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ON THE BREZIS-NIRENBERG PROBLEM IN A BALL

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Abstract. We study the following Brezis-Nirenberg type critical exponent problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^{2^*-1} \text{ in } B_R, \\ u > 0 \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R, \end{cases}$$

where B_R is a ball with radius R in $\mathbb{R}^N (N \geq 3)$, $\lambda > 0$, $1 \leq q < 2^* - 1$, and 2^* is the critical Sobolev exponent. We prove the uniqueness results of the least-energy solution when $3 \leq N \leq 5$ and $1 \leq q < 2^* - 1$. We give extremely accurate energy estimates of the least-energy solutions as $R \to 0$ for $N \geq 4$ and $q = 1$.

1. Introduction

In 1983, Brezis-Nirenberg [4] studied the following semilinear problem:

$$\begin{cases} -\Delta u = \lambda |u|^{q-1} u + |u|^{2^*-2} u \text{ in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$, $1 \leq q < 2^* - 1$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, and $\lambda > 0$. Without loss of generality, we assume $0 \in \Omega$. First we consider the case of $1 < q < 2^* - 1$, and let Ω be a ball in $\mathbb{R}^N$. Define

$$B_R := B(0, R) := \{x \in \mathbb{R}^N : |x| < R\}.$$

Then we may assume $\Omega = B_1$ without loss of generality. We consider only positive solutions, so (1.1) turns out to be

$$\begin{cases} -\Delta u = \lambda u^q + u^{2^*-1} \text{ in } B_1, \\ u > 0 \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1. \end{cases} \tag{1.2}$$

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When $N \geq 4$, Brezis-Nirenberg [4] proved that (1.2) has a least-energy solution u_λ (a mountain-pass type solution) for every $\lambda > 0$. When $N = 3$, they proved that (1.2) has a least-energy solution u_λ only for large values of $\lambda > 0$, but one does not know whether or not the least-energy solution is unique. Moreover, it is suggested in [4] by numerical computations, and later proved by Atkinson-Peletier [3] that, when $N = 3$ and $1 < q < 3$, there is some $\bar{\mu} > 0$ such that (1.2) has at least two solutions for $\lambda > \bar{\mu}$, a unique solution for $\lambda = \bar{\mu}$, and no solution for $\lambda < \bar{\mu}$.

An interesting open problem raised by Brezis-Nirenberg [4] was whether or not the solution of (1.2), whose existence is ensured in [4], is unique (except the case $N = 3$ and $1 < q < 3$).

Erbe-Tang [12] gave a partial answer to this open problem. They [12] proved that (1.2) has a unique solution for every $\lambda > 0$ when $N \geq 6$. There are also some kinds of papers studying the uniqueness of solutions of the related subcritical problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p \ \text{ in } B_1, \\ u > 0 \ \text{ in } B_1, \quad u = 0 \ \text{ on } \partial B_1, \end{cases}$$

where $N \geq 3$ and $1 < q < p < 2^* - 1$; see [14, 20] for example.

To the best of our knowledge, the uniqueness of solutions of (1.2) in the case of $N \leq 5$ seems unknown.

Since the uniqueness of solutions of (1.2) can not hold when $N = 3$ and $1 < q < 3$, we turn to considering the uniqueness of least-energy solutions for $N \leq 5$. To be precise, we will prove the following theorem.

Theorem 1.1. *Let $3 \leq N \leq 5$ and $1 < q < 2^* - 1$.*

(1) *If $N = 4, 5$, then problem (1.2) has a unique least-energy solution u_λ for almost every $\lambda > 0$.*
(2) *If $N = 3$, then there exists $\mu_0 > 0$ such that (1.2) has a unique least-energy solution u_λ for almost every $\lambda > \mu_0$.*

Remark 1.1. As for the existence of least-energy solutions in Theorem 1.1, we refer the readers to [4]. The novelty is that we can prove the uniqueness of the least-energy solution for almost every λ.

Let us turn to the case of $q = 1$ and (1.1) becomes

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

It is well known that solutions of problem (1.3) are the critical points of the C^2 functional $I_{\lambda,\Omega} : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_{\lambda,\Omega}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx. \tag{1.4}$$

Let $\lambda_n(\Omega)$ be the n-th Dirichlet eigenvalue of $-\Delta$ on Ω counted with multiplicity. Then Brezis-Nirenberg [4] showed that for $N \geq 4$ and $\lambda \in (0, \lambda_1(\Omega))$ problem (1.3) has at least one positive solution. Later, Capozzi-Fortunato-Palmieri [5] and Zhang [21] studied the case of $\lambda \geq \lambda_1(\Omega)$ independently. They showed that, if $N = 4$ and $\lambda \neq \lambda_n(\Omega)$ for every $n \geq 1$, (1.3) has a nontrivial solution; if $N \geq 5$ (1.3) has a nontrivial solution for every $\lambda \geq \lambda_1(\Omega)$. In 2005, Clapp-Weth [8] got finitely many solutions to (1.3) for each $\lambda > 0$ and $N \geq 4$. The main result of [8] says that if $0 < \lambda < \lambda_1(\Omega)$, then (1.3) has at least $\lceil \frac{N+2}{2} \rceil$ pairs of nontrivial solutions; if $\lambda_n(\Omega) < \lambda < \lambda_{n+1}(\Omega)$, then (1.3) has at least $\lceil \frac{N+1}{2} \rceil$ pairs of nontrivial solutions; if $\lambda = \lambda_{n+1}(\Omega) = \cdots = \lambda_{n+m}(\Omega)$ is an eigenvalue of multiplicity $m < N+2$, then (1.3) has at least $\lceil \frac{N+1-m}{2} \rceil$ pairs of nontrivial solutions. Here, $\lceil a \rceil$ is the least integer n satisfying $n \geq a$ for $a > 0$. Recently, Chen-Shioji-Zou [10] improved Clapp-Weth's result by showing that (1.3) has at least $\lceil \frac{N+1}{2} \rceil$ pairs of nontrivial solutions for all $\lambda \geq \lambda_1(\Omega)$ if $N \geq 5$. Remark that all these solutions obtained in [4, 5, 21] satisfy $I_{\lambda,\Omega}(u) < \frac{1}{N} S_N^{N/2}$, and solutions obtained in [8, 10] satisfy $I_{\lambda,\Omega}(u) < \frac{2}{N} S_N^{N/2}$, where S_N is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$,

$$S_N \Big(\int_{\mathbb{R}^N} |u|^{2^*}\,dx \Big)^{2/2^*} \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

Note that S_N is achieved by the function $U_N(x) = A_N(1+|x|^2)^{-\frac{N-2}{2}}$, where A_N is a normalization constant (cf. [1, 19]). There are also many other papers studying (1.3), and related results can be seen in [2, 6, 7, 11, 17] and references therein.

Note that the existence of a nontrivial solution in the case of $N = 4$ and $\lambda = \lambda_1(\Omega)$ is ensured by the multiple result in [8], but we do not know whether this solution satisfies $I_{\lambda,\Omega}(u) < \frac{1}{N} S_N^{N/2}$. Besides, the existence of a nontrivial solution in the case of $N = 4$ and $\lambda = \lambda_n(\Omega)$ for $n \geq 2$ is still unknown. Remark that, in the case of $\lambda \geq \lambda_1(\Omega)$, solutions with $I_{\lambda,\Omega}(u) < \frac{1}{N} S_N^{N/2}$ are very important for the existence of the least-energy solutions of (1.3), since it was proved in [10] that (1.3) has a least-energy solution if there exists a solution with $I_{\lambda,\Omega}(u) < \frac{1}{N} S_N^{N/2}$. Therefore, *the existence* *of a nontrivial solution with* $I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}$ *in the case of* $N = 4$ *and* $\lambda = \lambda_n(\Omega)$ *for* $n \geq 1$ *remains an important open problem.*

From the results in [5, 21] as mentioned above, we see that, if $N \geq 5$, the existence of a nontrivial solution with $I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}$ holds for $\lambda = \lambda_n(\Omega)$. So there is a natural question about this open problem: Why is it that the very same ideas for proving the case of $N \geq 5$ can not be applied to solve the case of $N = 4$ and $\lambda = \lambda_n(\Omega)$ for $n \geq 1$?

To answer this question, we would like to give a very brief proof of the existence of nontrivial solutions in the case of $N \geq 5$. Fix any a $\lambda = \lambda_n(\Omega)$ for some $n \geq 1$. Let $\eta_R \in C_0^\infty(B_R, [0,1])$ be cut-off functions with $\eta_R \equiv 1$ in $B_{R/2}$. We choose a sequence of L^2-normalized orthonormal eigenfunctions e_k corresponding to Dirichlet eigenvalues $\lambda_k(\Omega)$, $k \in \mathbb{N}$, that is, $-\Delta e_k = \lambda_k(\Omega)e_k$ as $x \in \Omega$ and $e_k = 0$ when $x \in \partial\Omega$. Define

$$V_R^- := \operatorname{span}\{(1-\eta_{2R})e_1, \dots, (1-\eta_{2R})e_n\}.$$

Then for sufficiently small $R > 0$, we have (see Lemma 2.6 in [10] for example) that

$$\sup_{u \in V_R^-} I_{\lambda,\Omega}(u) \leq C_3(N)R^{\frac{N(N-2)}{2}}, \tag{1.5}$$

where $C_3(N)$ is a constant independent of R. In order to seek solutions with $I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}$ as in [5, 21], the key idea is to prove that for sufficiently small $R > 0$ there holds

$$\max_{t>0} I_{\lambda,\Omega}(t\eta_R U_N) \leq \frac{1}{N}S_N^{N/2} - C(N)R^{\alpha(N)}, \tag{1.6}$$

with some positive constants $C(N)$ and $\alpha(N)$ independent of R. Since the proof of (1.6) heavily depends on the formula of U_N, (1.6) only actually holds for $N \geq 5$. Moreover, $\alpha(N) < \frac{N(N-2)}{2}$ when $N \geq 5$.

Let $N \geq 5$. Define $V_R := \mathbb{R}\eta_R U_N \oplus V_R^-$, then $\dim(V_R) = n+1$. Note that $\operatorname{supp}(\eta_R U_N) \cap \operatorname{supp}(u) = \emptyset$ for any $u \in V_R^-$, we see from (1.5) and (1.6) that

$$\sup_{v \in V_R} I_{\lambda,\Omega}(v) \leq \max_{t>0} I_{\lambda,\Omega}(t\eta_R U_N) + \sup_{u \in V_R^-} I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}$$

for $R > 0$ sufficiently small. Note that $I_{\lambda,\Omega}$ satisfies $(PS)_c$ condition for $c < \frac{1}{N}S_N^{N/2}$ (see [18] for example). Then by the linking theorem (cf. [16]), (1.3) has a nontrivial solution u with $I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}$, and the proof is complete.

Note that $B_R \subset \Omega$ and $\lambda < \lambda_1(B_R)$ for R sufficiently small. Then by [4] we see that

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} \text{ in } B_R, \\ u > 0 \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R, \end{cases} \tag{1.7}$$

has a least-energy solution u_R (a mountain-pass type solution) with least energy

$$c(R) := I_{\lambda,B_R}(u_R) = \inf_{u \in H_0^1(B_R)\setminus\{0\}} \max_{t>0} I_{\lambda,B_R}(tu). \tag{1.8}$$

This implies

$$I_{\lambda,B_R}(u_R) \leq \max_{t>0} I_{\lambda,\Omega}(t\eta_R U_N).$$

Then, a natural idea is generated that it seems possible to solve the existence of solutions for $N = 4$ just as for $N \geq 5$ above, if we may prove

$$I_{\lambda,B_R}(u_R) \leq \frac{1}{N} S_N^{N/2} - C(N) R^{\alpha(N)} \tag{1.9}$$

instead of (1.6) when $N = 4$. Here, we give a negative answer to the inequality (1.9). Precisely, we have the following interesting theorem.

Theorem 1.2. *Assume $N \geq 4$ and $\lambda > 0$. Let $R_0 > 0$ such that $\lambda \leq \lambda_1(B_{R_0})$. Then for any $0 < R < R_0$, there exists a least-energy solution u_R of* (1.7), *and*

(1) *if $N \geq 5$, then there exist $\widetilde{R} \in (0, R_0)$ and positive constants C_1 and C_2 independent of R, such that for any $0 < R \leq \widetilde{R}$ there holds*

$$\frac{1}{N} S_N^{N/2} - C_1 R^{\frac{2N-4}{N-4}} \leq I_{\lambda,B_R}(u_R) \leq \frac{1}{N} S_N^{N/2} - C_2 R^{\frac{2N-4}{N-4}}; \tag{1.10}$$

(2) *if $N = 4$, then there exist $\widetilde{R} \in (0, R_0)$ and positive constants C_1, C_2, C_3, and C_4 independent of R, such that for any $0 < R \leq \widetilde{R}$ there holds*

$$\frac{1}{N} S_N^{N/2} - C_1 e^{-C_3 R^{-2}} \leq I_{\lambda,B_R}(u_R) \leq \frac{1}{N} S_N^{N/2} - C_2 e^{-C_4 R^{-2}}.$$

Remark 1.2. For $N = 4$, we in fact get, for any $k \in \mathbb{N}$, that

$$\lim_{R\to 0} \frac{\frac{1}{N} S_N^{N/2} - I_{\lambda,B_R}(u_R)}{R^k} = 0. \tag{1.11}$$

It is easily seen from (1.11) that (1.9) cannot hold for any positive constants $C(N)$ and $\alpha(N)$. (1.10) gives extremely accurate energy estimates for least-energy solutions of (1.7) in the case of $N \geq 5$. Comparing (1.11) with (1.10), we see that the case of $N = 4$ is essentially different from the case of $N \geq 5$

for the Brezis-Nirenberg problem, and the ideas of proving the existence of solutions with $I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}$ in the case of $N \geq 5$ cannot work in the particular case of $N = 4$ and $\lambda = \lambda_n(\Omega)$ for some $n \geq 1$.

Remark 1.3. As has been pointed out above, the existence of solutions for (1.3) in the case of $N \geq 5$ and $\lambda \geq \lambda_1(\Omega)$ was proved in [5, 21], where the formula of U_N played a crucial role. As an application of Theorem 1.2, we can give a new proof, which seems to be the first proof of obtaining solutions with energy below $\frac{1}{N}S_N^{N/2}$ for critical exponent problems but without using any information of U_N. Assume $\lambda \in [\lambda_n(\Omega), \lambda_{n+1}(\Omega))$ for some $n \geq 1$. Define $\widetilde{V}_R := \mathbb{R}u_R \oplus V_R^-$, then $\dim(\widetilde{V}_R) = n+1$. Noting that $\mathrm{supp}(u_R) \cap \mathrm{supp}(u) = \emptyset$ for any $u \in V_R^-$ and $\frac{2N-4}{N-4} < \frac{N(N-2)}{2}$ for $N \geq 5$, we see from (1.5) and (1.10) that

$$\sup_{v \in \widetilde{V}_R} I_{\lambda,\Omega}(v) \leq I_{\lambda,B_R}(u_R) + \sup_{u \in V_R^-} I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}$$

for $R > 0$ sufficiently small. Then by the linking theorem (cf. [16]), (1.3) has a nontrivial solution u with

$$I_{\lambda,\Omega}(u) < \frac{1}{N}S_N^{N/2}.$$

This completes the proof.

We will use a new elementary approach to prove these two theorems, and we believe that this approach can be used on other elliptic problems with a parameter. In the sequel, we denote positive constants (possibly different) dependent only on λ and N by C or $C(\lambda, N)$. We will prove Theorem 1.1 in Section 2. Theorem 1.2 will be proved in Section 3.

2. Proof of Theorem 1.1

In this section, we assume $N \geq 3$ and $1 < q < 2^* - 1$. When $N = 3$, by [4] there is some $\mu_0 > 0$ such that (1.2) has a least-energy solution for all $\lambda > \mu_0$. Let $\mu_1 = 0$ if $N \geq 4$, $\mu_1 = \mu_0$ if $N = 3$. Then (1.2) has a least-energy solution u_λ for any $\lambda \in (\mu_1, +\infty)$. Denote $a(\lambda) := J_\lambda(u_\lambda)$, where $J_\lambda : H_0^1(B_1) \to \mathbb{R}$ is the corresponding functional of (1.2) given by

$$J_\lambda(u) := \frac{1}{2}\int_{B_1} |\nabla u|^2\,dx - \frac{\lambda}{q+1}\int_{B_1} |u|^{q+1}\,dx - \frac{1}{2^*}\int_{B_1} |u|^{2^*}\,dx.$$

Since u_λ is a mountain-pass type solution (see [4]), it is easily seen that

$$a(\lambda) = \inf_{u \in H_0^1(B_1)\setminus\{0\}} \max_{t>0} J_\lambda(tu). \tag{2.1}$$

For any $u \in H_0^1(B_1)\backslash\{0\}$, it is easy to see that

$$\max_{t>0} J_\lambda(tu) = J_\lambda(t_{\lambda,u}u) \tag{2.2}$$
$$= \Big(\frac{1}{2} - \frac{1}{q+1}\Big) t_{\lambda,u}^2 \int_{B_1} |\nabla u|^2\, dx + \Big(\frac{1}{q+1} - \frac{1}{2^*}\Big) t_{\lambda,u}^{2^*} \int_{B_1} |u|^{2^*}\, dx,$$

where $t_{\lambda,u} > 0$ satisfies $f(\lambda, u, t_{\lambda,u}) = 0$ and

$$f(\lambda, u, t) := \lambda t^{q-1} \int_{B_1} |u|^{q+1}\, dx + t^{2^*-2} \int_{B_1} |u|^{2^*}\, dx - \int_{B_1} |\nabla u|^2\, dx, \quad t > 0. \tag{2.3}$$

Since $f(\lambda, u, \cdot)$ is increasing with respect to $t > 0$, $t_{\lambda,u}$ is unique. Moreover, $t_{\lambda,u_\lambda} = 1$ since u_λ is a critical point of J_λ.

Lemma 2.1. *$a(\lambda)$ is strictly decreasing with respect to $\lambda \in (\mu_1, +\infty)$.*

Proof. For any $\mu_1 < \lambda < \mu < +\infty$, it follows from (2.3) that

$$f(\mu, u_\lambda, 1) = (\mu - \lambda) \int_{B_1} |u_\lambda|^{q+1}\, dx > 0,$$

so $t_{\mu,u_\lambda} < 1$. Combining this with (2.1) and (2.2), we have

$$a(\mu) \le \max_{t>0} J_\mu(tu_\lambda)$$
$$< \Big(\frac{1}{2} - \frac{1}{q+1}\Big) \int_{B_1} |\nabla u_\lambda|^2\, dx + \Big(\frac{1}{q+1} - \frac{1}{2^*}\Big) \int_{B_1} |u_\lambda|^{2^*}\, dx = a(\lambda).$$

This completes the proof. □

From Lemma 2.1, we see that $a'(\lambda) := \frac{da(\lambda)}{d\lambda} \le 0$ exists for almost every $\lambda \in (\mu_1, +\infty)$. The following lemma gives the relation between $a'(\lambda)$ and u_λ, which is inspired by authors' other paper [9], but the ideas of the proof in [9] cannot be used in this lemma.

Lemma 2.2. *Let $\lambda \in (\mu_1, +\infty)$ such that $a'(\lambda)$ exists; then*

$$a'(\lambda) = -\frac{1}{q+1} \int_{B_1} |u_\lambda|^{q+1}\, dx.$$

Proof. Fix a $\lambda \in (\mu_1, +\infty)$ such that $a'(\lambda)$ exists. Denote

$$A = \int_{B_1} |\nabla u_\lambda|^2\, dx, \quad B = \int_{B_1} |u_\lambda|^{q+1}\, dx, \quad C = \int_{B_1} |u_\lambda|^{2^*}\, dx$$

for convenience, and consider

$$F(\mu, t) := f(\mu, u_\lambda, t) = \mu B t^{q-1} + C t^{2^*-2} - A.$$

Note $F(\lambda,1)=0$, $\frac{\partial}{\partial t}F(\lambda,1)=(q-1)\lambda B+(2^*-2)C>0$, and $F(\mu,t_{\mu,u_\lambda})\equiv 0$. By the implicit function theorem, there is a small neighborhood $(\lambda-\delta,\lambda+\delta)$ of λ in $(\mu_1,+\infty)$, such that $t(\mu):=t_{\mu,u_\lambda}\in C^\infty((\lambda-\delta,\lambda+\delta),\mathbb{R})$ as a function of μ. By $F(\mu,t(\mu))\equiv 0$ we see that

$$t'(\lambda)=-\frac{B}{\lambda(q-1)B+(2^*-2)C}. \tag{2.4}$$

By a Taylor expansion, we see that $t(\mu)=1+t'(\lambda)(\mu-\lambda)+O((\mu-\lambda)^2)$, and so

$$t^2(\mu)=1+2t'(\lambda)(\mu-\lambda)+O((\mu-\lambda)^2),$$
$$t^{2^*}(\mu)=1+2^*t'(\lambda)(\mu-\lambda)+O((\mu-\lambda)^2).$$

Note that $A=\lambda B+C$. Combining these with (2.2) and (2.4), we have

$$\begin{aligned}a(\mu)\le \max_{t>0}J_\mu(tu_\lambda)&=\Big(\frac12-\frac{1}{q+1}\Big)At^2(\mu)+\Big(\frac{1}{q+1}-\frac{1}{2^*}\Big)Ct^{2^*}(\mu)\\&=\Big(\frac12-\frac{1}{q+1}\Big)A+\Big(\frac{1}{q+1}-\frac{1}{2^*}\Big)C\\&\quad+\Big[\Big(1-\frac{2}{q+1}\Big)A+\Big(\frac{2^*}{q+1}-1\Big)C\Big]t'(\lambda)(\mu-\lambda)+O((\mu-\lambda)^2)\\&=a(\lambda)-\frac{B}{q+1}(\mu-\lambda)+O((\mu-\lambda)^2)\quad\text{as}\quad \mu\to\lambda.\end{aligned}$$

It follows that $\frac{a(\mu)-a(\lambda)}{\mu-\lambda}\ge -\frac{B}{q+1}+O((\mu-\lambda))$, as $\mu\nearrow\lambda$, and so $a'(\lambda)\ge -\frac{B}{q+1}$. Similarly, we have

$$\frac{a(\mu)-a(\lambda)}{\mu-\lambda}\le -\frac{B}{q+1}+O((\mu-\lambda)),\quad\text{as}\quad \mu\searrow\lambda;$$

that is, $a'(\lambda)\le -\frac{B}{q+1}$. Hence,

$$a'(\lambda)=-\frac{B}{q+1}=-\frac{1}{q+1}\int_{B_1}|u_\lambda|^{q+1}\,dx.$$

This completes the proof. □

Proof of Theorem 1.1. Fix a $\lambda\in(\mu_1,+\infty)$ such that $a'(\lambda)$ exists. Assume that u_1 and u_2 are both least-energy solutions of (1.2) with λ. By Lemma 2.2 we have

$$\int_{B_1}|u_1|^{q+1}\,dx=-(q+1)a'(\lambda)=\int_{B_1}|u_2|^{q+1}\,dx.$$

By elliptic regularity theory, we have $u_1,u_2\in C^2(B_1,\mathbb{R})$. By [13] we know that u_1 and u_2 are radial and that $u_1'(r)<0$ and $u_2'(r)<0$ for $r\in(0,1]$, where we write $u_1(r) = u_1(x), u_2(r) = u_2(x)$, and $r = |x|$ for convenience. Using Pohozaev's identity (cf. [15]) we have

$$\int_{\partial B_1} (x\cdot\nu)\Big(\frac{\partial u_1}{\partial\nu}\Big)^2 = \Big(\frac{2N}{q+1} - N + 2\Big)\lambda\int_{B_1} |u_1|^{q+1}\,dx = \int_{\partial B_1} (x\cdot\nu)\Big(\frac{\partial u_2}{\partial\nu}\Big)^2.$$

This implies that $u_1'(1) = u_2'(1) < 0$, and that u_1 and u_2 are both solutions of

$$\begin{cases} -u'' - \frac{N-1}{r}u' = \lambda u^q + u^{2^*-1}, \\ u'(0) = u(1) = 0, \quad u'(1) = u_1'(1). \end{cases}$$

Therefore, $u_1 \equiv u_2$. This completes the proof. □

3. Proof of Theorem 1.2

In this section, we assume $N \geq 4$ and $q = 1$. First we consider the equation (1.7) in B_1:

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} \ \text{ in } B_1, \\ u > 0 \ \text{ in } B_1, \quad u = 0 \ \text{ on } \partial B_1. \end{cases} \tag{3.1}$$

By [4] we see that (3.1) has a least-energy solution U_λ for any $\lambda \in (0, \lambda_1(B_1))$. Denote $e(\lambda) := I_\lambda(U_\lambda)$, where $I_\lambda := I_{\lambda,B_1}$ is the corresponding functional of (3.1) and $I_{\lambda,\Omega}$ is defined in (1.4). Then $e(\lambda) < e(0) := \frac{1}{N}S_N^{N/2}$. Similarly, for any $\lambda \in [0, \lambda_1(B_1))$ there holds

$$e(\lambda) = \inf_{u\in H_0^1(B_1)\backslash\{0\}} \max_{t>0} I_\lambda(tu), \tag{3.2}$$

and for any $u \in H_0^1(B_1)\backslash\{0\}$, we have

$$\max_{t>0} I_\lambda(tu) = I_\lambda(\tau_{\lambda,u}u) = \frac{1}{N}\tau_{\lambda,u}^{2^*}\int_{B_1} |u|^{2^*}\,dx, \tag{3.3}$$

where $\tau_{\lambda,u} > 0$ satisfies

$$\tau_{\lambda,u}^{2^*-2} = \frac{\int_{B_1}(|\nabla u|^2 - \lambda|u|^2)\,dx}{\int_{B_1}|u|^{2^*}\,dx}. \tag{3.4}$$

Here, the conclusion that (3.2) holds for $\lambda = 0$ is guaranteed by the fact that S_N is also the best constant in the embedding $H_0^1(B_1) \hookrightarrow L^{2^*}(B_1)$. By proofs similar to those in Section 2, we have

Lemma 3.1. *$e(\lambda)$ is strictly decreasing with respect to $\lambda \in [0, \lambda_1(B_1))$, and*

$$e'(\lambda) := -\frac{1}{2}\int_{B_1} |U_\lambda|^2\, dx$$

holds for almost every $\lambda \in (0, \lambda_1(B_1))$.

Using Lemma 3.1, we can give the relation between $e(\lambda)$ and $e(0)$ in the following lemma.

Lemma 3.2. *For any $\lambda \in (0, \lambda_1(B_1))$, there holds*

$$e(\lambda) = e(0) - \frac{1}{2}\int_0^\lambda \Big(\int_{B_1} |U_\mu|^2\, dx\Big) d\mu. \tag{3.5}$$

Proof. Fix a $\lambda_0 \in (0, \lambda_1(B_1))$ and consider $\lambda \in [0, \lambda_0]$ only. For any $\lambda, \mu \in [0, \lambda_0]$, $\mu < \lambda$, we see from (3.4) that

$$\tau_{\mu,U_\lambda}^{2^*-2} = \frac{\int_{B_1}(|\nabla U_\lambda|^2 - \mu|U_\lambda|^2)\, dx}{\int_{B_1} |U_\lambda|^{2^*}\, dx} = 1 + (\lambda-\mu)\beta(\lambda),$$

where

$$\beta(\lambda) = \frac{\int_{B_1} |U_\lambda|^2\, dx}{\int_{B_1}(|\nabla U_\lambda|^2 - \lambda|U_\lambda|^2)\, dx} \le \frac{1}{\lambda_1(B_1)-\lambda}.$$

Thus, we have from (3.2) and the mean value theorem that

$$\begin{aligned}
0 &< e(\mu) - e(\lambda) \le \max_{t>0} I_\mu(tU_\lambda) - e(\lambda) = \tau_{\mu,U_\lambda}^{2^*} e(\lambda) - e(\lambda)\\
&= \Big[1 + (\lambda-\mu)\beta(\lambda)\Big]^{N/2} e(\lambda) - e(\lambda)\\
&\le \frac{N}{2}\Big[1 + \beta(\lambda)(\lambda-\mu)\Big]^{\frac{N-2}{2}} \beta(\lambda) e(\lambda)(\lambda-\mu)\\
&\le \frac{N}{2}\Big(1 + \frac{\lambda-\mu}{\lambda_1(B_1)-\lambda}\Big)^{\frac{N-2}{2}} \frac{1}{\lambda_1(B_1)-\lambda} e(\lambda)(\lambda-\mu)\\
&\le \frac{N}{2}\Big(\frac{\lambda_1(B_1)}{\lambda_1(B_1)-\lambda_0}\Big)^{\frac{N-2}{2}} \frac{1}{\lambda_1(B_1)-\lambda_0} e(0)(\lambda-\mu) = C(\lambda_0)(\lambda-\mu).
\end{aligned}$$

Similarly, for any $\lambda, \mu \in [0, \lambda_0]$, $\mu > \lambda$, we also have $0 < e(\lambda) - e(\mu) \le C(\lambda_0)(\mu-\lambda)$. Hence, for any $\lambda, \mu \in [0, \lambda_0]$ we have $|e(\mu) - e(\lambda)| \le C(\lambda_0)|\mu - \lambda|$. That is, $\lambda \mapsto e(\lambda)$ is Lipschitz continuous with respect to $\lambda \in [0, \lambda_0]$. In particular it is absolutely continuous and so it follows from Lemma 3.1 that (3.5) holds for all $\lambda \in [0, \lambda_0]$. This completes the proof. $\square$

Now, let us fix any a $\lambda > 0$ and consider equation (1.7):

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} \ \text{ in } B_R, \\ u > 0 \ \text{ in } B_R, \quad u = 0 \ \text{ on } \partial B_R. \end{cases}$$

There exists $R_0 > 0$ such that $\lambda \le \lambda_1(B_{R_0})$. Then by [4] we see that (1.7) has a least-energy solution u_R with least-energy $c(R) := I_{B_R}(u_R)$ for any $R \in (0, R_0)$, where $I_{B_R} := I_{\lambda, B_R}$ is the corresponding functional of (1.7).

By [13] we know that u_R must be radial, and we can write $u_R(r) = u_R(x)$ for convenience, where $r = |x|$. Recall that $u_R \in C^2([0, R])$; we denote $u'_R(R) := \frac{d}{dr} u_R(r)|_{r=R}$. Then we have the following lemma, which plays a crucial role in the proof of Theorem 1.2.

Lemma 3.3. *$c(R)$ is strictly decreasing with respect to R, and for any $R \in (0, R_0)$, there holds*

$$c(R) = I_{B_R}(u_R) = \frac{1}{N} S_N^{N/2} - \frac{|S^{N-1}|}{2} \int_0^R |u'_r(r)|^2 r^{N-1}\, dr.$$

Here, S^{N-1} is the unit sphere of $\mathbb{R}^N$ and $|S^{N-1}|$ is the Lebesgue measure of S^{N-1}.

Proof. By Pohozaev's identity (cf. [15]) we have

$$\lambda \int_{B_R} |u_R|^2\, dx = \frac{1}{2} \int_{\partial B_R} (x \cdot \nu) \Big(\frac{\partial u_R}{\partial \nu} \Big)^2 = \frac{|S^{N-1}|}{2} |u'_R(R)|^2 R^N. \tag{3.6}$$

Let $v_R(x) := R^{\frac{N-2}{2}} u_R(Rx)$, then v_R satisfies

$$-\Delta v = \lambda R^2 v + v^{2^*-1}, \qquad x \in B_1.$$

That is, v_R is a least-energy solution of (3.1) with λR^2. Recall that $U_{\lambda R^2}$ is also a least-energy solution of (3.1) with λR^2; we see from Lemma 3.1 that

$$\int_{B_1} |v_R|^2\, dx = \int_{B_1} |U_{\lambda R^2}|^2\, dx = -2e'(\lambda R^2) \tag{3.7}$$

holds for almost every $R \in (0, R_0)$. It is easily seen that $c(R) = I_{B_R}(u_R) = I_{\lambda R^2}(v_R) = e(\lambda R^2)$, which implies that $c(R)$ is strictly decreasing with respect to R. From (3.6) we have

$$\int_{B_1} |v_R|^2\, dx = R^{-2} \int_{B_R} |u_R|^2\, dx = \frac{|S^{N-1}|}{2\lambda} |u'_R(R)|^2 R^{N-2}.$$

Combining this with Lemma 3.2 and (3.7), we have

$$\begin{aligned} c(R) = e(\lambda R^2) &= e(0) - \frac{1}{2}\int_0^{\lambda R^2} \Big(\int_{B_1} |U_\mu|^2\,dx\Big)d\mu \\ &= e(0) - \frac{1}{2}\int_0^R \Big(\int_{B_1} |v_r|^2\,dx\Big)2\lambda r\,dr \quad (\text{ let } \mu = \lambda r^2) \\ &= \frac{1}{N}S_N^{N/2} - \frac{|S^{N-1}|}{2}\int_0^R |u_r'(r)|^2 r^{N-1}\,dr. \end{aligned}$$

This completes the proof. □

Recall the fact that

$$\lim_{R\to 0} u_R(0) = +\infty. \tag{3.8}$$

Actually, if there exists $R_n \to 0$ such that $u_{R_n}(0) \le M$ for some constant $M > 0$, then by the Dominated Convergence Theorem we have

$$0 < Ne(0) = \lim_{R_n\to 0} Nc(R_n) = \lim_{R_n\to 0}\int_{B_{R_n}} |u_{R_n}|^{2^*}\,dx = 0,$$

which is a contradiction. Note that $u_R(r)$ is the unique solution of

$$\begin{cases} -u'' - \frac{N-1}{r}u' = \lambda u + u^{2^*-1}, \\ u'(0) = 0, \quad u(0) = u_R(0). \end{cases} \tag{3.9}$$

As in [3], we define

$$y(t) := \lambda^{-\frac{N-2}{4}} u\Big((N-2)\lambda^{-\frac{1}{2}} t^{-\frac{1}{N-2}}\Big), \tag{3.10}$$

then $-u'' - \frac{N-1}{r}u' = \lambda u + u^{2^*-1}$ becomes

$$y''(t) + t^{-\frac{2(N-1)}{N-2}}\Big(y(t) + y^{2^*-1}(t)\Big) = 0.$$

Denote $k = \frac{2(N-1)}{N-2} > 2$; then $2^* - 1 = 2k - 3$. Let us consider

$$\begin{cases} y''(t) + t^{-k}\big(y(t) + y^{2k-3}(t)\big) = 0, \\ \lim_{t\to+\infty} y(t) = \gamma > 0. \end{cases} \tag{3.11}$$

Since $k > 2$, the existence of a unique positive solution $y_\gamma(t)$ for (3.11) is ensured for t large (cf. [3]). Define

$$T(\gamma) := \inf\{t > 0 : y_\gamma(\tau) > 0,\ \forall\,\tau > t\}, \tag{3.12}$$

and

$$z_\gamma(t) := \gamma t\Big(t^{k-2} + \frac{1+\gamma^{2k-4}}{k-1}\Big)^{-\frac{1}{k-2}}. \tag{3.13}$$

Then (see Lemma 1 and Remark 1 in [3]) $y_\gamma(t) \le z_\gamma(t)$ for any $t \ge T(\gamma)$ and $z_\gamma(t)$ satisfies

$$z''(t) + t^{-k}\frac{1+\gamma^{2k-4}}{\gamma^{2k-4}} z^{2k-3}(t) = 0. \tag{3.14}$$

We recall some results from [3].

Lemma 3.4. (See Theorem 3 in [3]). *Suppose $k > 2$. Then there exist positive constants C_1, C_2, and γ_0, which depend on k, such that the following inequalities holds for all $\gamma \ge \gamma_0$.*

(1) *If $k = 3$, then $C_1 \log\gamma < T(\gamma) < C_2 \log\gamma$.*
(2) *If $k < 3$, then $C_1\gamma^{6-2k} < T(\gamma) < C_2\gamma^{6-2k}$.*

Using Lemma 3.4, we can prove the following lemma.

Lemma 3.5. *Suppose $2 < k \le 3$. Then there exist positive constants C_3 and $\gamma_1 \ge \gamma_0$, which depend on k, such that*

$$y'_\gamma(T(\gamma)) := \frac{d}{dt}y_\gamma(t)|_{t=T(\gamma)} \le \frac{C_3}{\gamma}$$

holds for all $\gamma \ge \gamma_1$.

Proof. From (3.13) we have $z_\gamma(t) \le (k-1)^{\frac{1}{k-2}}\frac{t}{\gamma}$ and

$$z'_\gamma(t) = \gamma\frac{1+\gamma^{2k-4}}{k-1}\Big(t^{k-2} + \frac{1+\gamma^{2k-4}}{k-1}\Big)^{-\frac{1}{k-2}-1} < (k-1)^{\frac{1}{k-2}}\frac{1}{\gamma}$$

for all $t \ge T(\gamma)$. Combining these with (3.11) and (3.14), we have

$$\begin{aligned}
y'_\gamma(T(\gamma)) := \frac{d}{dt}y_\gamma(t)|_{t=T(\gamma)} &= \int_{T(\gamma)}^{+\infty} \frac{y_\gamma(t) + y_\gamma^{2k-3}(t)}{t^k}\,dt \\
&\le \int_{T(\gamma)}^{+\infty}\frac{z_\gamma(t)}{t^k}\,dt + \int_{T(\gamma)}^{+\infty}\frac{z_\gamma^{2k-3}(t)}{t^k}\,dt \\
&\le (k-1)^{\frac{1}{k-2}}\frac{1}{\gamma}\int_{T(\gamma)}^{+\infty} t^{1-k}\,dt - \frac{\gamma^{2k-4}}{1+\gamma^{2k-4}}\int_{T(\gamma)}^{+\infty} z''_\gamma(t)\,dt \\
&= (k-1)^{\frac{1}{k-2}}\frac{1}{k-2}T(\gamma)^{2-k}\frac{1}{\gamma} + \frac{\gamma^{2k-4}}{1+\gamma^{2k-4}}z'_\gamma(T(\gamma))
\end{aligned}$$

$$\leq \Big(\frac{(k-1)^{\frac{1}{k-2}}}{k-2}T(\gamma)^{2-k} + (k-1)^{\frac{1}{k-2}}\Big)\frac{1}{\gamma}.$$

From Lemma 3.4 we see that $T(\gamma) \to +\infty$ as $\gamma \to +\infty$, which completes the proof. □

Lemma 3.6. *There exists $R_1 \in (0, R_0)$ and positive constants C_4 and C_5, which depend only on N and λ, such that for any $R \leq R_1$ there holds*

$$\frac{C_4}{u_R(0)R^{N-1}} \leq |u_R'(R)| \leq \frac{C_5}{u_R(0)R^{N-1}}. \tag{3.15}$$

Proof. Denote $y_R(t) := \lambda^{-\frac{N-2}{4}} u_R\left((N-2)\lambda^{-\frac{1}{2}} t^{-\frac{1}{N-2}}\right)$ in (3.10). Then y_R satisfies (3.11) with $k = \frac{2(N-1)}{N-2}$ and

$$\gamma = \lambda^{-\frac{N-2}{4}} u_R(0), \quad T(\gamma) = (N-2)^{N-2}\lambda^{-\frac{N-2}{2}} R^{2-N}, \tag{3.16}$$

and so

$$y_R'(T(\gamma)) = -\lambda^{-\frac{N}{4}} u_R'(R) T(\gamma)^{-\frac{N-1}{N-2}} = (N-2)^{1-N}\lambda^{\frac{N-2}{4}} |u_R'(R)| R^{N-1}.$$

Then by (3.8) and Lemma 3.5, there exists $R_1 \in (0, R_0)$ and a constant C_5, which depend only on N and λ, such that for any $R \leq R_1$ the right-hand side of (3.15) holds. On the other hand, from (3.9) and Lemma 3.3 we have

$$\begin{aligned} R^{N-1}|u_R'(R)| &= \int_0^R r^{N-1}\left(\lambda u_R(r) + u_R^{2^*-1}(r)\right) dr \\ &> \frac{1}{u_R(0)} \int_0^R r^{N-1} u_R^{2^*}(r)\, dr = \frac{1}{u_R(0)} \frac{1}{|S^{N-1}|} \int_{B_R} u_R^{2^*}(x)\, dx \\ &= \frac{Nc(R)}{|S^{N-1}|} \frac{1}{u_R(0)} \geq \frac{Nc(R_1)}{|S^{N-1}|} \frac{1}{u_R(0)} = \frac{C_4}{u_R(0)}, \end{aligned}$$

for $R \leq R_1$, so the left-hand side of (3.15) holds. This completes the proof. □

Lemma 3.7. *There exists $R_2 \in (0, R_1]$ and positive constants C_6, C_7, C_8, and C_9, which depend only on N and λ, such that the following inequalities hold for any $R \leq R_2$.*

(1) *If $N = 4$, then $\lambda^{1/2} e^{C_6 R^{-2}} < u_R(0) < \lambda^{1/2} e^{C_7 R^{-2}}$.*

(2) *If $N \geq 5$, then $C_8 R^{-\frac{(N-2)^2}{2N-8}} < u_R(0) < C_9 R^{-\frac{(N-2)^2}{2N-8}}$.*

Proof. Note that $k = \frac{2(N-1)}{N-2} = 3$ when $N = 4$, and $k < 3$ when $N \geq 5$; this lemma follows directly from Lemma 3.4 and (3.16). □

Proof of Theorem 1.2. Fix an $R \in (0, R_2]$. First we assume $N \geq 5$. Then we see from Lemma 3.6 and Lemma 3.7 that

$$\int_0^R |u_r'(r)|^2 r^{N-1}\,dr \geq C_4^2 \int_0^R \frac{1}{|u_r(0)|^2 r^{N-1}}\,dr \geq C(N,\lambda)\int_0^R r^{\frac{(N-2)^2}{N-4}-N+1}\,dr \geq C_1(N,\lambda) R^{\frac{2N-4}{N-4}}.$$

Similarly,

$$\int_0^R |u_r'(r)|^2 r^{N-1}\,dr \leq C_2(N,\lambda) R^{\frac{2N-4}{N-4}}.$$

Thus, by Lemma 3.3 we see that (1.10) in Theorem 1.2 holds. Now, we assume $N = 4$; then

$$\int_0^R |u_r'(r)|^2 r^{N-1}\,dr \geq C_4^2 \int_0^R \frac{1}{|u_r(0)|^2 r^3}\,dr \geq \frac{C_4^2}{\lambda}\int_0^R e^{-2C_7 r^{-2}} r^{-3}\,dr = \frac{C_4^2}{2\lambda}\int_{R^{-2}}^{+\infty} e^{-2C_7 t}dt = \frac{C_4^2}{4\lambda C_7} e^{-2C_7 R^{-2}}.$$

Similarly,

$$\int_0^R |u_r'(r)|^2 r^{N-1}\,dr \leq \frac{C_5^2}{4\lambda C_6} e^{-2C_6 R^{-2}}.$$

That is, when $N = 4$, then

$$\frac{1}{N} S_N^{N/2} - \frac{|S^{N-1}| C_5^2}{8\lambda C_6} e^{-2C_6 R^{-2}} \leq I_{\lambda, B_R}(u_R) \leq \frac{1}{N} S_N^{N/2} - \frac{|S^{N-1}| C_4^2}{8\lambda C_7} e^{-2C_7 R^{-2}}$$

holds for any $R \in (0, R_2]$, which implies that (1.11) holds. This completes the proof. □

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