$\frac{\sup_{(x,t) \in \mathbb{R}^2 \times [0, 1]} k(x, t)}{\sup_{(x,t) \in \mathbb{R}^2 \times [0, 1]} k(x, t)a + 1} < \inf_{(x,t) \in \mathbb{R}^2 \times [0, 1]} k(x, t), \quad (1.2)$$

*then equation (1.1) possesses at least two simple solutions.*

To illustrate the pinching condition (1.2) we note that the assumptions of Theorem 1.1 are satisfied, if

$$\frac{1}{2}a^{-1} < \inf_{\mathbb{R}^2 \times [0, 1]} k \text{ and } \sup_{\mathbb{R}^2 \times [0, 1]} k < a^{-1}.$$

The small solution is found in the set

$$\begin{aligned} M_{small} := & \left\{ \gamma \in C^2([0, 1], \mathbb{R}^2) : \gamma(0) = (a, 0), \gamma(1) = (-a, 0), \right. \\ & \gamma \oplus [-a, a] \text{ is simple, } |\dot{\gamma}(0)|^{-1} \dot{\gamma}(0) \in \{e^{i\theta} : \pi/2 < \theta < \pi\}, \\ & \left. |\dot{\gamma}(1)|^{-1} \dot{\gamma}(1) \in \{e^{i\theta} : \pi < \theta < 3\pi/2\} \right\}, \end{aligned}$$

whereas the large solution belongs to

$$M_{large} := \left\{ \gamma \in C^2([0, 1], \mathbb{R}^2) : \gamma(0) = (a, 0), \gamma(1) = (-a, 0), \right.$$

$$\begin{aligned} \gamma \oplus [-a, a] \text{ is simple, } & \frac{\dot{\gamma}(0)}{|\dot{\gamma}(0)|} \in \{e^{i\theta} : -\pi/2 < \theta < \pi\}, \\ & \frac{\dot{\gamma}(1)}{|\dot{\gamma}(1)|} \in \{e^{i\theta} : \pi < \theta < 5\pi/2\}, \text{ and} \\ & \left( \frac{\dot{\gamma}(0)}{|\dot{\gamma}(0)|} \in \{e^{i\theta} : -\pi/2 < \theta < \pi/2\} \text{ or} \right. \\ & \left. \frac{\dot{\gamma}(1)}{|\dot{\gamma}(1)|} \in \{e^{i\theta} : 3/2\pi < \theta < 5\pi/2\} \right), \end{aligned}$$

where we define for a curve  $\gamma \in C^0([0, L], \mathbb{R}^2)$  connecting  $(a, 0)$  and  $(-a, 0)$  the closed curve  $\gamma \oplus [-a, a] \in C([0, L + 2a], \mathbb{R}^2)$  by

$$\gamma \oplus [-a, a](t) := \begin{cases} \gamma(t) & 0 \leq t \leq L \\ (-a + t - L, 0) & L \leq t \leq L + 2a. \end{cases}$$

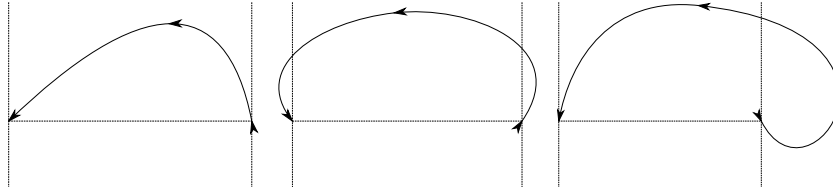


FIGURE 1. Examples of a small solution and two large solutions

The existence result is proved by using the Leray-Schauder degree and suitable a priori estimates; i.e., we show that the degree of (1.1) with respect to  $M_{small}$  equals 1 and is given by  $-1$ , when computed in the set  $M_{large}$ . The existence result then follows, since a non-vanishing degree gives rise to a solution. The degree approach is interesting in itself and yields the flexibility to deal with functions  $k$  that depend on  $x$  and  $t$ , for instance if  $k$  does only depend on  $t$ , then the existence result shows that in contrast to the four-vertex theorem for simple closed curves of prescribed curvature (see [8, 9]) there is no additional condition on  $k$  besides the  $L^\infty$ -bound for the corresponding boundary value problem. Moreover, the degree argument gives the perspective to be applied to the higher-dimensional case as well, e.g., to surfaces in  $\mathbb{R}^3$  with prescribed mean curvature.

## 2. A PRIORI ESTIMATES

**Lemma 2.1.** *Let  $\gamma \in C^2([0, L], \mathbb{R}^2)$  be a unit speed curve with positive geodesic curvature connecting  $(a, 0)$  and  $(-a, 0)$ , such that the closed curve  $\gamma \oplus [-a, a] \in C([0, L + 2a], \mathbb{R}^2)$  is simple. If*

$$\begin{aligned}\dot{\gamma}(0) &= e^{i\theta_0} \text{ for some } \pi/2 \leq \theta_0 < \pi \text{ and} \\ \dot{\gamma}(L) &= e^{i\theta_L} \text{ for some } \pi < \theta_L \leq \frac{3}{2}\pi,\end{aligned}$$

*then  $\gamma$  is a graph over the  $x_1$ -axis and there is a strictly decreasing  $C^2$ -function  $\theta : [0, L] \rightarrow [\theta_L, \theta_0]$  such that  $\dot{\gamma}(t) = e^{i\theta(t)}$ . If*

$$\begin{aligned}\dot{\gamma}(0) &= e^{i\theta_0} \text{ for some } -1/2\pi \leq \theta_0 < \pi \text{ and} \\ \dot{\gamma}(L) &= e^{i\theta_L} \text{ for some } \pi < \theta_L \leq \frac{5}{2}\pi,\end{aligned}$$

*then there are a strictly increasing  $C^2$ -function  $\theta : [0, L] \rightarrow [\theta_0, \theta_L]$  such that  $\dot{\gamma}(t) = e^{i\theta(t)}$  and  $0 \leq t_0 < t_1 \leq L$  such that  $\gamma$  restricted to  $[0, t_0]$ ,  $[t_0, t_1]$ , and  $[t_1, L]$  is a graph over the  $x_1$ -axis.*

**Proof.** We define the tangent angle  $\theta : [0, L] \rightarrow \mathbb{R}$  of  $\gamma$  as the unique continuous map such that  $\theta(0) = \theta_0$  and  $\dot{\gamma}(t) = e^{i\theta(t)}$  for all  $t \in [0, L]$ . Since the curvature of  $\gamma$  is positive, the tangent angle  $\theta$  is strictly increasing. We apply Hopf's rotation angle theorem [13, 12] to the simple positive-oriented curve  $\gamma \oplus [-a, a]$  and find that the rotation angle of  $\gamma \oplus [-a, a]$  is exactly  $2\pi$ . Consequently,  $2\pi = \theta(L) + (2\pi - \theta_0)$ , such that  $\theta(L) = \theta_0$ . The curve  $\gamma$  fails to be a graph over the  $x_1$ -axis, if  $\theta(t)$  crosses  $\pi/2$  or  $3\pi/2$ . Since  $\theta$  is strictly increasing, this can happen at most two times in the interval  $(0, L)$ . This yields the claim.  $\square$

**Lemma 2.2.** *Let  $\gamma \in C^2([0, L], \mathbb{R}^2)$  be a unit speed curve with positive geodesic curvature connecting  $(a, 0)$  and  $(-a, b)$ , such that  $\dot{\gamma}(t) = e^{i\theta(t)}$ , for some strictly increasing function  $\theta \in C^0([0, L], \mathbb{R})$  satisfying  $\pi/2 \leq \theta(0) < \pi$  and  $\pi < \theta(L) \leq 3\pi/2$ . Then*

$$\min\{k_\gamma(t) : t \in [0, L]\} \leq a^{-1}.$$

**Proof.** Consider the upper half of the ball centered at  $(0, 0)$  and radius  $a$

$$B_a^+ := \{(x, y) \in \mathbb{R}^2 : |x| \leq a, y \geq 0, x^2 + y^2 \leq a^2\},$$

$$C_a^+ := \{(x, \sqrt{a^2 - x^2}) \in \mathbb{R}^2 : |x| \leq a\},$$

$$s_1 := \sup\{s \in \mathbb{R} : (0, s) + \gamma \cap B_a^+ \neq \emptyset\}.$$

Obviously, there holds  $s_1 \geq \max\{0, -b\}$ . If  $s_1 > \max\{0, -b\}$ , then  $s_1 + \gamma$  and  $B_a^+$  intersect in a point  $(s_1, 0) + \gamma(t_0)$  with  $t_0 \in (0, L)$  and  $s_1 + \gamma$  lies above  $B_a^+$ . From the maximum principle the curvature of  $\gamma$  at  $\gamma(t_0)$  is smaller than  $a^{-1}$ . If  $s_1 = 0$ , then  $\gamma$  lies above  $B_a^+$  and  $\theta(0)$  has to be  $\pi/2$ , such that the slope of  $\gamma$  and  $C_a^+$  coincide at  $(a, 0)$ . Writing  $\gamma$  and  $C_a^+$  as graphs over the  $x_2$ -axis the maximum principle shows that the curvature of  $\gamma$  at  $(a, 0)$  is smaller than  $a^{-1}$ . If  $s_0 = -b > 0$ , then  $\theta(L) = 3\pi/2$  and as above we deduce  $k_\gamma(L) \leq a^{-1}$ .  $\square$

**Lemma 2.3.** *Let  $\gamma \in C^2([0, L], \mathbb{R}^2)$  be a unit speed curve with positive geodesic curvature connecting  $(a, 0)$  and  $(-a, b)$ , such that  $\dot{\gamma}(t) = e^{i\theta(t)}$ , for some strictly increasing function  $\theta \in C^0([0, L], \mathbb{R})$  satisfying  $\pi/2 = \theta(0)$ . Moreover, if  $b > 0$ , we assume that  $\theta(L) = 3\pi/2$ , and if  $b \leq 0$ , we assume that  $\pi < \theta(L) \leq 3\pi/2$ . Then*

$$\max\{k_\gamma(t) : t \in [0, L]\} \geq a^{-1}.$$

**Proof.** The curve  $\gamma$  may be written as a graph over the interval  $[-a, a]$  for some function  $g \in C^0([-a, a], \mathbb{R}) \cap C^2((-a, a), \mathbb{R})$ . Let  $G$  be set defined by  $G := \{(x, y) \in \mathbb{R}^2 : -a \leq x \leq a, y \leq g(x)\}$ . Due to the positive curvature of  $\gamma$  the set  $G$  is convex and

$$G \cap \{(x, y) \in \mathbb{R}^2 : x \in \{\pm a\}, y > g(x)\} = \emptyset.$$

As in the proof of Lemma 2.2 we consider  $C_a^+$  and

$$s_0 := \sup\{s \in \mathbb{R} : (0, s) + C_a^+ \cap G \neq \emptyset\}.$$

Obviously, there holds  $s_0 \geq \max\{0, b\}$ . If  $s_0 > \max\{0, b\}$ , then  $(0, s_0) + C_a^+$  and  $G$  intersect in a point  $(t_0, g(t_0))$  with  $|t_0| < a$  and  $(0, s_0) + C_a^+$  lies above  $G$ . From the maximum principle the curvature of  $\gamma$  at  $(t_0, g(t_0))$  is bigger than  $a^{-1}$ . If  $s_0 = 0$ , then  $C_a^+$  lies above  $G$ . Since  $\theta(0) = \pi/2$  the slope of  $\gamma$  and  $C_a^+$  coincide at  $(a, 0)$ . From the maximum principle we deduce that  $k_\gamma(0) \geq a^{-1}$ . If  $s_0 = b > 0$ , then the slope of  $\gamma$  and  $(0, b) + C_a^+$  coincide at  $(-a, b)$  and the maximum principle shows that  $k_\gamma(L) \geq a^{-1}$ .  $\square$

**Lemma 2.4.** *Let  $\gamma \in C^2([0, L], \mathbb{R}^2)$  be a unit speed curve with positive geodesic curvature connecting  $(a, 0)$  and  $(-a, 0)$ , such that the closed curve  $\gamma \oplus [-a, a]$  is simple and  $\dot{\gamma}(L) \in \{e^{i\theta} : \pi < \theta \leq 5/2\pi\}$ . If  $\dot{\gamma}(0) = e^{-i\pi/2}$ , then the maximum  $k_{max}$  and the minimum  $k_{min}$  of the geodesic curvature of  $\gamma$  satisfy*

$$k_{min} \leq \frac{k_{max}}{k_{max}a + 1}.$$

**Proof.** We apply Lemma 2.1, write  $\dot{\gamma}(t) = e^{i\theta(t)}$ ,  $-\pi/2 < \theta(t) \leq 5/2\pi$ , and denote by  $t_0$  the point such that  $t_0 := \sup\{t \in [0, L] : \theta(s) \leq \pi/2 \text{ for all } 0 \leq s \leq t\}$ . By Lemma 2.1 there holds  $t_0 < L$ ,  $\theta(t_0) = \pi/2$ , and  $\theta(\cdot)$  is strictly increasing. Consequently, after a rotation by  $\pi$ , we may apply Lemma 2.3 and deduce that  $\gamma(t_0) = (x_0, y_0)$  with  $x_0 \geq a + 2k_{max}^{-1}$ .

We denote by  $t_1$  the point

$$t_1 := \sup\{t \in [t_0, L] : \theta(s) < 3/2\pi \text{ for all } t_0 \leq s \leq t\}.$$

Since  $\gamma \oplus [a, -a]$  is simple, we have  $\gamma(t_1) = (x_1, y_1)$  for some  $x_1 \leq -a$  (if  $t_1 < L$ , then  $x_1 < -a$ ). From Lemma 2.2 applied to  $\gamma$  restricted to  $[t_0, t_1]$  we see that  $k_{min} \leq (a + k_{max}^{-1})^{-1}$ , which yields the claim.  $\square$

We define for a given curvature function  $k \in C(\mathbb{R}^2 \times [0, 1], \mathbb{R})$  the set of small solutions  $L_{small}(k)$  and large solutions  $L_{large}(k)$  by

$$\begin{aligned} L_{small}(k) &:= \{\gamma \in M_{small} : \gamma \text{ solves (1.1)}\}, \\ L_{large}(k) &:= \{\gamma \in M_{large} : \gamma \text{ solves (1.1)}\}. \end{aligned}$$

**Lemma 2.5.** *Let  $\{k_s \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^+)$  :  $s \in [0, 1]$  be a continuous family of prescribed curvature functions, such that*

$$\begin{aligned} \sup\{k_s(x, t) : (x, t, s) \in \mathbb{R}^2 \times [0, 1]^2\} a &< 1, \\ \inf\{k_s(x, t) : (x, t, s) \in \mathbb{R}^2 \times [0, 1]^2\} &> 0. \end{aligned}$$

*Then the set*

$$L_{small} := \{\gamma \in M_{small} : \gamma \text{ solves (1.1) for some } k \in \{k_s\}\}$$

*is compact in  $C^2([0, 1], \mathbb{R}^2)$ . If, moreover, for all  $s \in [0, 1]$*

$$\frac{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_s(x, t)\}}{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_s(x, t)\} a + 1} < \inf_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_s(x, t)\},$$

*then*

$$L_{large} := \{\gamma \in M_{large} : \gamma \text{ solves (1.1) for some } k \in \{k_s\}\}$$

*is compact in  $C^2([0, 1], \mathbb{R}^2)$ .*

**Proof.** We first show that that  $L_{large}$  and  $L_{small}$  are closed. To this end we observe that any  $\gamma \in L_{large} \cup L_{small}$  is parameterized proportional to its arclength.

Let  $(\gamma_n)$  be a sequence in  $L_{small}$  converging to  $\gamma_0$  in  $C^2([0, 1], \mathbb{R}^2)$ . Choosing a subsequence, we may assume that  $\gamma_n$  is a solution to (1.1) with  $k = k_{s_n}$  for some sequence  $(s_n)$  converging to  $s_0 \in [0, 1]$ . Thus,  $\gamma_0$  solves (1.1) with

$k = k_{s_0}$ . Using the maximum principle and the positive curvature of  $\gamma_0$  it is easy to see that the curve  $\gamma_0$  cannot touch itself or the straight line  $[-a, a]$  tangentially, such that  $\gamma_0 \oplus [-a, a]$  remains simple as a limit of simple curves and

$$\begin{aligned} |\dot{\gamma}_0(0)|^{-1}\dot{\gamma}_0(0) &\in \{e^{i\theta} : 1/2\pi \leq \theta < \pi\}, \\ |\dot{\gamma}_0(1)|^{-1}\dot{\gamma}_0(1) &\in \{e^{i\theta} : \pi < \theta \leq 3/2\pi\}. \end{aligned}$$

Since  $\sup\{k_{s_0}(x, t) : (x, t) \in \mathbb{R}^3\}a < 1$  by Lemma 2.3 it is impossible that

$$|\dot{\gamma}_0(0)|^{-1}\dot{\gamma}_0(0) = e^{i\pi/2} \text{ or } |\dot{\gamma}_0(1)|^{-1}\dot{\gamma}_0(1) = e^{i3\pi/2}.$$

Consequently,  $\gamma_0$  is contained in  $L_{small}$ .

Let  $(\gamma_n)$  be a sequence in  $L_{large}$  converging to  $\gamma_0$  in  $C^2([0, 1], \mathbb{R}^2)$ . As above, we may deduce that  $\gamma_0$  is a solution to (1.1) with  $k = k_{s_0}$  for some  $s_0 \in [0, 1]$ ,  $\gamma_0 \oplus [-a, a]$  is simple, and satisfies

$$\begin{aligned} |\dot{\gamma}_0(0)|^{-1}\dot{\gamma}_0(0) &\in \{e^{i\theta} : -\pi/2 \leq \theta < \pi\}, \\ |\dot{\gamma}_0(1)|^{-1}\dot{\gamma}_0(1) &\in \{e^{i\theta} : \pi < \theta \leq 5/2\pi\}, \end{aligned}$$

and at least one of the following two conditions holds:

$$\begin{aligned} \dot{\gamma}(0)|\dot{\gamma}(0)|^{-1} &\in \{e^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\}, \\ \dot{\gamma}(1)|\dot{\gamma}(1)|^{-1} &\in \{e^{i\theta} : 3/2\pi \leq \theta \leq 5\pi/2\}. \end{aligned}$$

Using Lemma 2.4 and the fact that

$$\frac{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_{s_0}(x, t)\}}{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_{s_0}(x, t)\}a + 1} < \inf_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_{s_0}(x, t)\}$$

we exclude the possibility that  $\dot{\gamma}(0)|\dot{\gamma}(0)|^{-1} = e^{-i\pi/2}$  or  $\dot{\gamma}(1)|\dot{\gamma}(1)|^{-1} = e^{i5\pi/2}$ . If neither  $\dot{\gamma}(0)|\dot{\gamma}(0)|^{-1} \in \{e^{i\theta} : \theta < \pi/2\}$  nor  $\dot{\gamma}(1)|\dot{\gamma}(1)|^{-1} \in \{e^{i\theta} : 3/2\pi < \theta\}$ , then Lemma 2.3 leads to a contradiction. Thus,  $\gamma_0$  belongs to  $L_{large}$ .

To show the compactness of  $L_{large}$  and  $L_{small}$  we fix a sequence  $(\gamma_n)$  of solutions in  $L_{large} \cup L_{small}$ . Since  $\gamma_n \oplus [-a, a]$  is simple we may apply the Gauß-Bonnet formula and get

$$2\pi = \alpha_{1,n} + \alpha_{2,n} + \int_{\gamma_n} k_{\gamma_n},$$

where  $\alpha_{1,n}, \alpha_{2,n} \in (-\pi/2, \pi)$  are the outward angles at  $t = 0$  and  $t = 1$  of the piecewise- $C^2$  curve  $\gamma_n \oplus [-a, a]$ . Consequently,

$$L(\gamma_n) \inf_{(x,t,s) \in \mathbb{R}^2 \times [0,1]^2} \{k_s(x,t)\} \leq \int_{\gamma_n} k_{\gamma_n} \leq 3\pi,$$

where  $L(\gamma_n)$  denotes the length of  $\gamma_n$ . Hence,  $L(\gamma_n)$  is uniformly bounded, which yields a uniform  $C^1$ -bound of  $\gamma_n$ . Using the equation (1.1) and the Arzela-Ascoli theorem we may extract a subsequence of  $(\gamma_n)$ , which converges in  $C^2([0, 1], \mathbb{R}^2)$ . This finishes the proof.  $\square$

### 3. THE LERAY-SCHAUDER DEGREE

For  $a > 0$  we consider the affine space

$$C^2_{a,-a}([0, 1], \mathbb{R}^2) := \left\{ \gamma \in C^2([0, 1], \mathbb{R}^2) : \gamma(0) = \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ and } \gamma(1) = \begin{pmatrix} -a \\ 0 \end{pmatrix} \right\}.$$

The operator  $L_k$  is defined by

$$L_k : C^2_{-a,a}([0, 1], \mathbb{R}^2) \rightarrow C^2_{-a,a}([0, 1], \mathbb{R}^2)$$

$$L_k(\gamma) := (-D_t^2)^{-1} \left( -\ddot{\gamma} + |\dot{\gamma}(\cdot)|k(\gamma(\cdot), \cdot)J(\dot{\gamma}(\cdot)) \right),$$

where the operator  $D_t^2$  is considered as an isomorphism

$$D_t^2 : C^2_{-a,a}([0, 1], \mathbb{R}^2) \rightarrow C^0([0, 1], \mathbb{R}^2).$$

Since  $|\dot{\gamma}(\cdot)|k(\gamma(\cdot), \cdot)J(\dot{\gamma}(\cdot)) \in C^0([0, 1], \mathbb{R}^2)$  depends only on  $\gamma$  and  $\dot{\gamma}$ , the map  $\gamma \mapsto (-D_t^2)^{-1} \left( |\dot{\gamma}(\cdot)|k(\gamma(\cdot), \cdot)J(\dot{\gamma}(\cdot)) \right)$  is compact from  $C^2_{-a,a}([0, 1], \mathbb{R}^2)$  to itself. Thus  $L_k$  is of the form  $Id$ -compact and the Leray-Schauder degree of  $L_k$  is defined.

Fix  $a > 0$  and a function  $k \in C(\mathbb{R}^2 \times [0, 1], \mathbb{R})$  satisfying

$$0 < \inf_{\mathbb{R}^2 \times [0,1]} k \leq \sup_{\mathbb{R}^2 \times [0,1]} k < a^{-1},$$

$$\frac{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} k(x,t)}{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} k(x,t)a + 1} < \inf_{(x,t) \in \mathbb{R}^2 \times [0,1]} k(x,t).$$

We define for  $s \in [0, 1]$  the function  $k_s \in C^0(\mathbb{R}^2 \times [0, 1], \mathbb{R})$  by

$$k_s(x,t) := (1 - s) \left( \sup_{(x,t) \in \mathbb{R}^3} k(x,t) \right) + sk(x,t).$$



Then the family  $\{k_s : s \in [0, 1]\}$  satisfies the assumptions of Lemma 2.5 and the sets  $L_{large}$  and  $L_{small}$  are compact. Thus, there is  $R > 0$  such that

$$L_{large} \cup L_{small} \subset \{\lambda \in C^2([0, 1], \mathbb{R}^2) : \|\lambda\|_{C^2([0,1],\mathbb{R}^2)} < R\}.$$

Consequently, if we define the open sets

$$\begin{aligned} M_{small,R} &:= \{\lambda \in M_{small} : \|\lambda\|_{C^2([0,1],\mathbb{R}^2)} < R\}, \\ M_{large,R} &:= \{\lambda \in M_{large} : \|\lambda\|_{C^2([0,1],\mathbb{R}^2)} < R\}, \end{aligned}$$

then from the homotopy invariance of the degree

$$\begin{aligned} \deg(L_k, M_{small,R}, 0) &= \deg(L_{k_0}, M_{small,R}, 0), \\ \deg(L_k, M_{large,R}, 0) &= \deg(L_{k_0}, M_{large,R}, 0). \end{aligned} \tag{3.1}$$

To compute the degree of  $L_{k_0}$  we note that solutions to (1.1) with a constant function  $k_0$  are given by curves with constant geodesic curvature  $k_0$ , i.e., subarcs of an  $n$ -fold iterate of a circle with radius  $k_0^{-1}$ . Thus the required simplicity and the bounds on the slope yield

$$\begin{aligned} L_{small}(k_0) &= \{\gamma_s(t) := k_0^{-1}e^{i(\alpha_0+\omega_s t)} - ik_0^{-1}\sin(\alpha_0)\}, \\ L_{large}(k_0) &= \{\gamma_b(t) := k_0^{-1}e^{i(-\alpha_0+\omega_b t)} + ik_0^{-1}\sin(\alpha_0)\}, \end{aligned}$$

where  $\alpha_0 := \arccos(k_0 a) \in (0, \pi/2)$ ,  $\omega_s := \pi - 2\alpha_0 \in (0, \pi)$ , and  $\omega_b := \pi + 2\alpha_0 \in (\pi, 2\pi)$ . Consequently, we have

$$\begin{aligned} \deg(L_k, M_{small,R}, 0) &= \deg_{loc}(DL_{k_0}|_{\gamma_s}, 0), \\ \deg(L_k, M_{large,R}, 0) &= \deg_{loc}(DL_{k_0}|_{\gamma_b}, 0). \end{aligned} \tag{3.2}$$

To compute the local degrees we note for  $V \in C^2_{0,0}([0, 1], \mathbb{R}^2)$  and  $* \in \{s, b\}$

$$\begin{aligned} DL_{k_0}|_{\gamma_*}(V) &= (-D_t^2)^{-1}(-\ddot{V} + \langle \dot{\gamma}_*, \dot{V} \rangle |\dot{\gamma}_*|^{-1} k_0 J(\dot{\gamma}_*) + |\dot{\gamma}_*| k_0 J(\dot{V})) \\ &= (-D_t^2)^{-1}(-\ddot{V} - \omega_* \langle ie^{i(\alpha_0+\omega_* t)}, \dot{V} \rangle e^{i(\alpha_0+\omega_* t)} + \omega_* J(\dot{V})). \end{aligned}$$

For  $\lambda \in [-1, 1]$  we consider the family of operators  $A_\lambda : C^2_{0,0}([0, 1], \mathbb{R}^2) \rightarrow C^2_{0,0}([0, 1], \mathbb{R}^2)$  defined by

$$\begin{aligned} A_\lambda(V) &:= \\ &(-D_t^2)^{-1}(-\ddot{V} - (1 - \lambda)\omega_* \langle ie^{i(\alpha_0+\omega_* t)}, \dot{V} \rangle e^{i(\alpha_0+\omega_* t)} + (1 + \lambda)\omega_* J(\dot{V})). \end{aligned}$$

Writing

$$V(t) = \alpha(t)e^{i(\alpha_0+\omega_* t)} + \beta(t)ie^{i(\alpha_0+\omega_* t)}, \tag{3.3}$$

for some  $\alpha, \beta \in C_0^2([0, 1], \mathbb{R})$  we find

$$A_{\lambda,*}(V) = (-D_t^2)^{-1} \left( (-\ddot{\alpha}(t) - \omega_*^2 \alpha(t)) e^{i(\alpha_0 + \omega_* t)} \right. \\ \left. + (-\ddot{\beta}(t) - (1 - \lambda)\omega_* \dot{\alpha}(t) - \lambda\omega_*^2 \beta(t)) i e^{i(\alpha_0 + \omega_* t)} \right).$$

The eigenvalues of the problem

$$\ddot{\varphi}(t) = \lambda \varphi(t) \text{ for } t \in [0, 1] \text{ and } \varphi(0) = \varphi(1) = 0$$

are given by

$$\{\pi^2 n^2 : n \in \mathbb{N}\}. \quad (3.4)$$

Since  $\omega_s < \pi$ , each  $A_{\lambda,s}$  is injective and due to its form, identity-compact,  $A_{\lambda,s}$  is invertible for each  $\lambda \in [0, 1]$ . By the homotopy invariance of the degree we obtain

$$\deg_{loc}(DL_{k_0}|_{\gamma_s}, 0) = \deg_{loc}(A_{1,s}, 0) = \deg_{loc}(id, 0) = 1, \quad (3.5)$$

where we used for the second equality the admissible homotopy  $\{B_\sigma : \sigma \in [0, 1]\}$  given by

$$B_\sigma(V) := (-D_t^2)^{-1} (-\ddot{V} + 2(1 - \sigma)\omega_* J(\dot{V})).$$

To compute the degree of  $DL_{k_0}|_{\gamma_b}$  we note that by the above analysis and the homotopy property we may replace  $k_0$  by some constant  $k_1$  close to  $a$  without changing the degree, such that we may assume

$$\pi < \omega_b < \sqrt{2}\pi. \quad (3.6)$$

Moreover, using the homotopy  $\{A_{\lambda,b} : \lambda \in [-1, 0]\}$ , we see that

$$\deg_{loc}(DL_{k_0}|_{\gamma_b}, 0) = \deg_{loc}(A_{-1,b}, 0).$$

To compute  $\deg_{loc}(A_{-1,b}, 0)$  we consider the decomposition  $C_{0,0}^2([0, 1], \mathbb{R}^2) = U_1 \oplus U_2$ , where

$$U_1 := \{V \in C_{0,0}^2([0, 1], \mathbb{R}^2) : \int_0^1 V(t) \cdot (\sin(\pi t) e^{i(\alpha_0 + \omega_b t)}) dt = 0\},$$

$$U_2 := \text{span}(\sin(\pi t) e^{i(\alpha_0 + \omega_b t)}).$$

Using the decomposition in (3.3) we fix  $V_1 \in U_1 \setminus \{0\}$  and  $V_2 \in U_2 \setminus \{0\}$ ,

$$V_1(t) = \alpha(t) e^{i(\alpha_0 + \omega_b t)} + \beta(t) i e^{i(\alpha_0 + \omega_b t)},$$

$$V_2(t) = \lambda \sin(\pi t) e^{i(\alpha_0 + \omega_b t)}.$$

From (3.4) and (3.6) we obtain

$$\begin{aligned} \langle D_t A_{-1,b}(V_1), D_t V_1 \rangle_{L^2([0,1], \mathbb{R}^2)} &= \langle -(D_t)^2 A_{-1,b}(V_1), V_1 \rangle_{L^2([0,1], \mathbb{R}^2)} \\ &= \int_0^1 (-\ddot{\alpha}(t) - \omega_b^2 \alpha(t)) \alpha(t) + (-\ddot{\beta}(t) - 2\omega_b \dot{\alpha}(t) + \omega_b^2 \beta(t)) \beta(t) dt \\ &= \int_0^1 (\dot{\alpha}(t))^2 - 2\omega_b^2 (\alpha(t))^2 + (\dot{\beta}(t) - \omega_b \alpha(t))^2 + \omega_b^2 (\beta(t))^2 dt \\ &\geq (4\pi^2 - 2\omega_b^2) (\alpha(t))^2 + \omega_b^2 (\beta(t))^2 > 0, \end{aligned}$$

$$\begin{aligned} \langle D_t A_{-1,b}(V_2), D_t V_2 \rangle_{L^2([0,1], \mathbb{R}^2)} &= \lambda^2 \int_0^1 (\pi^2 - \omega_b^2) (\sin(\pi t))^2 dt \\ &= \frac{1}{2} \lambda^2 (\pi^2 - \omega_b^2) < 0, \end{aligned}$$

and

$$\begin{aligned} \langle D_t A_{-1,b}(V_1), D_t V_2 \rangle_{L^2([0,1], \mathbb{R}^2)} &= \lambda \int_0^1 (-\ddot{\alpha}(t) - \omega_b^2 \alpha(t)) \sin(\pi t) dt \\ &= \lambda (\pi^2 - \omega_b^2) \int_0^1 \alpha(t) \sin(\pi t) dt = 0. \end{aligned}$$

Thus the following homotopy is admissible:  $[0, 1] \ni \sigma \mapsto \sigma C + (1 - \sigma)A_{-1,b}$ , where  $C \in \mathcal{L}(C_{0,0}^2([0, 1], \mathbb{R}^2), C_{0,0}^2([0, 1], \mathbb{R}^2))$  is given in the decomposition  $U_1 \oplus U_2$  by

$$C := \begin{pmatrix} id & 0 \\ 0 & -1 \end{pmatrix}.$$

From the above computations we finally see that

$$\text{deg}_{loc}(DL_{k_0}|_{\gamma_b}, 0) = \text{deg}_{loc}(C, 0) = -1,$$

which yields together with (3.2) the proof of Theorem 1.1 announced in the introduction.

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