THE ELLIPTIC KIRCHHOFF EQUATION IN $\mathbb{R}^N$ PERTURBED BY A LOCAL NONLINEARITY

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Abstract. In this paper we present a very simple proof of the existence of at least one nontrivial solution for a Kirchhoff-type equation on $\mathbb{R}^N$, for $N \geqslant 3$. In particular, in the first part of the paper we are interested in studying the existence of a positive solution to the elliptic Kirchhoff equation under the effect of a nonlinearity satisfying the general Berestycki-Lions assumptions. In the second part we look for ground states using minimizing arguments on a suitable natural constraint.

Introduction

The multidimensional Kirchhoff equation is

$$\frac{\partial^2 u}{(\partial t)^2} - \Big(1 + \int_\Omega |\nabla u|^2\Big)\Delta u = 0, \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ satisfies some initial or boundary conditions. It arises from the following Kirchhoff nonlinear generalization (see [7]) of the well known d'Alembert equation

$$\rho \frac{\partial^2 u}{(\partial t)^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{(\partial x)^2} = 0. \tag{0.2}$$

Equation (0.2) describes a vibrating string, taking into account the changes in length of the string during the vibration. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the

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material, ρ is the mass density and P_0 is the initial tension. In [8] the problem was proposed in the following form:

$$\begin{cases} \frac{\partial^2 u}{(\partial t)^2} - M(\int_\Omega |\nabla u|^2)\Delta u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

where $M : [0, +\infty) \to \mathbb{R}$ is a continuous function such that $M(s) \geqslant c > 0$ for any $s \geqslant 0$, and Ω is a bounded set in $\mathbb{R}^N$, with smooth boundary. This hyperbolic problem has an elliptic version when we look for static solutions.

In [16], a class of problems has been considered among which the following elliptic Kirchhoff-type equation was included:

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

(here Ω is an open subset of $\mathbb{R}^N$).

Taking into account the original formulation of the equation given by Kirchhoff, we assume the following:

Definition 0.1. *If there exist two positive constants a and b such that $M : \mathbb{R}_+ \to \mathbb{R}$ can be written $M(s) = a + bs$, then M is called a Kirchhoff function.*

Recently, many authors have used variational methods to study the Kirchhoff equation perturbed by a local nonlinear term (see [9] for a short survey on the topic). By arguments based on the mountain-pass theorem, in [1] the problem

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

has been solved in a bounded domain of $\mathbb{R}^N$ under suitable growth conditions on $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $M : [0, +\infty) \to \mathbb{R}$. Taking $N = 1, 2$, or 3, the problem has been treated also in [12] where M is a Kirchhoff function, and the nonlinearity $f(x,t)$ has been supposed to behave linearly near 0 and like t^3 at infinity. The Yang index has been used to find a nontrivial solution. A similar growth at infinity has been assumed in [17] where the authors have looked for sign-changing solutions. They also have obtained a sign-changing solution when the nonlinearity f satisfies either the following growth condition,

$$|f(x,t)| \leqslant C(|t|^{p-1} + 1), \text{ uniformly in } x, \text{ for } p < 4,$$

or the following Ambrosetti-Rabinowitz condition,

$$\nu F(x,t) \leqslant t f(x,t), \text{ for } |t| \text{ large and } \nu > 4.$$

In [10], the equation has been studied assuming that the nonlinearity grows at infinity more than t^3, without introducing the Ambrosetti-Rabinowitz hypothesis. Using a variational approach, a multiplicity result has been showed in [5]. Finally we recall the recent result obtained in [13], where three solutions have been found for the Kirchhoff-type problem

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = \lambda f(x,u) + \mu g(x,t) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where λ and μ are two parameters.

The joining point of all these papers is that they consider the equation on a bounded domain, with Dirichlet conditions on the boundary.

In [2, 6] the problem has been studied in $\mathbb{R}^N$ in the presence of a potential and a nonlinearity satisfying the well-known Ambrosetti-Rabinowitz condition.

In this paper we study an autonomous Kirchhoff-type equation on $\mathbb{R}^N$, looking for the existence of positive solutions; namely, we consider the problem

$$\begin{cases} -M(\int_\Omega |\nabla u|^2)\Delta u = g(u) & \text{in } \mathbb{R}^N,\ N \geqslant 3, \\ u > 0. \end{cases} \tag{$\mathcal{K}$}$$

In this sense, the problem turns out to be a generalization of the well-known Schrödinger equation

$$-\Delta u = g(u), \quad \text{in } \mathbb{R}^N. \tag{$\mathcal{S}$}$$

In the first part of the paper we are interested in studying the problem $(\mathcal{K})$ in the presence of a Berestycki-Lions nonlinearity. In order to explain what this means, we provide the following definition

Definition 0.2. *A function $g : \mathbb{R} \to \mathbb{R}$ is called a Berestycki-Lions nonlinearity if it satisfies the following assumptions:*

- **(g1)** $g \in C(\mathbb{R}, \mathbb{R})$, $g(0) = 0$;
- **(g2)** $-\infty < \liminf_{s\to 0^+} g(s)/s \leqslant \limsup_{s\to 0^+} g(s)/s = -m < 0$;
- **(g3)** $-\infty \leqslant \limsup_{s\to+\infty} g(s)/s^{2^*-1} \leqslant 0$;
- **(g4)** *there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(s)\, ds > 0$.*

A function $g : \mathbb{R} \to \mathbb{R}$ is called a zero-mass Berestycki-Lions nonlinearity if it satisfies **(g1)**, **(g3)**, **(g4)** *and the following zero-mass assumption*

(g2)$'$ $-\infty < \liminf_{s\to 0^+} g(s)/s$, $\limsup_{s\to 0^+} g(s)/s^{2^*-1} \leqslant 0$.

Remark 0.3. Using the terminology inherited by [3], we refer to the constant m in **(g2)** calling it *mass*. This is the reason for which we say that a function g satisfying **(g2)**$'$ instead of **(g2)** is a *zero-mass* nonlinearity.

In the very celebrated paper [3], these types of nonlinearities appeared for the first time, and it was showed that the hypotheses (**g1**),. . .,(**g4**) are *almost* optimal to get an existence result for the Schrödinger equation.

In the first section of this paper, we use a simple rescaling argument to establish a sufficient condition for the existence of a solution to $(\mathcal{K})$. The first result we get is the following:

Theorem 0.4. *Let $M : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ be a continuous function such that*

$$\liminf_{t\to 0} tM(t^{\frac{2-N}{2}}) = 0 \tag{0.3}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be a (possibly zero-mass) Berestycki-Lions nonlinearity. Then the problem $(\mathcal{K})$ has a solution in $C^2(\mathbb{R}^N)$.

We point out that any Kirchhoff function satisfies our assumptions if $N = 3$. On the other hand, when M is a Kirchhoff-type function we are able to refine our estimates and we get the following result:

Theorem 0.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a (possibly zero-mass) Berestycki-Lions nonlinearity, $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function, and $M(s) = a + bf(s)$. Then for any $N \geqslant 3$ there exists a positive constant δ (depending on a) such that if $b \in (0, \delta)$, the problem $(\mathcal{K})$ has a solution in $C^2(\mathbb{R}^N)$. If moreover,*

$$\liminf_{t\to 0} t^{\frac{2}{2-N}} f(t) = 0, \tag{0.4}$$

then there exists a positive constant δ (depending on b) such that if $a \in (0, \delta)$, the problem $(\mathcal{K})$ has a solution in $C^2(\mathbb{R}^N)$.

We again remark that when M is a Kirchhoff function, then $f = id|_{\mathbb{R}_+}$ and assumption (0.4) is automatically satisfied for $N \geqslant 5$.

In the second part of the paper we study the existence of the so-called *ground-state* solutions to the Kirchhoff equation

$$-\Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2\Big)\Delta u = g(u). \tag{0.5}$$

We recall that a ground state is a solution which minimizes the functional of the action among all the other solutions.

The problem of finding such a type of solution is a very classical problem: it was introduced by Coleman, Glazer, and Martin in [4], and reconsidered by Berestycki and Lions in [3] for a class of nonlinear equations including the Schrödinger's one. Here we will use a minimizing argument based on an idea developed in [15] (recent applications can be found in [11] and in [14]). In that paper the author showed that the ground state for the Schrödinger equation can be found as the minimizer of the functional of the action restricted to a particular natural constraint. This natural constraint actually is a manifold which is constituted by all non-null functions satisfying the Pohozaev identity related to the Schrödinger equation.

Unfortunately, a similar manifold turns out to be a nice constraint in order to look for ground-state solutions of the Kirchhoff equation only if $N = 3$ or $N = 4$: when $N \geqslant 5$, we are no more able to establish if the restricted functional is bounded below. The final result we get is the following:

Theorem 0.6. *If $N = 3$ or $N = 4$ and g is a Berestycki-Lions nonlinearity, then equation* (0.5) *possesses a ground-state solution.*

The paper is so organized: In Section 1, we show our rescaling argument to get a solution for $(\mathcal{K})$ in the general case described in Theorem 0.4 and in the particular situation of a Kirchhoff-type function as in Theorem 0.5.

In Section 2, we study the problem of the existence of a ground-state solution for the Kirchhoff equation using a variational approach.

1. Bound-state solution

In the sequel, we denote by v a ground-state solution of $(\mathcal{S})$ (respectively a bound-state solution if g is a zero-mass Berestycki-Lions nonlinearity).

Proof of Theorem 0.4. Observe that, by hypothesis (0.3) and since

$$\lim_{t\to+\infty} tM(t^{\frac{2-N}{2}}) = +\infty,$$

by continuity we have that there exists $\bar{t} > 0$ such that

$$\bar{t}^2 M(\bar{t}^{2-N} \int_{\mathbb{R}^N} |\nabla v|^2) = 1.$$

The function $u : \mathbb{R}^N \to \mathbb{R}$ is defined as follows: $x \in \mathbb{R}^N \to v(\bar{t}x) \in \mathbb{R}$, satisfies the equalities

$$\begin{cases} M(\int_{\mathbb{R}^N} |\nabla u|^2) = \frac{1}{\bar{t}^2} \\ -\Delta u(x) = -\bar{t}^2 \Delta v(\bar{t}x) = \bar{t}^2 g(v(\bar{t}x)) = \bar{t}^2 g(u(x)), \end{cases}$$

and then it is a solution of $(\mathcal{K})$. □

Remark 1.1. We point out that, since the only moment in which we use hypothesis (0.3) is to determine the rescaling parameter $\bar{t}$, we can relax our assumption, just requiring that

$$\inf_{t>0} tM\Big(t^{\frac{2-N}{2}} \int_{\mathbb{R}^N} |\nabla v|^2\Big) < 1. \tag{1.1}$$

Proof of Theorem 0.5. As in the proof of Theorem 0.4 we look for a solution to the equation

$$t^2\Big(a+bf(t^{2-N}\int_{\mathbb{R}^N}|\nabla v|^2)\Big)=1.$$

Taking into account the previous remark, it is enough to prove that

$$\inf_{t>0}\Psi(t)<1,$$

where $\Psi(t):=t\left(a+bf(t^{\frac{2-N}{2}}\int_{\mathbb{R}^N}|\nabla v|^2)\right)$. Set

$$\bar{h}=f\Big((2a)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}|\nabla v|^2\Big)$$

and $\delta_1=\frac{a}{\bar{h}}$. It is easy to verify that, if $b<\delta_1$, then $\Psi(1/2a)<1$.

Now suppose that (0.4) holds. We deduce that

$$\liminf_{t\to+\infty} tf\Big(t^{\frac{2-N}{2}}\int_{\mathbb{R}^N}|\nabla v|^2\Big)=0.$$

Let $\bar{t}$ be such that $\bar{t}f\left(\bar{t}^{\frac{2-N}{2}}\int_{\mathbb{R}^N}|\nabla v|^2\right)\leqslant\frac{1}{2b}$ and choose $a<\delta_2=\frac{1}{2\bar{t}}$. Again we have that $\Psi(\bar{t})<1$. □

2. Ground-state solution

In this section we use a variational approach which requires some preliminaries. In the next subsection we will use the same arguments as in [3] to modify the nonlinearity g in such a way that we can study equation $(\mathcal{K})$ looking for critical points of a suitable functional.

2.1. Functional framework.

Define $s_0:=\min\{s\in[\zeta,+\infty):g(s)=0\}$ ($s_0=+\infty$ if $g(s)\neq 0$ for any $s\geqslant\zeta$). We set $\tilde{g}:\mathbb{R}\to\mathbb{R}$ to be the function such that

$$\tilde{g}(s)=\begin{cases} g(s) & \text{on } [0,s_0];\\ 0 & \text{on } \mathbb{R}_+\setminus[0,s_0];\\ (g(-s)-ms)^+-g(-s) & \text{on } \mathbb{R}_-.\end{cases} \tag{2.1}$$

By the strong maximum principle, if u is a nontrivial solution of $(\mathcal{K})$ with $\tilde{g}$ in the place of g, then $0<u<s_0$ and so it is a positive solution of $(\mathcal{K})$. Therefore we can suppose that g is defined as in (2.1), so that (**g1**), (**g2**), (**g4**) and the following limit,

$$\lim_{s\to\pm\infty}\frac{g(s)}{|s|^{2^*-1}}=0, \tag{2.2}$$

hold. We set

$$g_1(s) := \begin{cases} (g(s)+ms)^+, & \text{if } s \geqslant 0,\\ 0, & \text{if } s < 0,\end{cases}$$
$$g_2(s) := g_1(s) - g(s), \quad \text{for } s \in \mathbb{R}.$$

Since

$$\lim_{s\to 0} \frac{g_1(s)}{s} = 0,$$
$$\lim_{s\to \pm\infty} \frac{g_1(s)}{|s|^{2^*-1}} = 0, \tag{2.3}$$

and

$$g_2(s) \geqslant ms, \quad \forall s \geqslant 0, \tag{2.4}$$

by some computations, we have that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$g_1(s) \leqslant C_\varepsilon s^{2^*-1} + \varepsilon g_2(s), \quad \forall s \geqslant 0. \tag{2.5}$$

If we set

$$G_i(t) := \int_0^t g_i(s)\, ds, \quad i = 1, 2,$$

then, by (2.4) and (2.5), we have

$$G_2(s) \geqslant \frac{m}{2} s^2, \quad \forall s \in \mathbb{R} \tag{2.6}$$

and for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$G_1(s) \leqslant \frac{C_\varepsilon}{2^*} s^{2^*} + \varepsilon G_2(s), \quad \forall s \in \mathbb{R}. \tag{2.7}$$

We define the functional

$$I(u) := \frac{1}{2}\tilde{M}(\|u\|^2) - \int_{\mathbb{R}^3} G(u),$$

where $\tilde{M}(s) = \int_0^s M(t)\, dt$ and we are denoting by $\|\cdot\|$ the norm $\left(\int_{\mathbb{R}^3} |\nabla \cdot|^2\right)^{\frac{1}{2}}$ of the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, which is the closure of the compactly supported smooth functions with respect to the norm $\|\cdot\|$. The previous functional is C^1 in $H^1(\mathbb{R}^N)$, $H^1(\mathbb{R}^N)$ being the closure of the compactly supported smooth functions with respect to the norm

$$\|\cdot\|^2_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^3} |\nabla \cdot|^2 + \int_{\mathbb{R}^3} |\cdot|^2.$$

We will look for critical points of the functional I inside

$$H^1_r(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radial}\},$$

which is a natural constraint for the functional I by Palais's principle of symmetric criticality. By standard variational arguments, it is easy to prove that any critical point of I corresponds to a weak solution of the equation. By the maximum principle we will get a positive solution.

2.2. **Existence of a ground-state solution.** We look for a ground-state solution to

$$-(a+b\|u\|^2)\Delta u = g(u), \quad u:\mathbb{R}^N \to \mathbb{R},\ N=3,4.$$

A ground state of (0.5) is a nontrivial solution $\bar{u} \in H^1(\mathbb{R}^N)$ such that, if $v \in H^1(\mathbb{R}^N)$ is another nontrivial solution of (0.5), then $I(\bar{u}) \leqslant I(v)$, where $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ is the functional of the action related with (0.5), namely

$$I(u) = \frac{1}{2}\Big(a + \frac{b}{2}\int_{\mathbb{R}^N} |\nabla u|^2\Big)\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u).$$

Usually a standard technique to find a ground state consists in looking for minimizers of the functional of the action restricted to a natural constraint which contains all the possible solutions. A candidate to play this role is the following Pohozaev set,

$$\mathcal{P} = \{u \in H^1(\mathbb{R}^N)\setminus\{0\} : P(u) = 0\},$$

where for any $u \in H^1(\mathbb{R}^N)$

$$P(u) = a\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u|^2 + b\frac{N-2}{2N}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} G(u).$$

Actually the equality $P(u) = 0$ is nothing but the Pohozaev identity related with the equation.

We will prove Theorem 0.6 following this scheme:

Step 1: we show that $\mathcal{P}$ is a C^1 manifold containing all the possible solutions of equation (0.5);

Step 2: we prove that $\mathcal{P}$ is a natural constraint, in the sense that every critical point of I restricted to $\mathcal{P}$ is a critical point of I;

Step 3: we show that $I|_{\mathcal{P}}$ is bounded below and

$$\mu = \inf_{u\in\mathcal{P}} I(u) = \inf_{u\in\mathcal{P}} \frac{1}{N}\Big(a\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{(4-N)b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\Big)^2\Big)$$

is achieved.

It is easy to see that P is a C^1 functional. Moreover $\mathcal{P}$ is nondegenerate in the following sense:

$$\forall u \in \mathcal{P} : P'(u) \neq 0$$

so that $\mathcal{P}$ is a C^1 manifold of codimension one. Indeed, suppose for the sake of contradiction that $u \in \mathcal{P}$ and $P'(u) = 0$, namely, u is a solution of the equation

$$-\Big(a\frac{N-2}{N} + 2b\frac{N-2}{N}\|u\|^2\Big)\Delta u = g(u). \tag{2.8}$$

As a consequence, u satisfies the Pohozaev identity referred to in (2.8), that is,

$$a\frac{(N-2)^2}{2N^2}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{(N-2)^2}{N^2}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2 = \int_{\mathbb{R}^N} G(u). \tag{2.9}$$

Since $P(u) = 0$, by (2.9) we get

$$-2a\int_{\mathbb{R}^N}|\nabla u|^2 + b(N-4)\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2 = 0$$

and we conclude that $u = 0$, which is absurd since $u \in \mathcal{P}$. So $\mathcal{P}$ is a C^1 manifold. It obviously contains all the solutions to (0.5) since every solution satisfies the Pohozaev identity $P(u) = 0$.

Now we pass to proving that $\mathcal{P}$ is a natural constraint for I. Suppose that $u \in \mathcal{P}$ is a critical point of the functional $I|_{\mathcal{P}}$. Then there exists $\lambda \in \mathbb{R}$ such that $I'(u) = \lambda P'(u)$, that is,

$$-(a + b\|u\|^2)\Delta u - g(u) = -\lambda(a\frac{N-2}{N} + 2b\frac{N-2}{N}\|u\|^2)\Delta u - \lambda g(u).$$

As a consequence, u satisfies the following Pohozaev identity:

$$P(u) = \lambda a\frac{(N-2)^2}{2N^2}\int_{\mathbb{R}^N}|\nabla u|^2 + \lambda b\frac{(N-2)^2}{N^2}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2 - \lambda\int_{\mathbb{R}^N} G(u),$$

which, since $P(u) = 0$, can be written

$$\lambda\Big(-2a\int_{\mathbb{R}^N}|\nabla u|^2 + b(N-4)\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2\Big) = 0.$$

Since $u \neq 0$, we deduce that $\lambda = 0$, and we conclude.

Now it remains to show that μ is achieved.

By the well-known properties of the Schwarz symmetrization, we are allowed to work on the functional space $H^1_r(\mathbb{R}^N)$ as shown by the following:

Lemma 2.1. *For any $u \in \mathcal{P}$ there exists $\tilde{u} \in \mathcal{P} \cap H^1_r(\mathbb{R}^N)$ such that $I(\tilde{u}) \leqslant I(u)$.*

Proof. Let $u \in \mathcal{P}$ and set $u^* \in H^1_r(\mathbb{R}^N)$ to be its symmetrization. It is easy to see that there exists $0 < \tilde\theta \leqslant 1$ such that $\tilde u := u^*(\cdot/\tilde\theta) \in \mathcal{P} \cap H^1_r(\mathbb{R}^N)$ and

$$\begin{aligned} I(\tilde u) &= \frac{1}{N}\Big(a\int_{\mathbb{R}^N}|\nabla \tilde u|^2 + \frac{(4-N)b}{4}\Big(\int_{\mathbb{R}^N}|\nabla \tilde u|^2\Big)^2\Big) \\ &= a\frac{\tilde\theta^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla u^*|^2 + b\frac{(4-N)\tilde\theta^{2(N-2)}}{4N}\Big(\int_{\mathbb{R}^N}|\nabla u^*|^2\Big)^2 \\ &\leqslant \frac{a}{N}\int_{\mathbb{R}^N}|\nabla u^*|^2 + \frac{(4-N)b}{4N}\Big(\int_{\mathbb{R}^N}|\nabla u^*|^2\Big)^2 \leqslant I(u). \qquad \square \end{aligned}$$

Before we proceed with the proof of the main result, another preliminary result is required:

Lemma 2.2. $\mu := \inf\{I(v) : v \in \mathcal{P}\} > 0$.

Proof. If $u \in \mathcal{P}$, then, by (2.7), we have

$$\begin{aligned} C\|u\|^2 &\leqslant a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{N-2}{2}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2 + N(1-\varepsilon)\int_{\mathbb{R}^N}G_2(u) \\ &\leqslant NC_\varepsilon\int_{\mathbb{R}^N}|u|^{2^*} \leqslant C'\|u\|^{2^*}, \end{aligned}$$

where $\varepsilon < 1$ and C_ε, C, and C' are suitable positive constants. We deduce that there exists a positive constant C'' such that $\|u\| \geqslant C''$ for any $u \in \mathcal{P}$. The conclusion then follows once one observes that $I|_{\mathcal{P}}(u) \geqslant \tilde C\|u\|^2$. $\square$

Now let $(u_n)_n$ be a minimizing sequence for $I|_{\mathcal{P}}$ in $H^1_r(\mathbb{R}^N)$, namely,

$$\{u_n\}_n \subset \mathcal{P} \cap H^1_r(\mathbb{R}^N), \quad I(u_n) \to \mu. \tag{2.10}$$

Obviously $\|u_n\|$ is bounded. Moreover, since $\{u_n\}_n \subset \mathcal{P}$, certainly, by (2.7), there exist $0 < \varepsilon < 1$ and $C_\varepsilon > 0$ such that

$$a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{N-2}{2}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2 + N(1-\varepsilon)\int_{\mathbb{R}^N}G_2(u) \leqslant C_\varepsilon N\|u_n\|_{2^*}^{2^*},$$

and then we deduce also the boundedness of the L^2-norm of $\{u_n\}_n$ by the continuous Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and (2.6).

Let $u \in H^1_r(\mathbb{R}^N)$ be the function such that, up to subsequences,

$$u_n \rightharpoonup u, \text{ weakly in } H^1(\mathbb{R}^N). \tag{2.11}$$

We are going to prove that there exists $\bar\theta > 0$ such that

$$\bar u \in \mathcal{P} \text{ and } I(\bar u) = \mu$$

where $\bar u := u(\cdot/\bar\theta)$.

Actually, by compactness due to the radial symmetry, from the weak convergence (2.11) we deduce

$$\lim_n \int_{\mathbb{R}^N} G_1(u_n) = \int_{\mathbb{R}^N} G_1(u). \tag{2.12}$$

Of course, $u \neq 0$. Otherwise, by (2.12) and since $u_n \in \mathcal{P}$ for any $n \geqslant 1$, we should have that

$$0 \leqslant \limsup_n a\frac{N-2}{2}\|u_n\|^2 \leqslant N \lim_n \int_{\mathbb{R}^N} G_1(u_n) = 0,$$

which, by (2.10), contradicts Lemma 2.2.

By (2.12), the lower semicontinuity of the $\mathcal{D}^{1,2}(\mathbb{R}^N)$-norm and the Fatou lemma, we have

$$\begin{aligned}
&a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{N-2}{2}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2 + N\int_{\mathbb{R}^N} G_2(u)\\
&\leqslant \liminf_n \Big(a\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 + b\frac{N-2}{2}\Big(\int_{\mathbb{R}^N}|\nabla u_n|^2\Big)^2 + N\int_{\mathbb{R}^N} G_2(u_n)\Big)\\
&= \lim_n N\int_{\mathbb{R}^N} G_1(u_n) = N\int_{\mathbb{R}^N} G_1(u).
\end{aligned}$$

Let $0 < \bar\theta \leqslant 1$ be such that $\bar u = u(\cdot/\bar\theta) \in \mathcal{P}$. Using the lower semicontinuity of the $\mathcal{D}^{1,2}(\mathbb{R}^N)$-norm, we infer that

$$\begin{aligned}
I(\bar u) &= a\frac{\bar\theta^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla u|^2 + b\frac{(4-N)\bar\theta^{2(N-2)}}{4N}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\Big)^2\\
&\leqslant \liminf_n \frac{a}{N}\int_{\mathbb{R}^N}|\nabla u_n|^2 + \frac{b(4-N)}{4N}\Big(\int_{\mathbb{R}^N}|\nabla u_n|^2\Big)^2 = \lim_n I(u_n) = \mu,
\end{aligned}$$

and then we conclude.

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