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RESONANCE AT THE FIRST EIGENVALUE FOR FIRST-ORDER SYSTEMS IN THE PLANE: VANISHING HAMILTONIANS AND THE LANDESMAN-LAZER CONDITION

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Abstract. The concept of resonance with the first eigenvalue ($\lambda = 0$) of the scalar T-periodic problem

$$x'' + \lambda x = 0, \quad x(0) = x(T),\ x'(0) = x'(T)$$

is considered for first-order planar systems, by dealing with positively homogeneous Hamiltonians which can vanish at some points on $\mathbb{S}^1$. By means of degree methods, an existence result at double resonance for a planar system of the kind

$$Ju' = F(t, u), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is then proved, under the assumption that $F(t, u)$ is controlled from below by the gradient of one of such Hamiltonians described above, complementing the main theorem in [7] and including some classical results for the scalar case.

1. Introduction

In the recent paper [7], a Landesman-Lazer existence result was given for nonlinear first-order planar systems, assuming that the nonlinearity is controlled by the gradients of two positively homogeneous Hamiltonians at resonance. To recall the precise statement, let us denote by J the standard symplectic matrix, namely

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

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and by $\mathcal{P}$ the set of C^1-functions $H : \mathbb{R}^2 \to \mathbb{R}$, with locally Lipschitz continuous gradient, which are positively homogeneous of degree 2 and positive; i.e.,

$$0 < H(\lambda u) = \lambda^2 H(u), \quad \lambda > 0,\, u \in \mathbb{R}^2 \setminus \{0\}.$$

It is essential to recall that, given a Hamiltonian $H \in \mathcal{P}$, all the nontrivial solutions to the system $Ju' = \nabla H(u)$ are periodic with the same minimal period τ; that is, the origin is an isochronous center for the system.

Here is the statement of [7, Theorem 2.1].

Theorem 1.1. *Consider the following T-periodic problem in $\mathbb{R}^2$:*

$$Ju' = F(t,u), \quad u(0) = u(T), \tag{1.1}$$

with $F(t,u)$ an L^2-Carathéodory function. Assume that there exist $H_1, H_2 \in \mathcal{P}$, satisfying $H_1 \leq H_2$, such that

$$F(t,u) = (1-\gamma(t,u))\nabla H_1(u) + \gamma(t,u)\nabla H_2(u) + r(t,u), \tag{1.2}$$

$\gamma(t,u)$ and $r(t,u)$ being two L^2-Carathéodory functions such that $0 \leq \gamma(t,u) \leq 1$, and $|r(t,u)| \leq \partial(t)$, for a suitable $\partial \in L^2(0,T)$. Denoting by τ_i the minimal period of the nontrivial solutions to $Ju' = \nabla H_i(u)$, $i = 1,2$, assume that there exists $N \in \mathbb{N}$ such that

$$\frac{T}{N+1} \leq \tau_2 < \tau_1 \leq \frac{T}{N}\,. \tag{1.3}$$

Finally, fixed nonzero functions φ and ψ satisfying, respectively,

$$J\varphi' = \nabla H_1(\varphi), \qquad \text{and} \qquad J\psi' = \nabla H_2(\psi),$$

assume that, for every $\theta \in [0,T]$, the following Landesman-Lazer conditions are fulfilled:

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \left[\langle F(t,\lambda\varphi(t+\omega)) | \varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t))\right] dt > 0, \tag{1.4}$$

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \left[2\lambda H_2(\psi(t)) - \langle F(t,\lambda\psi(t+\omega)) | \psi(t+\omega)\rangle\right] dt > 0, \tag{1.5}$$

where $\langle\cdot|\cdot\rangle$ denotes the scalar product in $\mathbb{R}^2$. Then, problem (1.1) *has a solution.*

Theorem 1.1 includes various existence results for scalar second-order equations, obtained through the years as generalizations of the pioneering theorem by Lazer and Leach [11] (extended in [10] to the PDE setting).

The Landesman-Lazer condition was introduced to provide existence when the nonlinearity is a perturbation of a noninvertible linear operator, or, as is commonly said borrowing the terminology from the linear setting, is *at resonance*. In this case, without further assumptions on the nonlinearity, the existence of a solution to the considered problem is not ensured. Focusing on the T-periodic problem associated with a scalar second-order differential equation like

$$x'' + g(t,x) = 0, \tag{1.6}$$

this concept of (nonlinear) resonance can easily be expressed, comparing the asymptotic behavior of the ratio $\frac{g(t,x)}{x}$ with the elements of the spectrum of the T-periodic problem, namely

$$\Sigma^P = \left\{ \lambda_m = \left(\frac{2\pi m}{T}\right)^2 : m = 0, 1, 2, \dots \right\}.$$

In particular, if $g(t,x)$ asymptotically behaves like a line $\lambda_m x$, for a nonnegative integer m, then equation (1.6) may not have a T-periodic solution (see [13] for an interesting survey about resonance).

Beginning in the seventies, several authors provided many original results of Landesman-Lazer type, under resonance assumptions which step by step became more general, and in more general abstract frameworks (suitable also for other kinds of problems, see, e.g., the references in [7, 12]). In particular, such theorems were proved in a situation of double resonance, namely when, referring to (1.6), $\frac{g(t,x)}{x}$ can asymptotically assume, at some time instants, the value λ_m and, at other ones, the value λ_{m+1}, i.e., two consecutive eigenvalues of the T-periodic problem (see [5], and [4] for an asymmetric version). In this case, two Landesman-Lazer conditions are needed to ensure existence, keeping the nonlinearity "far" from each eigenvalue.

Theorem 1.1 was born with the aim of extending these Landesman-Lazer-type results when double resonance occurs in the setting of planar systems. As a starting point, the concept of resonance had been generalized to systems in $\mathbb{R}^2$ in [6], where an organic analysis of forced positively homogeneous systems of the kind

$$Ju' = \nabla H(u) + f(t),$$

with $H \in \mathcal{P}$, was performed. In particular, it was shown that a situation of resonant type can arise when the minimal period associated with ∇H is a submultiple of T. Roughly speaking, thus, the gradients of positively homogeneous (of degree 2) Hamiltonians play in $\mathbb{R}^2$ the same role, as comparison terms, as the lines $\lambda_m x$ do in $\mathbb{R}$ (in fact, the deriving concept of resonance extends both the linear and the asymmetric ones of the scalar setting). Indeed, conditions (1.2) and (1.3) just express the concept that the nonlinearity $F(t,u)$ interacts with the gradients of two "consecutive"

Hamiltonians belonging to $\mathcal{P}$, both at resonance. Consequently, the two Landesman-Lazer conditions (1.4) and (1.5) are needed in order to reach the desired conclusion.

However, as also pointed out in [8], referring, for instance, to (1.6), Theorem 1.1 does not include the case when resonance occurs with the first eigenvalue of the T-periodic problem, $\lambda_0 = 0$. This situation is of interest and has been widely investigated by many authors. For instance, this happens when $g(t,x)$ is bounded (simple resonance), or, in more generality, when it is of the form

$$g(t,x) = \omega(t,x)\lambda_1 x + r(t,x),$$

with $0 \leq \omega(t,x) \leq 1$ and $r(t,x)$ bounded (double resonance with λ_0 and λ_1). Indeed, asking that $H_1 > 0$ in (1.2) implies that we are not considering, as a comparison term, the gradient of the Hamiltonian $H(x,y) = \frac{1}{2}y^2$, naturally associated with the scalar equation $x'' = 0$.

The aim of this paper is to complement Theorem 1.1 in order to include also this situation. To do this, it will be necessary to permit the considered Hamiltonians to vanish along some directions. However, referring to the autonomous system

$$Ju' = \nabla H(u), \tag{1.7}$$

there is an important qualitative difference with respect to the previously considered case.

The remarkable feature of a Hamiltonian $H \in \mathcal{P}$, as already mentioned, is that the origin is an *isochronous center* for (1.7). In particular, the level curves of H, on which the motion takes place, are strictly star-shaped Jordan curves around the origin. For this reason, they can be used to introduce some kind of modified polar coordinates, often simplifying the computations.

On the contrary, when $H(u) = 0$ for some $u \neq 0$, it is not true that all the solutions to the Cauchy problems associated with (1.7) are periodic, as is immediately seen considering the scalar equation $x'' = 0$. Moreover, it can be proved, by homogeneity, that the level curves of H are unbounded, so that no modified polar coordinates can easily be used. However, as we will see, the preservation of the energy will provide sufficient information about (1.7), which will be enough to treat also this case. Nevertheless, some care will be necessary, and the Landesman-Lazer condition related to the first eigenvalue will undergo some slight modifications, because of this lack of modified polar coordinates.

The contents of the article are the following. In Section 2, we introduce the class $\mathcal{P}^*$ of positively homogeneous Hamiltonians which can vanish at some points in $\mathbb{R}^2$, and we analyze their main qualitative features, as well as the dynamics of the associated autonomous Hamiltonian systems. In Section 3, we give and discuss our main result of existence, Theorem 3.2, while in Section 4 we briefly examine the scalar case, comparing Theorem 3.2 with some known results in this framework.

2. Vanishing Hamiltonians: a qualitative analysis

We give the following definition.

Definition 2.1. *We set*

$$\mathcal{P}^* = \left\{ H : \mathbb{R}^2 \to \mathbb{R} \ : \ \begin{array}{l} H \in C^1, \nabla H \text{ is locally Lipschitz continuous} \\ 0 \leq H(\lambda u) = \lambda^2 H(u),\, \lambda > 0,\, u \in \mathbb{R}^2. \end{array} \right\}$$

The elements of $\mathcal{P}^*$ are thus nonnegative C^1-functions which have a locally Lipschitz continuous gradient and are positively homogeneous of degree 2. Obviously, $\mathcal{P} \subset \mathcal{P}^*$; notice, moreover, that $\mathcal{P}^* + \mathcal{P} \subset \mathcal{P}$.

Some remarks are in order. First, if $\bar{u} \neq 0$ is such that $H(\bar{u}) = 0$, with $H \in \mathcal{P}^*$, then H vanishes on the whole half-line passing through the origin and $\bar{u}$. For this reason, to determine unambiguously the set where H is zero, it will be sufficient to compute H on $\mathbb{S}^1 = \{u \in \mathbb{R}^2 : |u| = 1\}$. Secondly, notice that, if $H \in \mathcal{P}^*$, for every $u_0 \in \mathbb{R}^2 \setminus \{0\}$ such that $H(u_0) = 0$, we have $\nabla H(u_0) = 0$, since u_0 is a minimum point of the C^1-function H. Conversely, by Euler's formula for functions which are homogeneous of degree 2, namely

$$\langle \nabla H(u_0) | u_0 \rangle = 2H(u_0),$$

every zero of ∇H is a zero of H. Therefore, from now on we will not distinguish between zeros of H and zeros of ∇H.

Let us consider the autonomous Hamiltonian system

$$Ju' = \nabla H(u), \quad u \in \mathbb{R}^2, \tag{2.1}$$

with $H \in \mathcal{P}^* \setminus \mathcal{P}$ (the case $H \in \mathcal{P}$ has been extensively treated in [6]). We will search for classical solutions to (2.1). To this aim, observe first that the uniqueness for the Cauchy problems associated with (2.1) is guaranteed, since ∇H is locally Lipschitz continuous. The following proposition asserts that the only periodic solutions to (2.1) are given by the constant functions which are equal to the zeros of H.

Proposition 2.2. *Let $H \in \mathcal{P}^* \setminus \mathcal{P}$. Denote by $\{u_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathbb{S}^1$ ($\mathcal{A}$ being a set of indices) the (nonempty) set where H vanishes, namely*

$$H(u) = 0,\, u \in \mathbb{R}^2 \quad \Longleftrightarrow \quad u = k u_\alpha,\, \alpha \in \mathcal{A},\, k \geq 0.$$

Then, the only periodic solutions to equation (2.1) *are given by* $u(t) \equiv k u_\alpha$, $\alpha \in \mathcal{A}$, $k \geq 0$.

Proof. Observe first that, by homogeneity, H vanishes at every point of the form $k u_\alpha$, for every $\alpha \in \mathcal{A}$, $k \geq 0$, so that the constants $k u_\alpha$, $\alpha \in \mathcal{A}$, $k \geq 0$, are T-periodic solutions to (2.1), for every $T > 0$. Moreover, if $u(t)$ solves (2.1),

$$\frac{d}{dt} H(u(t)) = \langle \nabla H(u(t)) | u'(t) \rangle = \langle J u'(t) | u'(t) \rangle = 0;$$

i.e., the energy is preserved along the solutions to (2.1). Consider now one such solution, say $u(t)$.

Assume that $H(u(0)) = 0$. Then, $u(0) = k u_\alpha$ for some $\alpha \in \mathcal{A}$, $k > 0$ (if $k = 0$, by uniqueness $u(t) \equiv 0$). By the preservation of the energy, $H(u(t)) = 0$ for every t. Since $u(t)$ cannot reach 0 by uniqueness, it is possible to write it in polar coordinates, $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, and from the expression of $\theta'(t)$,

$$-\theta'(t) = \frac{\langle J u'(t) | u(t) \rangle}{\rho(t)^2} = \frac{2H(u(t))}{\rho(t)^2}, \tag{2.2}$$

we immediately see that the motion is radial. Since $u(t) \neq 0$ for every t, the motion will take place along the open half-line starting from the origin and passing through u_α, namely $u(t) = \rho(t) u_\alpha$, for a suitable positive function $\rho(t)$ such that $\rho(0) = k$. By computing $\rho'(t)$, we see that

$$\rho'(t) = \frac{\langle J u'(t) | J u(t) \rangle}{\rho(t)} = \rho(t) \langle \nabla H(u_\alpha) | J u_\alpha \rangle = 0,$$

since $\nabla H(u_\alpha) = 0$. It follows that $u(t)$ is constantly equal to $k u_\alpha$.

Assume, on the contrary, that $H(u(0)) > 0$. Then, $u(0) \neq k u_\alpha$ for every $\alpha \in \mathcal{A}$, $k \geq 0$. Again by the preservation of the energy, it follows that $H(u(t)) > 0$ for every $t \in [0, T]$, and so $u(t) \neq k u_\alpha$ for every $k \geq 0$, $\alpha \in \mathcal{A}$ and $t \in [0, T]$. However, if $u(t)$ is periodic, this cannot happen since, in view of (2.2), $\theta'(t)$ has constantly strictly negative sign. Hence, no periodic solutions to (2.1) with positive energy can exist, concluding the proof. □

Remark 2.1. From the first part of the proof, we can deduce partial information about the behavior of the solutions to the Cauchy problems associated with (2.1). In particular, if we start with 0-energy, i.e., in a point u_0 such that $H(u_0) = 0$, then, thanks to the homogeneity of H and ∇H, the only possible motions are the constant ones. If we start with positive energy, we only know that the angle associated with our solution will strictly decrease, but different behaviors may be possible, in principle (see the considerations below).

Example 2.1. For $u = (x, y) \in \mathbb{R}^2$ and h a positive constant, consider $H_1(x, y) = \frac{1}{2}hy^2$, $H_2(x, y) = \frac{1}{2}hx^2$, vanishing, respectively, in $(s, 0)$ and $(0, s)$, for $s \in \mathbb{R}$. These Hamiltonians correspond, respectively, to the systems

$$\begin{cases} x' = hy \\ y' = 0, \end{cases} \qquad \begin{cases} x' = 0 \\ y' = -hx \end{cases}$$

(the first one obviously equivalent to the scalar equation $x'' = 0$). It is well-known that the only periodic solutions to such systems are the constant ones $(x(t), y(t)) \equiv (s, 0)$ in the first case and $(x(t), y(t)) \equiv (0, s)$ in the second one, $s \in \mathbb{R}$.

Let now $H \in \mathcal{P}^* \setminus \mathcal{P}$. We are now going to make some qualitative considerations about the level curves of H. From now on, we denote by $u(t; u_0)$ the solution to the Cauchy problem

$$Ju' = \nabla H(u), \quad u(0) = u_0;$$

moreover, we set

$$\hat{H}(\theta) = H(\cos\theta, \sin\theta). \tag{2.3}$$

Once $\hat{H}(\theta)$ is defined for every $\theta \in [0, 2\pi)$, by homogeneity H is unambiguously determined on the whole plane. Assume first that there is only one direction, determined by an angle $\bar{\theta} \in [0, 2\pi)$ with respect to the horizontal axis, along which H vanishes: $\hat{H}(\bar{\theta}) = 0$. For simplicity, assume $0 < \bar{\theta} < \pi/2$, but analogous considerations hold in the other cases. As a notation, for $c \geq 0$ set $\gamma_c = \{(x, y) \in \mathbb{R}^2 : H(x, y) = c\}$; it is clear that $\gamma_{c_1} \cap \gamma_{c_2} = \emptyset$ for $c_1 \neq c_2$, and $\cup_{c \geq 0} \gamma_c = \mathbb{R}^2$.

The homogeneity of the continuous function H has some important consequences on the shape of γ_c, $c > 0$. Indeed, the curve γ_c is such that every ray emanating from the origin, apart from the one corresponding to the slope $\tan\bar{\theta}$, intersects it exactly once (so that, in particular, γ_c is "star-shaped" with respect to the origin). Moreover, by continuity, since H vanishes on the unbounded set $\{k(\cos\bar{\theta}, \sin\bar{\theta}) : k \geq 0\}$, when the angle θ on γ_c approaches $\bar{\theta}$ (in both directions), the curve γ_c has to become unbounded, otherwise the star-shapedness would be violated. Taking into account, moreover, that θ is strictly decreasing on γ_c in view of (2.2), for a fixed a point $u_0 \in \mathbb{R}^2$ such that $H(u_0) = c$ it will be $\gamma_c = \{u(t; u_0)\}_{t \in \mathbb{R}}$. As a consequence of this discussion, we have

$$\lim_{|t| \to +\infty} |u(t; u_0)| = +\infty. \tag{2.4}$$

Writing the solution (which never reaches the origin by uniqueness) in polar coordinates, namely

$$u(t;u_0) = |u(t;u_0)|(\cos\theta(t;\theta_0), \sin\theta(t;\theta_0)),$$

since

$$H(u(t;u_0)) = |u(t;u_0)|^2 H(\cos\theta(t;\theta_0), \sin\theta(t;\theta_0))$$

we will have, in view of the preservation of the energy and (2.4),

$$\hat{H}(\theta(t;\theta_0)) \to 0 \quad \text{for } |t| \to +\infty.$$

It follows that, if $\theta_0 > \bar{\theta}$,

$$\theta(t;\theta_0) \to \bar{\theta} \ \text{ for } t \to +\infty, \quad \theta(t;\theta_0) \to \bar{\theta} - 2\pi \ \text{ for } t \to -\infty, \tag{2.5}$$

and, if $\theta_0 < \bar{\theta}$,

$$\theta(t;\theta_0) \to \bar{\theta} + 2\pi \ \text{ for } t \to +\infty, \quad \theta(t;\theta_0) \to \bar{\theta} \ \text{ for } t \to -\infty, \tag{2.6}$$

as was also possible to see in view of the strict monotonicity of the angle θ along the level curves of H. As an idea, with respect to a Hamiltonian belonging to $\mathcal{P}$, one can think as if the level curves were "broken" in correspondence with the angle $\bar{\theta}$ and then spread, in some sense, along the line having $\tan\bar{\theta}$ as slope. Anyway, (2.5) and (2.6) do not necessarily imply that the level curves of H approach asymptotically the half-lines where H is zero; to explain this, let us discuss the following example. Consider, for $(x,y) \in \mathbb{R}^2$, the Hamiltonian

$$H(x,y) = \tfrac{1}{2}y^2 + \tfrac{1}{4}(x^+)^2,$$

where $x^+ = \max\{x,0\}$. This function vanishes along the half-line $\mathcal{L} = \{(x,y) \in \mathbb{R}^2 : x \leq 0,\ y = 0\}$. The picture in Figure 1, plotted with MAPLE software, gives an idea of the level curves of H, for different values of c. We see that such curves are definitively parallel to the line $\mathcal{L}$, in the sense that there exists a sector, determined, on $\mathbb{S}^1$, by a neighborhood of $\bar{\theta} = \pi$, such that the level curves γ_c, while lying in that sector, are parallel to $\mathcal{L}$.

One can thus wonder if a behavior of definitive parallelism between the level curves of H is always possible. To understand when this situation can occur, it suffices to notice that, if the level curve γ_c, $c > 0$, written in polar coordinates (ρ,θ), is a line having slope equal to $\tan\bar{\theta}$ for θ belonging to a (possibly only right or left) neighborhood of $\bar{\theta}$, then it is of the kind

$$\rho\sin\theta = (\tan\bar{\theta})\rho\cos\theta + m, \tag{2.7}$$

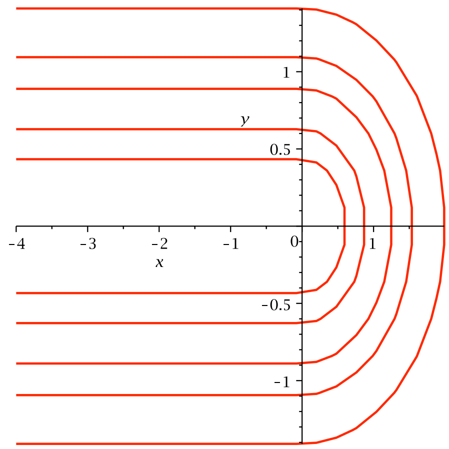

FIGURE 1. The level curves of $H(x, y) = \frac{1}{2}y^2 + \frac{1}{4}(x^+)^2$ for $c = 0.1, 0.2, 0.4, 0.6, 1$.

for a suitable $m \neq 0$. Since $H(\rho\cos\theta, \rho\sin\theta) = \rho^2 \hat{H}(\theta)$, finding explicitly ρ in (2.7) and inserting it into this last relation gives, on γ_c,

$$\frac{m^2}{(\sin\theta - \tan\bar{\theta}\cos\theta)^2}\hat{H}(\theta) = c.$$

We thus deduce the following: if there exist a neighborhood $\mathcal{U}$ of $\bar{\theta}$ and a constant $\bar{m} > 0$ such that, for every $\theta \in \mathcal{U}$,

$$\hat{H}(\theta) = \bar{m}(\sin\theta - \tan\bar{\theta}\cos\theta)^2, \tag{2.8}$$

at least as long as the level curves of H lie in the sector determined by $\mathcal{U}$ in the corresponding quadrant, they will be parallel to the half-line where H vanishes. By standard geometric considerations, this condition is necessary and sufficient. In the scalar case, for instance, defining $\hat{H}(\theta) = \sin^2\theta$ produces the Hamiltonian $H(x, y) = y^2$, whose level curves are horizontal lines, parallel to $\{y = 0\}$.

Notice that, as a consequence, a mixing behavior at infinity for the level curves (asymptotic on one side of $\bar{\theta}$ and definitively parallel on the other one) is possible, according to how $\hat{H}$ is defined in a neighborhood of $\bar{\theta}$. The same considerations hold in the case when H vanishes on a finite number of points on $\mathbb{S}^1$, repeating the arguments for every zero of $\hat{H}$; in particular, if $\bar{\theta}_1, \bar{\theta}_2$ are such that $\hat{H}(\bar{\theta}_1) = \hat{H}(\bar{\theta}_2) = 0$, and $\hat{H}(\theta) > 0$ for $\theta \in (\bar{\theta}_1, \bar{\theta}_2)$, then the level curves of H "in between" will be asymptotic or definitively parallel to the lines having $\bar\theta_1$ and $\bar\theta_2$ as slopes.

Lastly, assume that $\hat H$ vanishes on an infinite number of points. In this case, if $\hat\theta$ is such that $\hat H(\hat\theta) > 0$, by continuity $\hat H$ is positive in a neighborhood of $\hat\theta$, so that there exist $\bar\theta_1$, $\bar\theta_2$ with $\hat\theta \in (\bar\theta_1, \bar\theta_2)$, and

$$\hat H(\bar\theta_1) = \hat H(\bar\theta_2) = 0, \quad \hat H(\theta) > 0 \text{ if } \theta \in (\bar\theta_1, \bar\theta_2).$$

Therefore, the solution starting from a point of the form $\rho(\cos\hat\theta, \sin\hat\theta)$, $\rho > 0$, will behave according to the previous discussion. The dynamics of the system $Ju' = \nabla H(u)$ is thus clear also in this case. The following figures try to give an idea of the different situations mentioned. In the first one, a case of parallelism between the level curves is represented, with $\hat H$ (vanishing at $\bar\theta = \pi/4$ and at $\bar\theta = 5\pi/4$) defined as in (2.8), with $\bar m = 1$, for every θ. In

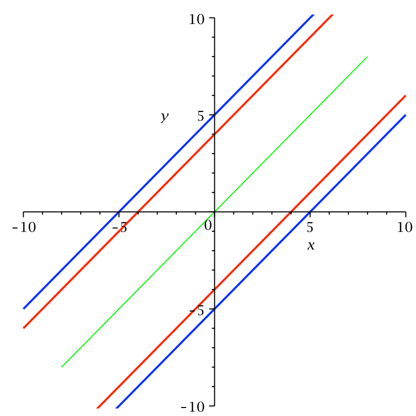

FIGURE 2. Two level curves of $H(x,y) = (x^2+y^2)(1-\sin(2\arctan(\frac{y}{x})))$.

the second one, the case of a mixing behavior is depicted; for $x > y$, the Hamiltonian is defined as the one for Figure 2, while for $x \le y$ it is (in polar coordinates) $H(\rho,\theta) = \rho^2(1-\cos^2 4\theta)$.

Remark 2.2. The arguments of the entire section would work for the more general class of systems having the form

$$Ju' = \zeta(t)\nabla H(u),$$

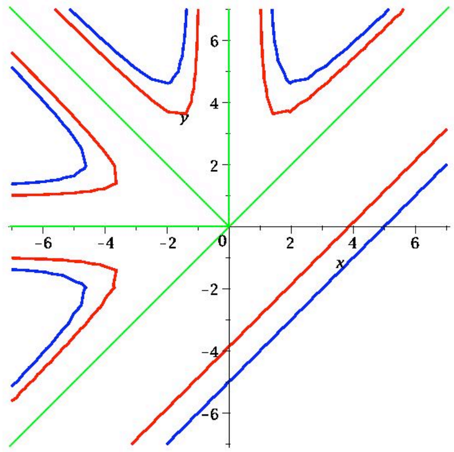

FIGURE 3. A mixing behavior: asymptotic and parallel.

with $\zeta : \mathbb{R} \to \mathbb{R}$ a nonnegative continuous function, since also in this case the function H is preserved along the solutions:

$$\frac{d}{dt}H(u(t)) = \langle \nabla H(u(t)) | u'(t) \rangle = \zeta(t) \langle J\nabla H(u(t)) | \nabla H(u(t)) \rangle = 0. \quad (2.9)$$

Indeed, if $u(t)$ solves $Ju' = \zeta(t)\nabla H(u)$, then, setting

$$Z(t) = \int_0^t \zeta(s)\, ds,$$

by uniqueness we have $u(t) = v(Z(t))$, for some suitable function $v(t)$ solving $Jv' = \nabla H(v)$. Thus, considering the further term $\zeta(t)$ corresponds only to changing the speed on the level curves of H.

3. THE MAIN RESULT

In this section, we will investigate the existence of a solution to the boundary-value problem

$$Ju' = F(t, u), \quad u(0) = u(T), \quad (3.1)$$

with $F : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ having the form

$$F(t, u) = \gamma(t, u)\nabla H_0(u) + (1 - \gamma(t, u))\nabla H_1(u) + r(t, u), \quad (3.2)$$

where $\gamma(t,u)$ and $r(t,u)$ are L^2-Carathéodory functions such that $0 \le \gamma(t,u) \le 1$, and

$$\lim_{|u|\to+\infty} \frac{r(t,u)}{|u|} = 0. \tag{3.3}$$

We will assume $H_0 \in \mathcal{P}^* \setminus \mathcal{P}$ and $H_1 \in \mathcal{P}$, with

$$H_0(u) \le H_1(u), \qquad \text{for every } u \in \mathbb{R}^2. \tag{3.4}$$

The interest in this situation arises from scalar second-order differential equations. In particular, (3.1) includes the case of double resonance involving the first eigenvalue of the linear problem (or, in the asymmetric case, the positive semi-axes in the plane where the Fučik spectrum is usually represented). As we have already remarked in the Introduction, the case when also H_0 belongs to $\mathcal{P}$ has been treated in [7].

We denote by τ_1 the minimal period associated with H_1, and we assume $\tau_1 = T$. Moreover, we fix φ satisfying $J\varphi' = \nabla H_1(\varphi)$ and $H_1(\varphi(t)) \equiv \frac{1}{2}$. Notice that, since $H_0 \in \mathcal{P}^* \setminus \mathcal{P}$, the strict inequality in (3.4) holds in a strip around each half-line passing through the origin and a zero of H.

We prove the following lemma.

Lemma 3.1. *Let $\alpha, \beta \in L^2(0,T)$ be such that, for almost every $t \in [0,T]$, $\alpha(t) \ge 0$, $\beta(t) \ge 0$, $\alpha(t) + \beta(t) \le 1$. Moreover, assume that $\alpha(t) + \beta(t) > 0$ for t belonging to a subset of $[0,T]$ having positive measure. Then, if $u(t)$ is a nontrivial solution to*

$$Ju' = \alpha(t)\nabla H_0(u) + \beta(t)\nabla H_1(u), \quad u(0) = u(T), \tag{3.5}$$

we have either $u(t) \equiv \xi$, with $H_0(\xi) = 0$ (and so $u(t)$ solves $Ju' = \nabla H_0(u)$), or $Ju' = \nabla H_1(u)$.

Proof. Since a nontrivial solution of (3.5) never reaches the origin, it is possible to pass to polar coordinates, writing $u(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$. We then have, by (3.5),

$$-\theta'(t) = 2\alpha(t)H_0(\cos\theta(t), \sin\theta(t)) + 2\beta(t)H_1(\cos\theta(t), \sin\theta(t)), \tag{3.6}$$

and, in particular, $\theta'(t) \le 0$ for almost every $t \in [0,T]$. This implies that $u(t)$ performs a nonnegative number of clockwise turns around the origin in the time T. On the other hand, in view of (3.4), (3.6) yields

$$\frac{-\theta'(t)}{2H_1(\cos\theta(t), \sin\theta(t))} \le 1. \tag{3.7}$$

Recalling that

$$\int_0^{2\pi} \frac{d\theta}{2H_1(\cos\theta, \sin\theta)} = \tau_1 = T,$$

and taking into account that $u(t)$ is T-periodic, integrating (3.7) from 0 to T we see that $u(t)$ performs at most one clockwise turn around the origin in the time T. Since $u(t)$ is T-periodic, the number of such revolutions around the origin is thus necessarily 0 or 1. In the second case, we pass to generalized polar coordinates by writing $u(t) = r(t)\varphi(t + \omega(t))$, and get, similarly as in [7, Lemma 2.3], the equations for $r'(t)$ and $\omega'(t)$:

$$r'(t) = -\alpha(t)r(t)\langle \nabla H_0(\varphi(t + \omega(t)))|\varphi'(t + \omega(t))\rangle,$$

$$\omega'(t) = 2\alpha(t)H_0(\varphi(t + \omega(t))) + \beta(t) - 1.$$

In view of (3.4) and the fact that $H_1(\varphi(t)) = \frac{1}{2}$ for every $t \in [0, T]$, $\omega'(t) \leq 0$ for almost every $t \in [0, T]$. Since, on the other hand, $\omega(0) = \omega(T)$ (otherwise $u(t)$ would wind around the origin a number of times different from 1), it follows that $\omega'(t) = 0$ almost everywhere. Notice that this necessarily implies $\beta(t) = 1 - \alpha(t)$ for almost every $t \in [0, T]$, otherwise this case is excluded. With the same reasoning as in [7, Lemma 2.3], one now concludes that $r'(t) = 0$ almost everywhere, implying that $u(t)$ satisfies $Ju' = \nabla H_1(u)$. Let us now analyze the first case. If $u(t)$ performs 0 turns around the origin, recalling (3.6) and the fact that $\theta'(t) \leq 0$ for almost every $t \in [0, T]$, we have

$$2\alpha(t)H_0(\cos\theta(t), \sin\theta(t)) + 2\beta(t)H_1(\cos\theta(t), \sin\theta(t)) = 0.$$

It immediately follows that $\beta(t) = 0$ almost everywhere (remember that $H_0 \geq 0$, $H_1 > 0$), and

$$-\theta'(t) = 2\alpha(t)H_0(\cos\theta(t), \sin\theta(t)) = 0 \tag{3.8}$$

for almost every $t \in [0, T]$. The function $u(t)$ is thus a solution of $Ju' = \alpha(t)\nabla H_0(u(t))$; by (2.9), H_0 is preserved along $u(t)$. Consequently, as already mentioned in Remark 2.2, it is possible to perform the same reasoning as in Proposition 2.2. In particular, if $H_0(u(0)) = 0$, then $u(t) \equiv u(0)$ (and thus $u(t)$ satisfies $Ju' = \nabla H_0(u)$); on the other hand, if $H_0(u(0)) > 0$, then $H_0(u(t)) > 0$ for every $t \in [0, T]$, so that (3.8) implies $\alpha(t) = 0$ for almost every $t \in [0, T]$, against the hypothesis. □

We now state our main result, which can be seen as a complement to the main theorem in [7].

Theorem 3.2. *In the previous setting* (*assuming, in particular,* (3.2)), *assume that for almost every* $t \in [0, T]$ *and every* $u \in \mathbb{R}^2$, *with* $|u| \leq 1$, *and for every* $\lambda \geq 1$,

$$\langle F(t, \lambda u)|u\rangle \geq \eta(t), \tag{3.9}$$

for a suitable $\eta \in L^2(0, T)$. *Moreover, suppose that the following two conditions hold:*

1) *for every* $\xi \in \mathbb{S}^1$ *satisfying* $H_0(\xi) = 0$,

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle F(t, \lambda\eta)|\eta\rangle \, dt > 0; \tag{3.10}$$

2) *for every* $\theta \in [0, T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} [\langle F(t, \lambda\varphi(t + \omega))|\varphi(t + \omega)\rangle - 2\lambda H_1(\varphi(t))] \, dt < 0. \tag{3.11}$$

Then, problem (3.1) *has a solution.*

Notice that condition (3.10) is invariant with respect to the dilatation $\xi \mapsto r\xi$, for $r > 0$, and this is the reason why it is sufficient to consider only the points $\xi \in \mathbb{S}^1$ such that $H_0(\xi) = 0$ and not the whole half-line emanating from the origin and passing through each of them. Wishing to make a comparison with condition (3.11), it is not strange that no correction terms are added to F under the integral sign in (3.10), since H_0 vanishes along the T-periodic solutions of $Ju' = \nabla H_0(u)$. On the other hand, since this time we do not have a natural star-shaped curve around the origin which permits us to pass to modified polar coordinates, when approaching the limit problem we need to control the behavior of the solutions separately in each direction, thus considering a "triple" inferior limit, over the three-dimensional variable $(\lambda, \eta) \in \mathbb{R} \times \mathbb{R}^2$.

Proof. Let us first observe that the problem

$$Ju' = \frac{1}{2}(\nabla H_0(u) + \nabla H_1(u)), \quad u(0) = u(T)$$

has only the trivial solution, thanks to Lemma 3.1 and to the fact that $\hat{H}_0(\theta) < \hat{H}_1(\theta)$ (using the notation in (2.3)) for θ belonging to a subset of $[0, 2\pi)$ having positive measure. Moreover, since $H_0 + H_1 \in \mathcal{P}$, for any open subset $\Omega \subset \mathbb{R}^2$

$$\deg_B(\nabla H_0 + \nabla H_1, \Omega, 0) = 1,$$

where $\deg_B$ denotes the Brouwer degree (see, for instance, [9, Lemma II.6.5]). Therefore, in view of [3, Theorem 2] it suffices to show that, uniformly in

$\sigma \in (0,1)$, it is possible to find an a priori bound in $L^\infty(0,T)$ for the solutions to

$$\begin{cases} Ju' = \sigma(\gamma(t,u)\nabla H_0(u) + (1-\gamma(t,u))\nabla H_1(u) + r(t,u)) \\ \qquad\qquad + \dfrac{(1-\sigma)}{2}(\nabla H_0(u) + \nabla H_1(u)) \\ u(0) = u(T). \end{cases} \tag{3.12}$$

By contradiction, assume that there exists $(u_n)_n \subset L^\infty(0,T)$ and $(\sigma_n)_n \subset (0,1)$ such that u_n satisfies (3.12) for $\sigma = \sigma_n$, and $\|u_n\|_\infty \to +\infty$. We can assume that $\sigma_n \to \bar\sigma \in [0,1]$. Setting $v_n(t) = u_n(t)/\|u_n\|_\infty$, for every n the function $v_n(t)$ satisfies

$$Jv_n' = \sigma_n \frac{F(t,u_n)}{\|u_n\|_\infty} + \frac{(1-\sigma_n)}{2}(\nabla H_0(v_n) + \nabla H_1(v_n)), \quad v_n(0) = v_n(T). \tag{3.13}$$

In view of (3.3), it follows that v_n is bounded in $H^1(0,T)$, so that there exists a nonzero $v \in H^1(0,T)$ such that, for instance, $v_n \rightharpoonup v$ (weakly) in $H^1(0,T)$ and $v_n \to v$ uniformly. Moreover, since $\|v_n\|_\infty = 1$ for every n, we have that v is nonzero. On the other hand, the sequence $(\gamma(\cdot, u_n(\cdot)))_n$ is bounded in $L^2(0,T)$, so there exists $\Gamma \in L^2(0,T)$ such that $\gamma(\cdot, u_n(\cdot)) \rightharpoonup \Gamma(t)$ in $L^2(0,T)$. Passing to the weak L^2-limit in (3.13), we thus reach, recalling (3.2), and in view of (3.3),

$$Jv' = \bar\sigma(\Gamma(t)\nabla H_0(v) + (1-\Gamma(t))\nabla H_1(v)) + \frac{1-\bar\sigma}{2}(\nabla H_0(v) + \nabla H_1(v));$$

that is,

$$Jv' = \Big(\bar\sigma\Gamma(t) + \frac{1-\bar\sigma}{2}\Big)\nabla H_0(v) + \Big(\bar\sigma(1-\Gamma(t)) + \frac{1-\bar\sigma}{2}\Big)\nabla H_1(v).$$

Notice that the case $\bar\sigma = 0$ is excluded, in view of the beginning of the proof. By Lemma 3.1, we either have v constant, satisfying $Jv' = \nabla H_0(v)$, or $Jv' = \nabla H_1(v)$. The second case can be treated exactly as in [7, Theorem 2.1], exploiting, in particular, the assumptions $H_0(u) \le H_1(u)$ and $H_1(\varphi(t)) \equiv \frac{1}{2}$. We thus reach a contradiction with condition (3.11).

We now focus on the first case, namely $v(t) \equiv \xi \in \mathbb{R}^2 \setminus \{0\}$, with $\nabla H_0(\xi) = 0$. In this situation, we have $\bar\sigma\Gamma(t) + \frac{1-\bar\sigma}{2} = 1$, which in turns implies $\bar\sigma = 1$ and $\Gamma(t) \equiv 1$. Passing to polar coordinates (thanks to the elastic property), so that $u_n(t) = \rho_n(t)(\cos\theta_n(t), \sin\theta_n(t))$, the expression of θ_n' is given by

$$-\theta_n'(t) = \frac{\sigma_n\langle F(t,u_n(t)) | u_n(t)\rangle}{\rho_n(t)^2} + \frac{1-\sigma_n}{\rho_n(t)^2}(H_0(u_n) + H_1(u_n)). \tag{3.14}$$

However, since $v_n \to \xi \neq 0$ uniformly, for n large $v_n(t)$ will not be able to turn around the origin, so that integrating (3.14) from 0 to T gives 0, for every n large. Recalling that $H_0 \geq 0$ and $H_1 > 0$, for every n large it follows that

$$\int_0^T \sigma_n \frac{\|u_n\|_\infty}{\rho_n(t)^2} \langle F(t, \|u_n\|_\infty v_n(t)) | v_n(t)\rangle \, dt < 0, \tag{3.15}$$

and, multiplying by $\|u_n\|_\infty$,

$$\int_0^T \sigma_n \frac{\|u_n\|_\infty^2}{\rho_n(t)^2} \langle F(t, \|u_n\|_\infty v_n(t)) | v_n(t)\rangle \, dt < 0. \tag{3.16}$$

Using Fatou's lemma thanks to (3.9), since $\frac{\|u_n\|_\infty^2}{\rho_n(t)^2}$ converges to a positive constant and $\sigma_n \to 1$, we have

$$\int_0^T \liminf_{n\to+\infty} \langle F(t, \|u_n\|_\infty v_n(t)) | v_n(t)\rangle \, dt \leq 0,$$

contradicting (3.10). □

Remark 3.1. As shown in [1, Section 4], Theorem 3.2 holds, in the same way, in the L^1-Carathéodory setting.

Remark 3.2. Theorem 3.2 includes also the case of simple resonance with a Hamiltonian $H_0 \in \mathcal{P}^* \setminus \mathcal{P}$ (see, as a guide, [7, Theorem 5.1], and, as a comparison term, [2, Section III.4]). Obviously, in this situation, only the Landesman-Lazer condition (3.10), relative to H_0, will be assumed. It is worth noticing that, while the Hamiltonians belonging to $\mathcal{P}$ are resonant with respect to the T-periodic problem only if the associated minimal period is a submultiple of T, a Hamiltonian belonging to $\mathcal{P}^* \setminus \mathcal{P}$ gives always rise to resonance (at least in the sense that $Ju' = \nabla H_0(u)$, with $H_0 \in \mathcal{P}^* \setminus \mathcal{P}$, always has a T-periodic solution which is nonzero).

Remark 3.3. Condition (3.10) can be rephrased in terms of the so-called *recession function*, according to the terminology used by Brézis and Nirenberg in [2]. For a function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$, the recession function of f is defined by

$$J_f(t,\xi) = \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle f(t,\lambda\eta) | \eta\rangle.$$

With this position, assumption 1) in the statement of Theorem 3.2 requires that $\int_0^T J_F(t,\xi)\, dt > 0$ for every ξ satisfying $\nabla H_0(\xi) = 0$. Speaking about *simple* resonance, in [2, Section III.4] condition (3.10) was used to give an existence result, for instance, for abstract systems of the kind $Au + g(t,u) =$

f, asking its validity for every ξ in the kernel of the linear operator A. In connection with the previous remark, considering the T-periodic problem associated with the system

$$Ju' = \nabla H_0(u) + r(t,u), \tag{3.17}$$

with $H_0 \in \mathcal{P}^* \setminus \mathcal{P}$, if ∇H_0 is linear (i.e., $H_0(u) = \frac{1}{2}\langle Bu|u\rangle$, with B a positive semidefinite square matrix having nontrivial kernel), then our result for (3.17) coincides with the one by Brézis and Nirenberg. On the other hand, if ∇H_0 is only homogeneous, a case which was not investigated in [2], wishing to use [2, condition (3.41)] we would have to ask the validity of (3.10) for every $\xi \in \mathbb{R}^2$, while we have seen that it is sufficient to impose it only for the zeros of H_0. However, it is worth mentioning that the results in [2] were stated in a general abstract setting, and applicable in arbitrary dimensions.

Remark 3.4. Notice that Theorem 3.2 allows us to consider the case when $H_0(u) \equiv 0$. In view of Remark 3.2, in particular, a situation like

$$Ju' = R(t,u),$$

with $R : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ an L^2-bounded function, is included in our framework, as well. This kind of simple resonance seems to be different from the usual concept of resonance for scalar second-order equations. However, since in this case (3.10) has to be verified for every point of $\mathbb{S}^1$, assumption 1) of Theorem 3.2 turns out to be a sign assumption on the product $\langle R(t,\xi)|\xi\rangle$ for every $\xi \in \mathbb{R}^2$ with $|\xi|$ sufficiently large.

Remark 3.5. Referring to (3.16), to achieve the same conclusion as in Theorem 3.2 we could also propose a kind of sign condition which is weaker than (3.10) and can be formulated as follows: for every $\xi \in \mathbb{S}^1$ satisfying $H_0(\xi) = 0$, there exist λ_ξ, $\delta_\xi > 0$ and $h_\xi \in L^2(0,T)$ with

$$\int_0^T h_\xi(t)\,dt \geq 0$$

such that, for almost every $t \in [0,T]$, for every η such that $|\eta - \xi| \leq \delta_\xi$ and every $\lambda \geq \lambda_\xi$,

$$\frac{\langle F(t,\lambda\eta)|\eta\rangle}{|\eta|^2} \geq h_\xi(t). \tag{3.18}$$

This requirement is similar to hypothesis (H) of [4, Theorem 1'] (for the scalar problem), which, however, does not seem to imply it. Let us show that (3.10) implies (3.18). To this aim, let ξ be a point where $H_0(\xi) = 0$, and set

$$l(t) = \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle F(t,\lambda\eta)|\eta\rangle;$$

by (3.10), choose $\epsilon > 0$ such that $\int_0^T (l(t) - \epsilon)\,dt > 0$. Accordingly, it will be possible to find $\lambda_0, \delta_0 > 0$ such that, for almost every $t \in [0,T]$, every $\lambda \geq \lambda_0$, and η with $|\eta - \xi| \leq \delta_0$,

$$\frac{\langle F(t,\lambda\eta)|\eta\rangle}{|\eta|^2} \geq \frac{l^+(t) - l^-(t) - \epsilon}{|\eta|^2},$$

where, as usual, $l^+(t) = \max\{l(t), 0\}$, and $l^-(t) = \max\{-l(t), 0\}$. Set

$$h_\xi(t) = \frac{l^+(t)}{\max_{|\eta-\xi|\leq\delta_0} |\eta|^2} - \frac{l^-(t)}{\min_{|\eta-\xi|\leq\delta_0} |\eta|^2} - \frac{\epsilon}{\min_{|\eta-\xi|\leq\delta_0} |\eta|^2};$$

i.e.,

$$h_\xi(t) = \frac{l^+(t)}{(|\xi| + \delta_0)^2} - \frac{l^-(t)}{(|\xi| - \delta_0)^2} - \frac{\epsilon}{(|\xi| - \delta_0)^2}.$$

Shrinking δ_0, if necessary, it is not difficult to see that $\int_0^T h_\xi(t)\,dt > 0$.

We now state the following corollary of Theorem 3.2.

Corollary 3.1. *Assume* (3.2) *and suppose that for almost every* $t \in [0,T]$ *and every* $u \in \mathbb{R}^2$, *with* $|u| \leq 1$, *and for every* $\lambda \geq 1$,

$$\langle r(t,\lambda u)|u\rangle \geq \eta(t),$$

for a suitable $\eta \in L^2(0,T)$. *Moreover, assume that*

1') *for every* $\xi \in \mathbb{S}^1$ *satisfying* $H_0(\xi) = 0$,

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle r(t,\lambda\eta)|\eta\rangle\,dt > 0; \tag{3.19}$$

2') *for every* $\theta \in [0,T]$,

$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle r(t,\lambda\varphi(t+\omega))|\varphi(t+\omega)\rangle\,dt < 0. \tag{3.20}$$

Then, problem (3.1) *has a solution.*

The sufficiency of assumption 2'), as a stronger requirement than assumption 2) of Theorem 3.2, has been discussed in [7, Corollary 2.7]. On the other hand, writing explicitly the expression of $F(t,u)$ in (3.14) and (3.15), we immediately see that 1') is sufficient to perform the same proof as above.

Remark 3.6. It is interesting to notice that, when the function $r(t,u)$ appearing in (3.2) does not depend on u, condition (3.19) can be used to get existence for a forced autonomous system like

$$Ju' = \nabla H_0(u) + r(t),$$

with $H_0 \in \mathcal{P}^* \setminus \mathcal{P}$. In particular, in this case (3.19) reads as

$$\int_0^T \langle r(t)|\xi\rangle \, dt > 0,$$

for every $\xi \in \mathbb{S}^1$ satisfying $H_0(\xi) = 0$.

4. The scalar case

We conclude the paper by briefly examining two corollaries of Theorem 3.2 for the scalar second-order case. From now on, we will focus on condition (3.19), which is easier to check in the applications, and make some considerations about it.

Remark 4.1. Let us consider the equation

$$x'' + h(t,x) = 0,$$

with $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ bounded. In this case, it is well known that the classical Landesman-Lazer condition at the first eigenvalue $\lambda_0 = 0$ reads as

$$\int_0^T \limsup_{x\to-\infty} h(t,x)\, dt < 0 < \int_0^T \liminf_{x\to+\infty} h(t,x)\, dt. \tag{4.1}$$

Setting $u = (x,y)$, $H_0(u) = \frac{1}{2}y^2$, and $r(t,u) = (h(t,x), 0)$, assumption 1') in Corollary 3.1 requires the following: for every constant $c \neq 0$,

$$\int_0^T \liminf_{(\lambda,\eta_1)\to(\infty,c)} h(t,\lambda\eta_1)\eta_1 \, dt > 0.$$

With computations similar to the ones in the proof of [7, Proposition 3.1], and in [8, page 159], this can be checked to be equivalent to (4.1).

This leads us to the following well-known corollary of Theorem 3.2 (see, for instance, [4, 5]).

Corollary 4.1. *Assume that $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function such that*

$$g(t,x) = \gamma_1(t,x)x + r(t,x),$$

with $0 \le \gamma_1(t,x) \le \lambda_1$, *where* $\lambda_1 = \left(\frac{2\pi}{T}\right)^2$, *and*

$$\lim_{|x|\to+\infty} \frac{r(t,x)}{x} = 0, \quad |x| \to +\infty.$$

Moreover, assume (4.1) *and suppose that, for every* ϕ *satisfying* $\phi''+\lambda_1\phi = 0$, *the following condition is satisfied:*

$$\int_{\{\phi>0\}} \limsup_{x\to+\infty}(g(t,x)-\lambda_1 x)\phi(t)\,dt + \int_{\{\phi<0\}} \liminf_{x\to-\infty}(g(t,x)-\lambda_1 x)\phi(t)\,dt < 0.$$

Then, the equation $x''+g(t,x)=0$ *has a* T*-periodic solution.*

Remark 4.2. Consider now the equation

$$x'' - \nu_0 x^- + h(t,x) = 0, \quad (\text{respectively } x'' + \mu_0 x^+ + h(t,x) = 0),$$

with $\nu_0 > 0$ (respectively $\mu_0 > 0$), and $h : [0,T]\times\mathbb{R}\to\mathbb{R}$ bounded. This equation describes a nonlinear oscillator where a one-sided restoring force acts (see, for instance, [14, 15]). In the plane (μ,ν) where the Fučik spectrum is usually represented, here the occurrence of resonance can be "visualized" on the vertical (respectively, horizontal) semi-axis, because the parameter μ (respectively, ν) is equal to 0. In this situation, assumption 1') in Corollary 3.1 is equivalent, via computations similar to the ones mentioned in Remark 4.1, to

$$\int_0^T \liminf_{x\to+\infty} h(t,x)\,dt > 0, \quad \Big(\text{resp. } \int_0^T \limsup_{x\to-\infty} h(t,x)\,dt < 0\Big).$$

This natural condition is the corresponding one-sided analogous of (4.1). For example, in the particular situation of the equation

$$x'' - \nu_0 x^- = p(t),$$

such a Landesman-Lazer condition reads as

$$\int_0^T p(t)\,dt < 0,$$

recovering a well-known classical result of existence (we refer again, in a more general setting, to [14, 15]).

We are thus led to the following corollary of Theorem 3.2 for the asymmetric case.

Corollary 4.2. *Assume that* $g : [0,T]\times\mathbb{R}\to\mathbb{R}$ *is an* L^2*-Carathéodory function such that*

$$g(t,x) = \gamma_1(t,x)x^+ - \gamma_2(t,x)x^- + r(t,x),$$

with

$$\lim_{|x|\to+\infty} \frac{r(t,x)}{x} = 0, \quad |x| \to +\infty,$$

and

$$\begin{aligned} 0 \le \gamma_1(t,x) \le b_+, &\qquad \text{(respectively } a_+ \le \gamma_1(t,x) \le b_+ \text{)}, \\ a_- \le \gamma_2(t,x) \le b_-, &\qquad \text{(respectively } 0 \le \gamma_2(t,x) \le b_- \text{)}, \end{aligned}$$

for suitable positive constants a_-, b_-, b_+ *(respectively* a_+, b_+, b_-*) such that*

$$\frac{\pi}{\sqrt{b_+}} + \frac{\pi}{\sqrt{b_-}} = T.$$

Moreover, assume that

$$\int_0^T \liminf_{x\to+\infty} g(t,x)\,dt > 0, \quad \Big(\text{respectively } \int_0^T \limsup_{x\to-\infty} g(t,x)\,dt < 0\Big), \tag{4.2}$$

and suppose that, for every ϕ *satisfying* $\phi'' + b_+\phi^+ - b_-\phi^- = 0$*, the following condition is satisfied:*

$$\int_{\{\phi>0\}} \limsup_{x\to+\infty} (g(t,x) - b_+x)\phi(t)\,dt + \int_{\{\phi<0\}} \liminf_{x\to-\infty} (g(t,x) - b_-x)\phi(t)\,dt < 0. \tag{4.3}$$

Then, the equation $x'' + g(t,x) = 0$ *has a* T*-periodic solution.*

Similarly as in Remark 4.2, referring to (4.2) in terms of the visual representation of the Fučik spectrum, since only one edge of the rectangle $[0, b_+] \times [a_-, b_-]$ (respectively $[a_+, b_+] \times [0, b_-]$) lies on an axis, only a one-sided Landesman-Lazer condition is needed, on the side corresponding to the coordinate vanishing on that axis.

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